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A note on approximate controllability of differential inclusion with non-instantaneous impulses

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Abstract

The article explores the study of approximation controllability in a particular subset of differential inclusion systems that include non-instantaneous impulses in the Banach space X. By using non-linear alternatives for Kakutani mappings, semi-group theory, and fixed-point theorems. Our approach yields noteworthy conclusions. Furthermore, we provide a concrete example to clarify how the suggested theoretical framework might be used in practice. An investigation of the concept of approximation controllability is crucial for comprehending the dynamics of a system, especially in situations when attaining exact control is difficult. When we consider systems with impulses that do not happen instantly, we face complexity that requires advanced mathematical techniques for a comprehensive examination. Our study expands the theoretical comprehension of controllability in complex systems and provides valuable insights for both theoretical investigation and practical applications in control theory.

Keywords: Approximate controllability, Differential inclusions, Non-instantaneous impulses, Analytic semi-group
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1 Introduction

Controllability represents a fundamental concept within mathematical control theory, serving as a pivotal informal attribute of dynamical control systems. This notion holds paramount importance in the realm of control theory, signifying the system's capacity to transition from any given initial state to a desired final state through a set of admissible controls. Specifically, approximate controllability entails guiding the system from a primary state to a vicinity of the final state. Originally introduced by Kalman in 1960, the concept of controllability has since garnered significant attention within the research community. Klamka conducted a comprehensive survey on the controllability of dynamical systems, contributing to the elucidation of its principles. Additionally, Balachandran et al. extended the understanding of controllability by establishing results applicable to non-linear systems situated in Banach spaces[12]-[24]

The dynamics of various evolving processes often encounter blunt alterations, such as those induced by harvesting activities or natural disasters. These events introduce short-term perturbations into the system, typically manifesting

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as impulses within a piecewise continuous framework. Within the literature, differential equations accommodating such impulsive effects are categorized into two main types. Initially, there exist individuals typified by instantaneous impulses, whereby the temporal extent of these alterations is markedly concise when juxtaposed with the overarching time frame of the phenomenon. [18]. Secondly, there exist systems governed by non-instantaneous impulses [11, 22], where the impulsive action commences at a predetermined fixed point and persists over a finite duration. M. Guo et al. [10] and Benchohra et al.[5] have laid the groundwork for understanding instantaneous impulsive differential equations. Simultaneously, Malik et al. [17] have advanced the understanding of second-order non-linear differential equations incorporating non-instantaneous impulses, elucidating facets such as the existence, uniqueness, and stability of solutions. Controllable impulsive differential inclusions have arisen as a versatile modelling apparatus spanning diverse domains, encompassing physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmaceutics, and optimal control. Noteworthy scholarly interest has been directed towards abstract functional differential equations and differential inclusions [1, 9, 14], with several authors elucidating controllability results in both differential equations and inclusions [1, 9, 14, 16, 19, 21], as well as in integro-differential equations and inclusions [4, 3, 15, 26, 27].

In light of the absence of literature addressing the approximate controllability for a specific category of differential inclusion systems featuring non-instantaneous impulses, we endeavour to address this gap in knowledge. To this end, we scrutinize the following impulsive differential inclusion model:

$$
\begin{aligned}
&x'(l) \in Ax(l) + Bu(l) + F(l, x(l)), \;\; l \in (s_i, l_{i+1}],\, i = 1, 2, \cdots, m,\\
&x(l_i^+) = g_i(l, x(l_i^-)), \;\; i = 1, 2, \cdots, m,\\
&x(l) = g_i(l, x(l_i^-)), \;\; l \in (l_i, s_i],\, i = 1, 2, \cdots, m,
\end{aligned} \tag{1.1}
$$

The framework under consideration involves an analytic semi-group $\{S(l) : l \geq 0\}$ of bounded linear operators on a Banach space $(X, \|.\|_x)$, where A serves as the infinitesimal generator of this semi-group. Here, the time interval $[0, T]$ is discretized into subintervals $[l_i, s_i]$ with $0 = l_0 = s_0 < l_1 < s_1 < l_2 \cdots \cdots l_m < s_m < l_{m+1} = T < \infty$. The map $F(l, x(l))$ denotes a non-empty bounded, closed, and convex multi-valued function, while the control functions $u \in L^2(J, U)$ are elements of a Banach space of admissible control functions, where $J = (s_i, l_{i+1}]$. Additionally, B represents a bounded linear operator from the control space U to the Banach space X, and $u(l) \in U$ represents the control function. Furthermore, the presence of non-instantaneous impulses during the intervals $(l_i, s_i]$ is characterized by the function $g_i(l, x(l_i^-))$, where $i = 1, 2, \cdots, m$. The functions f and g_i are appropriately defined, with specific definitions to be provided subsequently.

2 Preliminaries and assumptions

In the following paragraphs, we will outline the initial notions and essential terminology that are the foundation for our future conversations. We are examining Banach spaces $(X, \|.\|_x)$ and $(Y, \|.\|_y)$. The notation $\pounds(Y, X)$ denotes the Banach space comprising bounded linear operators from Y to X, endowed with its corresponding topology. Consider A as the infinitesimal generator of an analytic semigroup $\{S(t) : t \geq 0\}$ consisting of bounded linear operators on X. Additionally, let $\{\rho(A)\}$ denote the spectrum of A, representing the set of complex values for which the linear operator A admits a resolvent. Here, $B_r(x, X)$ signifies the closed ball centered at x with radius $r \geq 0$ in the Banach space X. Moreover, the Banach space $C(J, X)$ denotes the collection of all continuous functions mapping the interval J to the space X, equipped with the supremum norm,

$$
\|x\|_c = \max_{l \in J} \|x(l)\|, \;\; \forall\, x \in C. \tag{2.1}
$$

In the given scenario, where A acts as the infinitesimal generator of a statistical semi-group comprising of bounded linear operators $\{S(l) : l \geq 0\}$ that operate on a separable Banach space X containing the norm $\|\cdot\|_X$, it can be proven that there exist non-negative constants M and ω that satisfy the inequality:

