Int. J. Nonlinear Anal. Appl. In Press, 1–6
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2025.34509.5153

Properties of multi subspace-supercyclic operators and C_0-semigroups

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(Communicated by Hamid Khodaei)

Abstract

In this paper, we describe multi-subspace-supercyclic operators and investigate them. We prove that any subspace-supercyclic operator and multi-supercyclic operator are multi-subspace-supercyclic. Also, we prove that an operator is multi-subspace-supercyclic if and only if any powers of it are multi-subspace-supercyclic. Furthermore, we define a subspace-supercyclic C_0-semigroup. We state that as a C_0-semigroup that contains a multi-subspace-supercyclic operator, it is multi-subspace-supercyclic.

Keywords: Subspace-supercyclic operators, Multi subspace-supercyclic operators, Subspace-supercyclic C_0-semigroups, Multi subspace-supercyclic C_0-semigroups
2020 MSC: Primary 47A16, Secondary 47B37

1 Introduction and Preliminaries

Let H be an infinite-dimensional and separable Hilbert space. We mention the set of all continuous and linear operators on H by $B(H)$. In the theory of dynamical systems, various types of operators are defined and investigated. If there is $h \in H$ such that $\overline{orb(T,h)} = H$, then T is hypercyclic, where

$$orb(T,h) = \{h, Th, T^2h, ...\}.$$

If for some $h \in H$,

$$\mathbb{C}orb(T,h) = \{\lambda T^n h, \lambda \in \mathbb{C} \text{ and } n \in \mathbb{N}_0\}$$

is dense in H, then T is called a supercyclic operator [4]. By definition, hypercyclic operators are supercyclic. Hypercyclicity, supercyclicity and related topics are considered for decades. One can see more about them in [4].

If there is $\{h_1, h_2, ..., h_n\} \subseteq H$ so that $\bigcup_{i=1}^{n} \mathbb{C}orb(T, h_i)$ is dense in H, then T is multi-supercyclic [10]. It is proved in [10, Theorem 4] that multi-supercyclic operators are supercyclic.

In [7] Subspace-hypercyclic operators were described. An operator $T \in B(H)$ is subspace-hypercyclic with respect to a closed and nontrivial subspace M of H if there is $h \in H$ such that $\overline{orb(T,x) \cap M} = M$. As it proved in [7],

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Received: June 2024 *Accepted:* February 2025

subspace-hypercyclic operators can be found only on infinite-dimensional spaces. There are some sufficient conditions for subspace-hypercyclicity in [1]. Bamerni, Kadets and Kilicman in [2] answered to a question that is mentioned in [7] by stating the following theorem.

Theorem 1.1. ([2]) Let $A \subseteq H$ be so that $\overline{A} = H$. Then, there exists a non-trivial closed subspace M of H such that $\overline{A \cap M} = M$.

By Theorem 1.1, authors stated in [2] that any hypercyclic operator is subspace-hypercyclic. Since when an operator T on H is hypercyclic, there exists $z \in H$ such that $\overline{\{z, Tz, T^2z, ..., T^nz, ...\}} = H$. Hence, if we consider $A := \{z, Tz, T^2z, ..., T^nz, ...\}$, by Theorem 1.1, there is a closed and nontrivial subspace Z of H such that $\overline{A \cap Z} = Z$. This means $\overline{\{z, Tz, T^2z, ..., T^nz, ...\} \cap Z} = Z$.

Subspace-supercyclic operators are introduced in [12] as follows.

Definition 1.2. An operator $T \in B(H)$ is called subspace-supercyclic with respect to a closed subspace M of H if $\overline{\mathbb{C}orb(T, h) \cap M} = M$ for some $h \in H$.

Authors in [12] constructed some examples of this type of operators. They also proved several theorems about them. Zhang and Zhou in [11] stated a subspace-supercyclicity criterion and some criteria equivalent to it. Furthermore, by Theorem 1.1 one can conclude that supercyclic operators are subspace-supercyclic. In [9] one can see some examples of operators that are subspace-supercyclic but they are not subspace-hypercyclic. Also, it is proved that subspace-supercyclic operators exist on finite-dimensional spaces. Moreover, we have the following theorem in [9]

Theorem 1.3. Let $T \in B(H)$ be an invertible operator. If T is subspace-supercyclic, then T^n and T^{-n} are subspace-supercyclic for any $n \in \mathbb{N}$.