$$
\|S(l)\| \leq Me^{\omega l}, \;\; \forall\; l \geq 0. \tag{2.2}
$$

This signifies that the norm of the operators within the semi-group is bounded by an exponential function of time, thereby demonstrating a fundamental property of the semi-group's behaviour over time. Given that the semi-group is bounded by M, denoted as $\|S(l)\| \leq M$, and that $0 \in \rho(A)$, indicating the invertibility of the operator A, it follows that A constitutes a closed linear operator on its domain $D(A)$ within the Banach space X [20]. Specifically, it holds that

$$\|x\| = \|Ax\| \tag{2.3}$$

for every x belonging to the set $D(A)$. In addition, the space $(D(A), \|\cdot\|_X)$ is a compact Banach space. The resolvent operator $R(\lambda, A) = (\lambda I + A)^{-1}$ associated with A is characterized as compact. Here, we define the Banach space of continuous functions equipped with a norm as $C(J, X)$. Specifically, for all $x \in C$, where J represents the domain of these functions, we express the norm $\|x\|_c$ as follows:

$$\|x\|_c = \sup_{l \in J} \|Ax(l)\| \tag{2.4}$$

Definition 2.1. A family of bounded linear operators $W(l)$ indexed by $l \in [0, T]$ is designated as a resolvent operator if it satisfies the following conditions:

1. Initialization: The operator $W(0)$ equals the identity operator (I), and its norm is bounded by $Ke^{\omega l}$ for some non-negative K and real ω, where l ranges over the interval $[0, T]$.
2. Strong Continuity: For every $x \in X$, the operator $W(l)$ exhibits strong continuity with respect to l over the interval I.
3. Controllability Operator Grammian: The resolvent operator $W(l)$ is defined as the integral $\int_0^l S(l-s)BB^*S^*(l-s)ds$, where S, B, and X represent relevant operators and spaces in the context of the problem.

Proposition 2.2. The family of bounded linear operator $W(l)$ is continuous for all $l > 0$ is the uniform operator topology of $\pounds(X, X)$.

Proposition 2.3. The family of bounded linear operator $AW(l)$ is continuous for all $l > 0$ is the uniform operator topology of $\pounds(X, X)$.

Definition 2.4. A multi-valued map $F : (J \times X) \to 2^X$ is termed Carathéodory if it satisfies the following conditions:

1. Measurability: For each $x \in X$, the mapping $F(l, x(l))$, where l varies, is measurable with respect to l.
2. Upper Semi-Continuity (u.s.c.): For each $l \in J$, the mapping $F(l, x(l))$, where x varies, is upper semi-continuous with respect to x.
3. Non-emptiness: For each fixed $x \in C(J, X)$, the set $S_{F,x}$ defined as $v \in L^1(J, X) : v(l) \in F(l, x(l))$ for almost every $l \in J$ is non-empty.

Let us consider a linear continuous mapping denoted as $\wp$, which operates from the space $L^1(J, X)$ to $C(J, X)$. Then, we define an operator as $\wp \ominus S_F : C(J, X) \to \wp(C(J, X))$, where for each x, $x \to \wp \ominus S_F(x) := \wp(S_{F,x})$. This operator is characterized by a closed graph.

Theorem 2.5. The linear operator $S(l - s_i)$ is characterized as an evolution operator. Furthermore, if $S(l, s_0)$ is compact for every $l \in [0, l_1]$, then it follows that $S(l, s_i)$ remains compact for all $l_i \le s_i \le l_{i+1}$, where $s_0 = l_0 = 0$.

Theorem 2.6. (Non-linear alternative for Kakutani maps) Consider a Banach space X and a closed, convex subset $Y \subseteq X$. Let $J \times X$ be an open subset of Y containing the origin $0 \in J \times X$. Suppose $\phi : J \times X \to 2^Y$ is an upper semi-continuous mapping, where 2^Y denotes the collection of nonempty, compact convex subsets of Y. Then, the theorem states that either:

1. ϕ possesses a fixed point in $J \times X$, or
2. There exist $v \in \partial(J \times X)$ and $\lambda \in (0, 1)$ such that $v \in \lambda\phi(v)$.

Definition 2.7. A multi-valued map Γ is termed upper semi-continuous (u.s.c.) on X provided that for each $x \in X$, the set $\Gamma(x, l)$ is a non-empty closed subset of X. Additionally, for any open set C in X containing $\Gamma(x, l)$, there exists an open neighborhood $\mathbf{N}$ of x_0 such that $\Gamma(\mathbf{N})$ is a subset of C.

Definition 2.8. The function set Γ is said to be totally continuous if the image of every bounded subset C of X under Γ is reasonably compact. If the non-empty multivalued map $\Gamma(x, l)$ is entirely continuous, then $\Gamma(x, l)$ is upper semi-continuous (u.s.c.) if and only if Γ has a closed graph. This implies that if two sequences $\{x_n\}$ and $\{\phi_n\}$ converge to x and ϕ respectively as n approaches infinity, and if $\{\phi_n\}$ belongs to the set $\Gamma x_n(l)$, then ϕ belongs to the set $\Gamma x(l)$. If there exists an element x in the set X such that x is a fixed point of the function Γ, then Γ has a fixed point.

The set $\mathcal{PC}([0,T],X)$ is defined as the collection of functions $x:[0,T]\to X$ that are continuous on each open interval $(l_i,l_{i+1}]$.For any i in the range 0 to m, the following condition must hold: there are limits $x(l_i^-)$ and $x(l_i^+)$ such that $x(l_i^-)=x(l_i)$.

Definition 2.9. A function $x(l)\in\mathcal{PC}([0,T],X)$ is said to be a mild solution of system (6.3) if

1. $x(0)=x_0$, for all $l\in[0,l_1]$.
2. $x(l_i^+)=g_i(l,x(l_i^-))$, for all $l\in(l_i,s_i]$, $i=1,2,\cdots,m$.
3. There exist a function $f\in L^1(J,X)$ such that $f(l)\in F(l,x(l))$ on J and

$$\begin{aligned}
x(l) =&S(l)x_0+\int_0^l S(l-\tau)f(\tau,x(\tau))d\tau+\int_0^l S(l-\tau)Bu(\tau)d\tau,\quad l\in[0,l_1],\\
x(l) =&S(l-s_i)g_i(s_i,x(l_i^-))+\int_{s_i}^l S(l-\tau)f(\tau,x(\tau))d\tau+\int_{s_i}^l S(l-\tau)Bu(\tau)d\tau,\qquad l\in(s_i,l_{i+1}],\\
x(l) =&g_i\Big(l,S(l_i-s_{i-1})g_i(s_{i-1},x(l_{i-1}^-))+\int_{s_{i-1}}^{l_i} S(l_i-\tau)f(\tau,x(\tau))d\tau+\int_{s_{i-1}}^{l_i} S(l_i-\tau)Bu(\tau)d\tau\Big),\ l\in(l_i,s_{i-1}].
\end{aligned}\tag{2.5}$$

Definition 2.10. System (6.3) is approximately controllable on J if $\overline{R(T,\phi)}$ is dense in X. Where

$$R(T,\phi)=\{x(T,\phi,u):u\in L^2(J,U)\}\tag{2.6}$$

$x(T,\phi,u)$ is mild solution of (6.3). We obtain the results, we introduce the set of resolvent operators and control operator Grammian is defined on X and the basic assumption on these operators :

1. $\Gamma_{s_i}^{l_{i+1}}=\int_{s_i}^{l_{i+1}}S(l_{i+1}-s)BB^*S^*(l_{i+1}-s)ds$.
2. $R(\lambda,\Gamma_{s_i}^{l_{i+1}})=(\lambda I+\Gamma_{s_i}^{l_{i+1}})^{-1}$.

where B^* and $S^*(.)$ are the adjoint operators of the B and $S(.)$ respectively. We also assume that the operator $R(\lambda,\Gamma_{s_i}^{l_{i+1}})$ satisfies the following hypothesis:

(V): $\lambda R(\lambda,\Gamma_{s_i}^{l_{i+1}})\to 0$ as $\lambda\to 0^+$ in the strong operator topology.

Hypothesis (**V**) holds if and only if the linear system (6.3) is approximately controllable on $[0,T]$.

3 Controllability for non-linear system

In this portion, we establish adequate criteria for the approximation controllability of a specific category of differential inclusions, expressed as equation (6.3), within Banach spaces. To accomplish this, we utilise the non-linear option offered by the Kakutani maps fixed point theorem. To provide evidence for our findings, we depend on the following hypotheses:

(V1): $S_0(l)$, $l>0$ is compact.

(V2): For B is bounded linear operator from U to X and $M_B>0$ then $\|B\|=M_B$.

(V3): For each positive number r and each x belonging to the set C such that $\|x\|_C\le r$, we require the existence of a function $L_{f,r}(.)$ in $L_1(J,\mathbb{R}^+)$ satisfying the condition:

$$\sup\{\|f\|:f(l)\in F(l,x(l))\}\le L_{f,r}(l),\ \text{ for } l\in J.$$

Here, $J=(s_i,l_{i+1}]$, where i ranges over non-negative integers.

(V4): The function $s_i\to L_{f,r}(s_i)\in L_1(J,\mathbb{R}^+)$, and there exists a $\gamma>0$, such that

$$\lim_{n\to\infty}\frac{\int_{s_i}^l L_{f,r}(s)ds}{r}=\gamma<+\infty.$$

(V5): There exists a positive constant L_{g_i} such that

$$\|g_i(l,x)-g_i(l,y)\|\le L_{g_i}\|x-y\|,$$

for all $x, y \in X$ and $l \in [l_i, s_i]$, $i = 1, 2, \cdots m$ Also, there exists a positive constant L_i, $i = 1, 2, \cdots, m$ such that

$$\|g_i(l, x)\| \leq L_i, \forall\, l \in J,\; x \in X,$$

Here, L_i is denoted as the impulsive constant. The claim posits that the system described by equation (6.3) exhibits approximate controllability under the condition that, for every $\lambda > 0$, there exists a continuous function denoted as $x(\cdot)$ such that

$$\begin{aligned} x(l) =& S(l - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l-\tau)f(\tau, x(\tau))d\tau + \int_{s_i}^{l} S(l-\tau)Bu(\tau)d\tau \qquad l \in (s_i, l_{i+1}]. \\ u(l) =& B^* S^*(l_{i+1} - s_i)R(\lambda, \Gamma_{s_i}^{l_{i+1}})p(x(l)), \end{aligned}$$

where

$$p(x(l)) = x(l_{i+1}) - S(l_{i+1} - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l_{i+1}} S(l_{i+1} - \tau)f(\tau, x(\tau))d\tau. \tag{3.1}$$

Theorem 3.1. Suppose that the hypotheses $(V1) - (V5)$ are satisfied. Assume also

$$M\left[1 + \frac{M^2 M_B^2 T}{\gamma}\right] < 1\cdot \tag{3.2}$$

where $\|B\| = M_B$, then system (6.3) has a solution on X.

Proof . In this section, our main focus is to delineate conditions conducive to the solvability of system (6.3) for $\lambda > 0$. We aim to illustrate that through the utilization of the control $u(x, l)$, the operator $\Gamma : C \to 2^C$ is defined as follows:

$$\Gamma(x) = \{\phi \in C : \phi(l) = S(l - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l-\tau)f(\tau, x(\tau))d\tau + \int_{s_i}^{t} S(l-\tau)Bu(\tau)d\tau\},$$

for $l \in (s_i, l_{i+1}]$, $i = 0, 1, 2, \cdots, m$, and $f \in S_{F,x}$, where $S_{F,x}$ represents the controllability operator Grammian. This operator Γ guarantees the existence of a fixed point $x(l)$, which serves as a mild solution of system (6.3). To establish that Γ fulfills all prerequisites outlined in Theorem (2.3), we delineate the proof into six distinct steps.