Now it is natural to define multi subspace-supercyclic operators. In Section 2, we describe and investigate multi subspace-supercyclic operators. We prove that any multi-supercyclic operator is multi subspace-supercyclic. Also, we prove that multi subsace-supercyclicity of an operator implies multi subspace-supercyclicity of any powers of it and vice versa.

A C_0-semigroup on H is a family of operators $(T_t)_{t\geq 0}$ so that $T_0 = I$ and for any $s, t \geq 0$, $T_{s+t} = T_sT_t$ and $\lim_{t\to s} T_th = T_sh$, for any $h \in H$ [4]. $(T_t)_{t\geq 0}$ on H is hypercyclic if $\overline{\{T_th : t \geq 0\}} = H$ for some $h \in H$ [4]. It is supercyclic if $\overline{\{\gamma T_th : t \geq 0, \gamma \in \mathbb{C}\}} = H$ for some $h \in H$ [5]. Also, supercyclicity of translation C_0-semigroups is investigated in [6]. Some criteria for supercyclicity of C_0-semigroups are presented in [5]. In [8], one can see more about subspace-hypercyclic C_0-semigroups. Furthermore, subspace-supercyclic C_0-semigroups is defined in [3] as follows.

Definition 1.4. ([3, Definition 2.1]) A C_0-semigroup $(T_t)_{t\geq 0}$ on H is subspace-supercyclic with respect to a closed and nontrivial subspace M of H so that $\{\gamma T_th : t \geq 0, \gamma \in \mathbb{C}\} \cap M$ is dense in M for some $h \in H$.

In Section 3, we describe multi subspace-supercyclic C_0-semigroups. This section states that as a C_0-semigroup is contains a multi subspace-supercyclic operator, it is multi subspace-supercyclic. Also, multi subspace-supercyclicity of some discretization of a C_0-semigroup implies its multi subspace-supercyclicity.

2 Multi Subspace-supercyclic Operators and their properties

We start this section by defining the multi subspace-supercyclic operators. In the rest of paper, M always denotes a closed and non-zero subspace of H.

Definition 2.1. Let $T \in B(H)$. T is multi subspace-supercyclic with respect to M or multi M-supercyclic if $\{h_1, h_2, ..., h_n\} \subseteq H$ can be found so that

$$\overline{\bigcup_{i=1}^{n} (\mathbb{C}orb(T, h_i) \cap M)} = M.$$

A set $F = \{h_1, h_2, ..., h_n\} \subseteq H$ is minimal for multi M-supercyclicity for $T \in B(H)$ if

$$\overline{\bigcup_{x_i \in E} (\mathbb{C}orb(T, h_i) \cap M)} \neq M,$$

for any $E \subset F$.

By Definition 2.1 one can conclude that subspace-supercyclic operators are multi subspace-supercyclic. As mentioned in the first section, supercyclic operators are subspace-supercyclic. So, we can conclude that supercyclic operators are multi subspace-supercyclic.

In the consequence theorem we prove that multi-supercyclic operators are multi subspace-supercyclic.

Theorem 2.2. Suppose $T \in B(H)$ is a multi-supercyclic operator. Then, there is a closed and non-trivial subspace M of H such that T is multi subspace-supercyclic.

Proof . Let $h_1, h_2, ..., h_n \in H$ such that $\overline{\bigcup_{1 \leq i \leq n} \mathbb{C}orb(T, h_i)} = H$. By Theorem 1.1, there exists a closed and non-trivial subspace M of H such that $(\bigcup_{1 \leq i \leq n} \mathbb{C}orb(T, h_i)) \cap M$ is dense in M. On the other hand,

$$\overline{\bigcup_{1 \leq i \leq n} (\mathbb{C}orb(T, h_i) \cap M)} = \overline{(\bigcup_{1 \leq i \leq n} \mathbb{C}orb(T, h_i)) \cap M}.$$

So, T is multi M-supercyclic. □

In the next theorem, we prove that multi subspace-supercyclicity of T^n can be concluded from multi subspace-supercyclicity of T for any $n \in \mathbb{N}$.