Step 1. Convexity of Γ for each $x \in C$: Given ϕ_1 and ϕ_2 belonging to $\Gamma(x)$, there exist f_1 and f_2 from $S_{F,x}$ such that for every $t \in J$, we can express the following:

$$\phi_i(l) \quad = \quad S(l - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l-\tau)f_i(\tau, x(\tau))d\tau + \int_{s_i}^{l} S(l-\tau)Bu(\tau)d\tau, \quad i = 1, 2.$$

Let $\lambda \in (0, 1)$ and for all $l \in J$, we get

$$\begin{aligned} \lambda\phi_1(l) + (1-\lambda)\phi_2(l) =& S(l - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l - s_i)[\lambda f_1(s_i, x(s_i))(1-\lambda)f_2(s_i, x(s_i))]ds \\ &+ \int_{s_i}^{l} S(l - s_i)BB^*S^*(l_{i+1} - s_i)R(\lambda, \Gamma_{s_i}^{l_{i+1}})[x(l_{i+1}) - S(l_{i+1} - s_i)g_i(s_i, x(l_i^-)) \\ &+ \int_{s_i}^{l} S(l_{i+1} - s_i)[\lambda f_1(s_i, x(s_i)) + (1-\lambda)f_2(s_i, x(s_i))]ds]ds. \end{aligned}$$

The convexity of $S_{F,x}$ is readily apparent due to the convex nature of the values assumed by the function F. Hence, we have:

$$\lambda f_1(s_i, x(s_i)) + (1-\lambda)f_2(s_i, x(s_i)) \in S_{F,x}.$$

Consequently,

$$(\lambda\phi_1(l) + (1-\lambda)\phi_2(l)) \in \Gamma(x), \quad l \in J.$$

Step 2. Let $B_r(x, X)$ be the set of elements x in X such that $\|x\|_{PC} < r$, where r is a positive number. Therefore, it may be concluded that B_r is a set that is both bounded, closed, and convex within the set C. Our claim is that there is a positive number r for which the set of all points within a distance r from a given point, denoted by $\Gamma(B_r)$, is completely contained within the set B_r. Assume that this statement is false. For any positive number r, there is a function x^r in B_r, but $\Gamma(x)^r$ is not in B_r, meaning that

$$\|x^r\|_C = \sup\{\|\phi_r\|_C : \phi_r \in (\Gamma(x_r))\} > r.$$

$$\phi_r(l) = S(l-s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l-s)f^r(s,x(s))ds + \int_{s_i}^{l} S(l-s)Bu^r(s)ds.$$

some $f^r(.\,,\,.) \in S_{F,x}$.

$$\begin{aligned}
r <&\|\Gamma(x^r(l))\| \leq \|S(l-s_i)g_i(s_i,x(l_i^-))\| + \int_{s_i}^{l} \|S(l-s)f^r(s,x(s))ds\| + \int_{s_i}^{l} \|S(l-s)Bu^r(s)ds\|. \\
<&\|\Gamma(x^r(l))\| \leq \|S(l-s_i)\|\|g_i(s_i,x(l_i^-))\| + \int_{s_i}^{l} \|S(l-s)\|\|f^r(s,x(s))\|\|ds\| \\
&+ \int_{s_i}^{l} \|S(l-s)\|\|B\|\|u^r(s)\|\|ds\|. \\
<&\left[ML_{g_i} + M\int_{s_i}^{l} L_{f^r(s)}ds\right] + \frac{1}{\lambda}M^2M_B^2\Bigg[\|x(l)_{i+1}\| + \|S(l_{i+1}-s_i)\|\|g_i(s_i,x(l_i^-))\| \\
&+ \int_{s_i}^{l} \|S(l_{i+1}-s)\|\|f(s,x(s))\|ds\|\Bigg]\|ds\| \\
<&\left[ML_{g_i} + M\int_{s_i}^{l} L_{f^r(s)}ds\right] + \frac{1}{\lambda}M^2M_B^2\left[\|x\| + ML_{g_i} + M\int_{s_i}^{l} L_{f^r(s)}ds\right].
\end{aligned}$$

By dividing both sides of the inequality by r and taking the limit as $r \to \infty$ using V_4, we obtain:

$$M\gamma\left[1 + \frac{1}{\lambda}M^2M_B^2T\right] \geq 1.$$

This contradicts condition (3.19). Therefore, there exists some $r > 0$ such that $\Gamma(B_r) \subseteq B_r$.

Step 3. The operator Γ demonstrates the property of mapping bounded sets to equicontinuous sets within C. For any given $x \in B_r$ and $\phi \in \Gamma(x)$, there exists an $f \in S_{F,x}$ such that

$$\phi(l) = S(l-s_i)g_i(s_i,x(l_i^-)) + \int_{s_i}^{l} S(l-s)f(s,x(s))ds + \int_{s_i}^{l} S(l-s)Bu(s)ds, \quad l \in (s_i, l_{i+1}]$$

Let $0 < \varepsilon < 0$ and $0 < s_i < l_{i+1} \leq T$ then

$$\begin{aligned}
\|\phi_1(l) - \phi_2(l)\| =&\|[S(l_1-s_i) - S(l_2-s_i)]g_i(s_i,x(l_i^-))\| + \|\int_{s_i}^{l_1-\varepsilon} [S(l_1-s_i) - S(l_2-s_i)]f(s,x(s))ds\| \\
&+ \|\int_{l_1-\varepsilon}^{l_1} [S(l_1-s_i) - S(l_2-s_i)]f(s,x(s))ds\| + \|\int_{l_1}^{l_2} S(l_2-s_i)f(s,x(s))ds\| \\
&+ \|\int_{s_i}^{l_1-\varepsilon} [S(l_1-s_i) - S(l_2-s_i)]Bu(s)ds\| \\
&+ \|\int_{l_1-\varepsilon}^{l_1} [S(l_1-s_i) - S(l_2-s_i)]Bu(s)ds\| + \|\int_{l_1}^{l_2} S(l_2-s_i)Bu(s)ds\| \\
\leq&\|S(l_1-s_i) - S(l_2-s_i)\|L_{g_i} + \int_{s_i}^{l_1-\varepsilon} \|S(l_1-s_i) - S(l_2-s_i)\|f_r(s)ds \\
&+ \int_{l_1-\varepsilon}^{l_1} \|S(l_1-s_i) - S(l_2-s_i)\|f_r(s)ds + M\int_{l_1}^{l_2} f_r(s)ds \\
&+ M_B\int_{s_i}^{l_1-\varepsilon} \|S(l_1-s_i) - S(l_2-s_i)\|\|u(s)\|ds \\
&+ M_B\int_{l_1-\varepsilon}^{l_1} \|S(l_1-s_i) - S(l_2-s_i)\|\|u(s)\|ds + MM_B\int_{l_1}^{l_2} \|u(s)\|ds.
\end{aligned}$$

As $(l_1 - l_2) \to 0$ and ε becomes sufficiently small, the right-hand side of the expression tends to zero regardless of $x \in B_r$. This phenomenon arises from the compactness exhibited by the operator $S(l, s)$, guaranteeing continuity within the uniform operator topology. As a result, $\Gamma(x_r)$ systematically transforms B_r into an equicontinuous array of functions.

Step 4. Consider the set $G(l) = \{\phi(l) : \phi \in \Gamma(B_r)\}$, which is deemed pre-compact within the space X. Let $l \in (s_i, l_{i+1}]$ be fixed, and let ϵ denote a real number satisfying $s_i < \epsilon < l_{i+1} \leq T$ for $x \in B_r$. Our objective is to establish that $\Gamma(x)$ systematically maps B_r into a pre-compact set within X. We define

$$\phi_\epsilon(l) = S(l - s_i - \epsilon)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l-\epsilon} S(l - s - \epsilon)f(s, x(s))ds + \int_{s_i}^{l-\epsilon} S(l - s - \epsilon)Bu(s)ds.$$

Since $S(l, s)$ is a compact operator, the set $Y_\epsilon(l) = \{\Gamma x(l) : x \in B_r\}$ is pre-compact in X for every ϵ, where $s_i < \epsilon < l_{i+1}$. Moreover, for every $x \in B_r$ we have:

$$\|\phi(l) - \phi_\epsilon(l)\| \leq M \int_{s_i}^{l-\epsilon} f_r(s)ds + MM_B \int_{s_i}^{l-\epsilon} \|u(s)\|ds.$$

Hence, there exist relatively compact sets in proximity to the set $Y_\epsilon(l) = \{\Gamma x(l) : x \in B_r\}$. This implies that the set $Y(l) = \{\Gamma x(l) : x \in B_r\}$ is compact within X, thereby affirming the completeness of the operator Γ.

Step 5. The operator $\Gamma x(l)$ exhibits upper semi-continuity (u.s.c.) if and only if Γ possesses a closed graph, i.e., if $\{x_n\} \to x$ as $n \to \infty$ and $\{\phi_n\} \in \Gamma x_n(l)$, then $\phi_n \to \phi x(l)$ as $n \to \infty$. To demonstrate that $\phi \in \Gamma x(l)$, we establish the existence of $f_n \in S_{F,x_n}$ such that

$$\begin{aligned}\phi_n(l) =& S(l - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l - s_i)f_n(s_i, x(s_i))ds \\ &+ \int_{s_i}^{l} S(l - s_i)BB^*S^*(l_{i+1} - s_i)R(\lambda, \Gamma_{s_i}^{l_{i+1}})[x(l_{i+1}) - S(l_{i+1} - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l_{i+1} - s)f_n(s, x(s))ds]ds.\end{aligned}$$

We must proof that there exist a $f \in S_{F,x}$ such that

$$\begin{aligned}\phi(l) =& S(l - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l - s_i)f(s_i, x(s_i))ds \\ &+ \int_{s_i}^{l} S(l - s_i)BB^*S^*(l_{i+1} - s_i)R(\lambda, \Gamma_{s_i}^{l_{i+1}})\Big[x(l_{i+1}) - S(l_{i+1} - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l_{i+1} - s)f(s, x(s))ds\Big]ds.\end{aligned}$$

Control sequence $u_{x_n}(l) \to u_x(l)$ as $n \to \infty$ and $l \in (s_i, l_{i+1}]$. Then

$$u_x(l) = B_*S_*(l_{i+1} - s_i)\left[x(l_{i+1}) - S(l_{i+1} - s_i)g_i(s_i, x(l_i^-))\right]$$

clearly, we have

$$\|f_n(l) - f(l)\| \to 0. \text{ as } n \to \infty.$$
$$\|u_{x_n}(l) - u_x(l)\| \to 0. \text{ as } n \to \infty.$$

Hence

$$\|\phi_n(l) - \phi(l)\| \to 0. \; as \; n \to \infty.$$

Consider the operator $\wp : L_1(J, X) \to C$,

$$(\wp f)(l) = \int_{s_i}^{l} S(l - s)\Big[f(s) - BB^*S^*(l_{i+1} - s_i)\Big(\int_{s_i}^{l_{i+1}} S(l_{i+1} - s_i)f(s)ds\Big)(s)\Big]ds.$$

The operator $\wp$ is linear and continuous. it follows that $(\wp f)(l)$ is a closed graph operator. Moreover,

$$\begin{aligned}&\Big[\phi_n(l) - S(l - s_i)g_i(s_i, x(l_i^-)) - \int_{s_i}^{l} S(l - s_i)f_n(s_i, x(s_i))ds - \int_{s_i}^{l} S(l - s_i)BB^*S^*(l_{i+1} - s_i) \\ &R(\lambda, \Gamma_{s_i}^{l_{i+1}})\Big[x(l_{i+1}) - S(l_{i+1} - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l_{i+1} - s)f_n(s, x(s))ds\Big]ds\Big] \in \wp(S_{F,x_n})\end{aligned}$$

Then the sequence $\{x_n\} \to x$ as $n \to \infty$. It follows that

$$(\phi(l) - S(l-s_i)g_i(s_i, x(l_i^-)) - \int_{s_i}^{l} S(l-s_i)f(s_i, x(s_i))ds - \int_{s_i}^{l} S(l-s_i)BB^*S^*(l_{i+1}-s_i)R(\lambda, \Gamma_{s_i}^{l_{i+1}})$$
$$[x(l_{i+1}) - S(l_{i+1}-s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l_{i+1}-s)f(s, x(s))ds]ds) \in \wp(S_{F,x})$$

Consequently, the operator $\Gamma x(l)$ satisfies the condition of possessing a closed graph. Therefore, Γ exhibits a closed graph.

Consequently, all the steps, coupled with the Arzela-Ascoli theorem, are duly satisfied. Hence, we can infer that Γ admits a fixed point $x(t)$, which serves as a mild solution to system (1.2). □

Theorem 3.2. Let X denote a Banach space, and let F represent a bounded subset of $L^1[(s_i, l_{i+1}), X]$. The equicontinuity of F holds if and only if F constitutes a relatively compact subset of $L^1[(s_i, l_{i+1}), X]$, equipped with the weak topology.