Theorem 2.3. Let $T \in B(H)$. If T is multi M-supercyclic, then for every $n \in \mathbb{N}$, T^n is multi M-supercyclic operator.

Proof . For $n = 1$, the proof is clear. Suppose $n \geq 2$. By multi M-supercyclicity of T, there are $h_1, h_2, ..., h_m$ in H such that $\bigcup_{1 \leq i \leq m} (\mathbb{C}orb(T, h_i) \cap M)$ is dense in M. Let $y_{i,j} = T^j h_i$, where $1 \leq i \leq m$ and $1 \leq j \leq n-1$. Consider that

$$\bigcup_{1 \leq i \leq m} (\mathbb{C}orb(T, h_i) \cap M) = \bigcup_{\substack{1 \leq i \leq m \\ 0 \leq j \leq n-1}} (\mathbb{C}orb(T^n, y_{i,j}) \cap M). \tag{2.1}$$

The left side of (2.1) is dense in M. So, the right is dense in M too. Therefore, T^n is multi subspace-supercyclic with respect to M. □

By using Theorem 2.2 and Theorem 2.3, the following corollary is concluded.

Corollary 2.4. Suppose $T \in B(H)$ is a multi-supercyclic operator. Then, there is a closed and non-trivial subspace M of H such that T^n is multi M-supercyclic for any $n \in \mathbb{N}$.

Proof . Let us assume T is multi-supercyclic. By theorem 2.2, there exists $M \subseteq H$ such that T is multi subspace-supercyclic with respect to it. By accordance of Theorem 2.3, T^n is multi subspace-supercyclic for any $n \in \mathbb{N}$.

□

In the following theorem, a sufficient condition for multi subspace-supercyclicity is stated.

Theorem 2.5. Let $T \in B(H)$. If T^n is multi M-hypercyclic for some $n \in \mathbb{N}$, then T is multi M-supercyclic.

Proof . Let n be a positive integer greater than or equal to 2 such that T^n is multi subspace-supercyclic with respect to M. So there are $x_1, x_2, ..., x_m \in H$ such that

$$\overline{\bigcup_{1 \leq i \leq m} (\mathbb{C}orb(T^n, x_i) \cap M)} = M. \tag{2.2}$$

But

$$\mathbb{C}orb(T^n, x_i) \cap M \subseteq \mathbb{C}orb(T, x_i) \cap M. \tag{2.3}$$

Now by (2.2) and (2.3), we conclude that $\bigcup_{1\leq i\leq m}(\mathbb{C}orb(T, x_i) \cap M)$ is dense in M. Therefore T is multi M-supercyclic. □

By Theorem 2.3 and Theorem 2.5 we can conclude the following corollary.

Corollary 2.6. Let $T \in B(H)$. Then T is multi M-supercyclic if and only if T^n is multi M-supercyclic for any $n \in \mathbb{N}$.

In the next theorem, we show that if T is subspace-supercyclic, then any power of it is multi subspace-supercyclic.

Theorem 2.7. Let $T \in B(H)$ be an M-supercyclic operator. Then T^n is multi M-supercyclic for any $n \in \mathbb{N}$.

Proof . When $n = 1$, the assertion is evident. Presume $n \geq 2$. Let y be an M-supercyclic vector for T. So, $\overline{\mathbb{C}orb(T, y) \cap M} = M$. Let $h_1 := y, h_2 := Ty, ..., h_n := T^{n-1}y$. Hence,

$$\begin{aligned}\bigcup_{j=1}^{n}(\mathbb{C}orb(T^n, T^{j-1}y) \cap M) &= (\mathbb{C}orb(T^n, y) \cup \mathbb{C}orb(T^n, Ty) \cup ... \cup \mathbb{C}orb(T^n, T^{n-1}y)) \cap M \\ &= \mathbb{C}\{y, Ty, ..., T^{n-1}y, T^n y, T^{n+1}y, ...\} \cap M \\ &= \mathbb{C}orb(T, y) \cap M.\end{aligned}$$

Therefore, T^n is multi M-supercyclic. □

The consequence example can be make by using Theorem 2.7.