Theorem 3.3. Suppose all the given assumptions hold. Then, there exists $\mathbb{K}^* \in L_1(J, \mathbb{R}^+)$ such that $\sup_{x\in X} \|F(l,x)\| \leq \mathbb{K}^*(l)$ for almost every $l \in (s_i, l_{i+1}]$, where $i = 0, 1, 2, \cdots, m$. Consequently, the non-linear differential inclusion (1.2) is approximately controllable on $(s_i, l_{i+1}]$ for each i.

Proof . Let $x^{\mho}$ denote a fixed point of Γ within B_r. According to the above theorem, any fixed point of Γ serves as a mild solution of (1.2) within the framework of the approximate control system utilizing the impulse condition, then

$$u^{\mho}(l) = B^*S^*(l_{i+1}-s_i)R(\lambda, \Gamma_{s_i}^{l_{i+1}})p(x^{\mho}(l)).$$

and satisfies the following

$$x^{\mho}(l_{i+1}) = x(l_{i+1}) + \lambda R(\lambda, \Gamma_{s_i}^{l_{i+1}})p(x^{\mho}(l)).$$

Moreover, leveraging the assumption regarding $F(x,l)$ and the Dunford-Pettis Theorem, we can establish the existence of a subsequence denoted as $\{f^{\mho}(s)\}$ that exhibits weak compactness within $L^1((s_i, l_{i+1}), X)$. Consequently, there emerges a subsequence, also labeled $\{f^{\mho}(s)\}$, which weakly converges to, for instance, $f(s)$ in $L^1((s_i, l_{i+1}), X)$. Defined

$$H = x(l_{i+1}) - S(l_{i+1}-s_i)g_i(s_i, x(l_i^-)) - \int_{s_i}^{l_{i+1}} S(l_{i+1}-\tau)f(\tau, x(\tau))d\tau.$$

We take norm of

$$\begin{aligned}\|p(x^{\mho}(l) - H\| =& \| \int_{s_i}^{l_{i+1}} S(l_{i+1}-s_i)\Big[f(s_i, x^{\mho}(s_i)) - f(s_i, x(s_i))\Big]ds_i\| \\ \leq& \sup_{l\in(s_i,l_{i+1}]} \| \int_0^l S(l-s_i)\Big[f(s_i, x^{\mho}(s_i)) - f(s_i, x(s_i))\Big]ds_i\|.\end{aligned}$$

By using the Ascoli-Arzela theorem, one can show that an operator

$$\ell(.) \to \int_0^{\alpha} S(\alpha - s_i)\ell(s_i)ds_i : L^1((s_i, l_{i+1}), X) \to C$$

is a compact. now we implies that $\|p(x^*(l) - H\| \to 0$ as $\lambda \to 0$. Then, we get

$$\begin{aligned}\|x^{\mho}(l_{i+1}) - x(l_{i+1})\| \leq& \|\lambda R(\lambda, \Gamma_{s_i}^{l_{i+1}})(H)\| + \|\lambda R(\lambda, \Gamma_{s_i}^{l_{i+1}})\|\|p(x^{\mho}(l) - H\| \\ \leq& \|\lambda R(\lambda, \Gamma_{s_i}^{l_{i+1}})(H)\| + \|p(x^*(l) - H\|.\end{aligned}$$

If the given condition (**V1**) satisfied then the above line $\|x^{\mho}(l_{i+1}) - x(l_{i+1})\| \to 0$ as $\lambda \to 0$. Hence proved that problems (1.2) is approximate controllability of differential inclusion with non-instantaneous impulses. □

4 Non-local problems

The investigation of non-local Cauchy problems holds significant relevance due to their prevalence in various physical phenomena. Byszewski [6] is credited as the pioneering scholar to delve into the examination of non-local initial value problems. Consideration is directed towards a specific class of controllable non-local differential inclusion problems characterized by non-instantaneous impulses. This can be succinctly represented by the following system of equations:

$$
\begin{aligned}
&x'(l) \in Ax(l) + Bu(l) + F(l, x(l)), \quad l \in (s_i, l_{i+1}],\ i = 1, 2, \ldots, m,\\
&x(l_i^+) = g_i(l, x(l_i^-)), \quad l \in (l_i, s_i],\ i = 1, 2, \ldots, m,\\
&x(l) = x_0 + q(x).
\end{aligned} \tag{4.1}
$$

In this formulation, the dynamics of the system are governed by a differential inclusion involving matrices A and B, along with a function $F(l, x(l))$. Additionally, the system incorporates impulses at specific time points l_i, where the behaviour of the solution $x(l)$ undergoes a prescribed transition represented by $g_i(l, x(l_i^-))$. Furthermore, the initial condition x_0 is augmented by a term $q(x)$, contributing to the overall state $x(l)$.

[**(V6):**] Given a continuous function $q : ([0, T], X) \to X$, it is asserted that there exists a positive constant K_q such that

$$\|q(x) - q(y)\| \leq K_q \|x - y\|.$$

To establish the approximate controllability of the system (4.1), it is required to demonstrate the existence of a continuous function $x(\cdot)$ for all $\lambda > 0$, satisfying the following relations:

$$
\begin{aligned}
&x(l) = S(l - s_i) g_i(s_i, x(l_i^-)) + \int_{s_i}^{l} S(l - \tau) f(\tau, x(\tau)) d\tau + \int_{s_i}^{l} S(l - \tau) Bu(\tau) d\tau, \quad l \in (s_i, l_{i+1}],\\
&u(l) = B^* S^* (l_{i+1} - s_i) R(\lambda, \Gamma_{s_i}^{l_{i+1}}) p(x(l)),
\end{aligned}
$$

where

$$p(x(l)) = x(l_{i+1}) - S(l_{i+1} - s_i) g_i(s_i, x(l_i^-)) + \int_{s_i}^{l_{i+1}} S(l_{i+1} - \tau) f(\tau, x(\tau)) d\tau. \tag{4.2}$$

Here, $S(\cdot)$ denotes the semi-group generated by A, B^* represents the ad-joint operator of B, and $R(\lambda, \Gamma_{s_i}^{l_{i+1}})$ is a certain operator determined by parameters λ and the evolution operator $\Gamma_{s_i}^{l_{i+1}}$.