Example 2.8. For a supercyclic operator $T \in B(H)$ if we consider $T \oplus I : H \oplus H \to H \oplus H$, then $T \oplus I$ is subspace-supercyclic with respect to $M := H \oplus \{0\}$. Theorem 2.7 asserts that $(T \oplus I)^n = T^n \oplus I$ is multi M-supercyclic for any $n \in \mathbb{N}$.

Corollary 2.9. Presume $T \in B(H)$. If $T^n = I$ for some $n \in \mathbb{N}$ with $n \geq 2$, then T is not subspace-supercyclic.

Proof . Consider that $T^n = I$ for some $n \in \mathbb{N}$ with $n \geq 2$. Suppose on contrary that T is subspace-supercyclic. By Theorem 2.7, T^n is multi subspace-supercyclic. But this is impossible since the identity operator is not multi subspace-supercyclic. □

3 Multi Subspace-supercyclic C_0-semigroups and their properties

In this section we define and investigate the concept of multi supercyclicity and multi subspace-supercyclicity for C_0-semigroups as follows.

Definition 3.1. A C_0-semigroup $(T_t)_{t\geq 0}$ on H is multi supercyclic if there exists $\{h_1, h_2, ..., h_n\} \subseteq H$ such that

$$\overline{\bigcup_{i=1}^{n} \mathbb{C}orb((T_t)_{t\geq 0}, h_i)} = H.$$

A set $F = \{h_1, h_2, ..., h_n\} \subseteq H$ is minimal set for multi supercyclicity for $(T_t)_{t\geq 0}$ if

$$\overline{\bigcup_{h_i\in E} \mathbb{C}orb((T_t)_{t\geq 0}, h_i)} \neq H,$$

for any $E \subset F$. It is evident that supercyclic C_0-semigroups are multi supercyclic. Anatural extention of the Definition 3.1 is the consequent definition.

Definition 3.2. A C_0-semigroup $(T_t)_{t\geq 0}$ on H is multi subspace-supercyclic with respect to M or multi M-supercyclic if there exists $\{h_1, h_2, ..., h_n\} \subseteq H$ such that

$$\overline{\bigcup_{i=1}^{n}(\mathbb{C}orb((T_t)_{t\geq 0}, h_i) \cap M)} = M.$$

We say $F = \{h_1, h_2, ..., h_n\} \subseteq H$ is a minimal set for multi M-supercyclicity for $(T_t)_{t\geq 0}$ if for any $E \subset F$ we have

$$\overline{\bigcup_{h_i \in E}(\mathbb{C}orb((T_t)_{t\geq 0}, h_i) \cap M)} \neq M.$$

A subspace-supercyclic C_0-semigroup is multi subspace-supercyclic by Definition 3.2. As the following theorem asserts, multi subspace-supercyclicity of an operator of a C_0-semigroup results to multi subspace-supercyclicity of the C_0-semigroup.

Theorem 3.3. Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on H. If there exists $s_0 > 0$ so that T_{s_0} is multi subspace-supercyclic, $(T_t)_{t\geq 0}$ is multi subspace-supercyclic.

Proof . By hyopothesis, there is $s_0 > 0$ so that $(T_t)_{t\geq 0}$ is multi subspace-supercyclic. So, there exists a closed and nontrivial subspace M of H and $\{h_1, h_2, ..., h_n\} \subseteq H$ so that

$$\overline{\bigcup_{i=1}^{n}(\mathbb{C}orb(T_{s_0}, h_i) \cap M)} = M.$$

Therefore,

$$M = \overline{\bigcup_{i=1}^{n}(\mathbb{C}orb(T_{s_0}, h_i) \cap M)} \subseteq \overline{\bigcup_{i=1}^{n}(\mathbb{C}orb((T_t)_{t\geq 0}, h_i) \cap M)} \subseteq M.$$

This means $\overline{\bigcup_{i=1}^{n}(\mathbb{C}orb((T_t)_{t\geq 0}, h_i) \cap M)} = M$ and hence, $(T_t)_{t\geq 0}$ is multi subspace-supercyclic. □

In accordance with Theorem 3.3 and Theorem 2.5, the next corollary is concluded.

Corollary 3.4. Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If there exist $s_0 > 0$ and $p \in \mathbb{N}$ so that $T_{s_0}^p$ is multi subspace-supercyclic, $(T_t)_{t\geq 0}$ is multi subspace-supercyclic.