Definition 4.1. A function $x(l) \in \mathcal{PC}([0, T], X)$ is termed a mild solution of system (4.1) under the following conditions:

1. At the initial time $l = 0$, the function satisfies $x(0) = x_0 + q(x)$ for all l in the interval $[0, l_1]$.
2. Across each interval $(l_i, s_i]$, the function adheres to the constraint $x(l_i^+) = g_i(l, x(l_i^-))$ for $i = 1, 2, \ldots, m$.
3. There exists a function $f \in L^1(J, X)$ such that $f(l) \in F(l, x(l))$ on J, and the function $x(l)$ satisfies the following integral equations:

$$
\begin{aligned}
&x(l) = S(l)(x_0 + q(x)) + \int_0^l S(l - \tau) f(\tau, x(\tau)) d\tau + \int_0^l S(l - \tau) Bu(\tau) d\tau, \quad l \in [0, l_1],\\
&x(l) = S(l - s_i) g_i(s_i, x(s_i)) + \int_{s_i}^{l} S(l - \tau) f(\tau, x(\tau)) d\tau + \int_{s_i}^{l} S(l - \tau) Bu(\tau) d\tau, \quad l \in (s_i, l_{i+1}],\\
&x(l) = g_i\Big(l, S(l_i - s_{i-1}) g_i(s_{i-1}, x(s_{i-1})) + \int_{s_{i-1}}^{l_i} S(l_i - \tau) f(\tau, x(\tau)) d\tau + \int_{s_{i-1}}^{l_i} S(l_i - \tau) Bu(\tau) d\tau\Big), \quad l \in (l_i, s_{i-1}].
\end{aligned}
$$

Theorem 4.2. Assuming that hypotheses $(V1) - (V6)$ hold, along with the additional condition

$$M \left[1 + \frac{M^2 M_B^2 T}{\gamma} \right] < 1\cdot \tag{4.3}$$

we can conclude that the system (4.1) admits a solution on X.

Proof . The proof methodology for establishing the existence of a solution to the non-local controllable differential inclusion system (4.1) mirrors that of system (6.3). □

5 Integro-differential equation

In this section, we examine an approximate control system characterized by an integro-differential inclusion within the domain:

$$\begin{aligned} &x'(l) \in Ax(l) + Bu(l) + F(l, x(l)) + \int_0^l k(l-s)h(s, x(s))ds, \quad l \in (s_i, l_{i+1}],\ i = 1, 2, \dots, m, \\ &x(l_i^+) = g_i(l, x(l_i^-)), \quad l \in (l_i, s_i],\ i = 1, 2, \dots, m, \\ &x(l) = x_0. \end{aligned} \tag{5.1}$$

This formulation describes the evolution of the system over intervals $(s_i, l_{i+1}]$, where A and B represent matrices governing the dynamics, $F(l, x(l))$ signifies a function of l and $x(l)$, and $k(l-s)h(s, x(s))$ captures the influence of an integral term over the interval. The system also incorporates transitions at specific times l_i, where the behaviour of the solution $x(l)$ is governed by $g_i(l, x(l_i^-))$. Furthermore, the initial condition is denoted by x_0.

(V7): It is assumed that the function $h : (J \times X) \to X$ is continuous, with a positive constant L_h satisfying the inequality:

$$\|h(l, x_1) - h(l, x_2)\| \leq L_h \|x_1 - x_2\|.$$

(V8): The function $K : J \to C$ is continuous such that

$$k_T = \int_0^T \|k(s)\| ds.$$

where C is set of continuous functions. We aim to demonstrate the approximate controllability of system (5.1). For any given $\lambda > 0$, the system is considered approximately controllable if there exists a continuous function $x(l)$ satisfying the following equations:

$$\begin{aligned} x(l) = {} & S(l - s_i)g_i(s_i, x(l_i^-)) + \int_{s_i}^l S(l-\tau)f(\tau, x(\tau))d\tau + \int_{s_i}^l S(l-\tau)Bu(\tau)d\tau \\ & + \int_{s_i}^l S(l-\tau)\left[\int_0^l k(l-s)h(s, x(s))ds\right]d\tau, \quad l \in (s_i, l_{i+1}]. \\ u(l) = {} & B^* S^*(l_{i+1} - s_i)R(\lambda, \Gamma_{s_i}^{l_{i+1}})p(x(l)), \end{aligned}$$

where

$$\begin{aligned} p(x(l)) = {} & x(l_{i+1}) - S(l_{i+1} - s_i)g_i(s_i, x(l_i^-)) - \int_{s_i}^{l_{i+1}} S(l_{i+1} - \tau)f(\tau, x(\tau))d\tau \\ & - \int_{s_i}^{l_{i+1}} S(l_{i+1} - \tau)\left[\int_0^l k(l-s)h(s, x(s))ds\right]d\tau. \end{aligned}$$

Definition 5.1. A function $x(l) \in \mathcal{PC}([0, T], X)$ is deemed a mild solution of system (5.1) if it satisfies the following conditions:

(a) $x(0) = x_0$ holds for all l within the interval $[0, l_1]$.

(b) At each transition time l_i, the function adheres to the constraint $x(l_i^+) = g_i(l, x(l_i^-))$ for $i = 1, 2, \dots, m$.

(c) There exists a function $f \in L^1(J, X)$ such that $f(l) \in F(l, x(l))$ on J and

$$\begin{aligned} x(l) = {} & S(l)x_0 + \int_0^l S(l-\tau)f(\tau, x(\tau))d\tau + \int_0^l S(l-\tau)Bu(\tau)d\tau \\ & + \int_0^l S(l-\tau)\Big[\int_0^l k(l-s)h(s, x(s))ds\Big]d\tau, \quad l \in [0, l_1], \end{aligned}$$

$$\begin{aligned} x(l) = {} & S(l - s_i)g_i(s_i, x(s_i)) + \int_{s_i}^l S(l-\tau)f(\tau, x(\tau))d\tau + \int_{s_i}^l S(l-\tau)Bu(\tau)d\tau \\ & + \int_{s_i}^l S(l-\tau)\Big[\int_0^l k(l-s)h(s, x(s))ds\Big]d\tau,\ l \in (s_i, l_{i+1}], \end{aligned}$$

$$\begin{aligned}x(l) =&g_i\bigg(l, S(l_i - s_{i-1})g_i(s_{i-1}, x(s_{i-1})) + \int_{s_{i-1}}^{l_i} S(l_i - \tau)f(\tau, x(\tau))d\tau \\ &+ \int_{s_{i-1}}^{l_i} S(l_i - \tau)Bu(\tau)d\tau + \int_{s_i}^{l} S(l - \tau)\Big[\int_0^l k(l - s)h(s, x(s))ds\Big]d\tau\bigg),\ l \in (l_i, s_{i-1}].\end{aligned}$$

Theorem 5.2. Suppose that the hypotheses $(V1) - (V5)$ and $(V7) - (V8)$ are satisfied.Then the system (5.1) is approximately controllable on z. Assume also

$$M\left[1 + \frac{M^2 M_B^2 T}{\gamma}\right] < 1\cdot \tag{5.2}$$

then the system (5.1) has a solution on $X\cdot$.

Proof . The proof of controllable integro-differential inclusion system is same as system (6.3). □

6 Application

Let us now consider the heat flow problem in finite length with using non-instantaneous impulse condition of the following :

$$\begin{aligned}&\frac{\partial}{\partial l}Z(l, y) \in \frac{\partial^2}{\partial y^2}Z(l, y) + b(y)W(l, y) + F(l, Z(l, y)), \quad y \in (0, \pi), \quad l \in (2i, 2i + 1], \ \ i \in \{0\} \cup \mathbb{N}, \\ &Z(l, 0) = 0, l \in [0, T],\ 0 < T < \infty, \\ &Z(l, \pi) = 0, l \in [0, T], \\ &Z(l, y) = (\sin il)Z((2i - 1)^-, y),\ l \in (2i - 1, 2i], i = 1, 2, \cdots, m, \\ &Z(0, y) = x_0 \quad y \in (0, \pi),\end{aligned} \tag{6.1}$$

Consider a domain defined by $0 = l_0 = s_0 < l_1 < s_1 < l_2 \cdots < l_m < s_m < l_{m+1} = T$. Within this domain, let $Z(l, y)$ denote the temperature at point y at time l. Additionally, let $\{F(l, x) : (2i, 2i + 1] \times X \to 2^X\}$ represent a non-empty, bounded, closed, and convex multivalued mapping. The function space X is characterized by $X = L^2[0, \pi]$. We define the operator A as follows:

$$Ax = x'',$$

where the domain $D(A)$ is specified as:

$$D(A) = \{x \in X \ \ : x'' \in X \text{ and } x(0) = x(\pi) = 0\}. \tag{6.2}$$

The operator A engenders a strongly continuous semi-group $\{S(l) : l \geq 0\}$ distinguished by traits of compactness, analyticity, and self-adjointness. Additionally, A manifests a discrete spectrum, characterized by eigenvalues denoted as $\{\lambda_n = n^2 : n \in \mathbb{N}\}$, alongside corresponding normalized eigenvectors $\eta_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. The semi-group $\{S(l)\}$ possesses certain pertinent properties:

(i) For $x \in D(A)$, the action of A on x is expressed as

$$Az = \sum_{n=1}^{\infty} n^2\langle x, \eta_n\rangle\eta_n.$$

(ii) For all $x \in X$, the semi-group $S(l)$ acts on x as

$$S(l)x = \sum_{n=1}^{\infty} e^{-n^2 l}\langle x, \eta_n\rangle\eta_n,$$

wherein $S(l)$ is uniformly stable, and $\|S(l)\|_{L^2[0,\pi]} \leq e^{-l}$ holds particularly true. These properties delineate fundamental aspects of the behaviour and structure of the semi-group generated by the operator A, contributing to its mathematical characterization and analytical significance. Consider the linear operator B belonging to the space of bounded linear operators from a Banach space U to a Hilbert space X, denoted as $L(U,X)$. The operator B is defined as follows:

$$Bu(l)(y) = b(y)W(l,y), \;\; 0 < y < \pi, \;\; b(y) \in L^2(0,\pi).$$

Equation (6.1) can be equivalently represented as an abstract equation in the Hilbert space $X = L^2(0,\pi)$:

$$\begin{cases} x'(l) \in Ax(l) + Bu(l) + F(l,x(l)), \;\; l \in (s_i, l_{i+1}], \, i = 1,2,\cdots,m, \\ x(l_i^+) = g_i(l, x(l_i^-)), \;\; i = 1,2,\cdots,m, \\ x(l) = g_i(l, x(l_i^-)), \;\; l \in (l_i, s_i], \, i = 1,2,\cdots,m, \end{cases} \tag{6.3}$$

Let $x(l) = Z(l,\cdot)$, where $x(l)(y) = Z(l,y)$ for $y \in (0,\pi)$. The functions $g_i(l, x(l_i^-)) = (\sin il)Z((2i-1)^-, y)$ represent non-instantaneous impulses occurring during the intervals $(l_i, s_i]$. The operator A remains consistent with the definition provided in equation (6.2). In the context of non-local conditions, the function $q(x)$ can be represented as:

$$q(x) = \sum_{k=1}^{n} \beta_k \, x(l_k),$$

where $l_k \in J$ for all $k = 1,2,3,\ldots,n$, and β_k denote constants. These formulations delineate the mathematical representation of the functions and operators involved, providing a systematic framework for analysing the non-local conditions within the given context.

7 Conclusion

This article examines the delicate domain of approximation controllability within a specific subset of differential inclusion systems involving non-instantaneous impulses within the Banach space X. Our technique, which uses non-linear alternatives such as Kakutani mappings, semi-group theory and fixed-point theorems, has revealed substantial discoveries. Furthermore, we provide a concrete illustration to clarify the actual use of the suggested theoretical framework.

The study of approximation controllability is crucial for comprehending the dynamics of mechanisms, especially in situations where gaining precise control is difficult. Complexity emerges with systems, including non-instantaneous impulses, demanding the application of advanced mathematical approaches for a complete analysis. Our study has enhanced the theoretical comprehension of controllability in complex systems, offering valuable insights for both theoretical investigation and practical implementation in control theory. Our research elucidates this complex idea, enhancing the progress of knowledge in this subject and facilitating further inquiry and innovation.

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