Proof . Assume there exist $s_0 > 0$ and $p \in \mathbb{N}$ so that $T_{s_0}^p$ is multi subspace-supercyclic. By Theorem 2.5, T_{s_0} is multi subspace-supercyclic. Now, Theorem 3.3 asserts that $(T_t)_{t\geq 0}$ is multi subspace-supercyclic.

□

Discretization of a C_0-semigroup is mentioned in the Section 1. The next theorem proves that multi subspace-supercyclicity of a discretization of a C_0-semigroup implies its multi subspace-supercyclicity.

Theorem 3.5. Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on H. If a discretization of $(T_t)_{t\geq 0}$ is multi subspace-supercyclic, then $(T_t)_{t\geq 0}$ is multi subspace-supercyclic.

Proof . Let $(T_{t_n})_{n\in\mathbb{N}}$ be a discretization of $(T_t)_{t\geq 0}$ so that $(T_{t_n})_{n\in\mathbb{N}}$ is multi subspace-supercyclic with respect to a closed and nontrivial subspace M. Hence, a set $\{h_1, h_2, ..., h_n\} \subseteq H$ exists so that

$$\overline{\bigcup_{i=1}^{n}(orb((T_{t_n})_{n\in\mathbb{N}}, h_i) \cap M)} = M.$$

Moreover,

$$M = \overline{\bigcup_{i=1}^{n}(orb((T_{t_n})_{n\in\mathbb{N}}, h_i) \cap M)} \subseteq \overline{\bigcup_{i=1}^{n}(\mathbb{C}orb((T_t)_{t\geq 0}, h_i) \cap M)} \subseteq M.$$

This means $(T_t)_{t\geq 0}$ is multi subspace-supercyclic. □

For a C_0-semigroup $(T_t)_{t\geq 0}$, $(T_{t_n})_{n\in\mathbb{N}}$ is titled an autonomous discretization of $(T_t)_{t\geq 0}$ if $t_n = nq$ for some $q > 0$. As a consequence of Theorem 3.5, we can state the next corollary.

Corollary 3.6. Presume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If an autonomous discretization of $(T_t)_{t\geq 0}$ is multi subspace-supercyclic, then $(T_t)_{t\geq 0}$ is multi subspace-supercyclic.

References

[1] A. Augusto and L. Pellegrini, *On the existence of subspace-hypercyclic operators and a new criteria for subspace-hypercyclicity*, Adv. Oper. Theory **5** (2020), no. 4, 1814–1824.

[2] N. Bamerni, V. Kadets, and A. Kilicman, *Hypercyclic operators are subspace-hypercyclic*, J. Math. Anal. Appl. **435** (2016), no. 2, 1812–1815.

[3] M. El Berrag and A. Tajmouati, *On subspace-supercyclic semigroup*, Commun. Korean Math. Soc. **33** (2018), 157–164.

[4] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*, Springer, 2011.

[5] A. Kumar and S. Srivastava, *Supercyclicity criteria for C_0-semigroups*, Adv. Oper. Theory **5** (2020), no. 4, 1646–1666.

[6] Y.X. Liang and Z.H. Zhou, *Supercyclic translation C_0-semigroup on complex sectors.* Dyn. Syst. **36** (2016), no. 1, 361–370.

[7] B.F. Madore and R.A. Martiınez-Avendano, *Subspace hypercyclicity*, J. Math. Anal. Appl. **373** (2011), 502–511.

[8] M. Moosapoor, *Some criteria for subspace-hypercyclicity of C_0-semigroups*, Aust. J. Math. Anal. Appl. **18** (2021), no. 1, Article 12, 8 pp.

[9] M. Moosapoor, *On subspace-supercyclic operators*, Aust. J. Math. Anal. Appl. **17** (2020), Art. 4, 8 pp.

[10] A. Peris, *Multi-hypercyclic operators are hypercyclic*, Math. Z. **236** (2001), 779–786.

[11] L. Zhang and Z.H. Zhou, *Notes about subspace-supercyclic operators*, Ann. Funct. Anal. **6** (2015), 60–68

[12] X.F. Zhao, Y.L. Shu, and Y.H. Zhou, *Subspace-supercyclicity and common subspace-supercyclic vectors*, J. East China Norm. Univ. Natural Sci. **1** (2012), 106.