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A certain coupled system of q-FDEs on two consecutive intervals under Dirichlet conditions via Krasnoselskii's theorem

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Abstract

In this study, we use certain mathematical tools to analyze the solutions of a system of fractional q-differential equation ${}^{C}\mathbb{D}_q^{\sigma_i}[\wp](\mathfrak{t}) = \mathfrak{w}_i(\mathfrak{t}, \wp(\mathfrak{t}), {}^{C}\mathbb{D}_q^{i\nu_j}[\wp](\mathfrak{t}), \mathbb{I}_q^{i\nu_j}[\wp](\mathfrak{t}))$, $i = 1$ whenever $\mathfrak{t} \in [0, \mathfrak{t}_0]$, and $i = 2$ whenever $\mathfrak{t} \in [\mathfrak{t}_0, 1]$, for $j = 1, 2$, such as fixed point theorem of Krasnoselskii and Banach contraction principle, under simultaneous Dirichlet boundary conditions. Here, we use standard definitions of the Liouville-Caputo fractional type $q-$derivative and Riemann-Liouville $q-$integral. Some illustrative examples with numerical results are discussed, too.

Keywords: nonlinear fractional equation, fractional $q-$differential equation, Dirichlet boundary conditions, Riemann–Liouville $q-$integral
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1 Introduction: Problem's formulation

Mathematical subjects in the analysis of the problems of today's world are really welcomed by researchers. Among others, the study of new mathematical models have been a growing filed of study due to its importance and applications in diverse discipline of science and engineering. In this respect, using factional or non-integer derivatives provides more insight for the description of natural phenomena in the language of mathematical modeling [16, 17]. Many interesting real-life models with fractional derivatives have been proposed and analyzed mathematically. Among others, we can refer to the ppopulation models [11, 14], the blood ethanol system [18], the viscolastic models [9], the Layla and Mojnun's love story [13], the HBV, HIV and SEIR infection models [10, 20, 29], and the human liver model [25], to name a few. For definitions of fractional derivatives and integrals and some related special functions we refer to the recently published papers on the subject [27, 28]. In the meantime, fractional differential and q-differential equations (FDE, FqDE) are significant, see [1, 3, 4, 15, 21, 30, 31].

The following FDE investigated by Ahmad *et al.* in 2014 as

$$\begin{cases} {}^{c}D_q^{\beta}({}^{c}D_q^{\gamma} + \lambda)u(t) = pf(t, u(t)) + kI_q^{\xi} g(t, u(t)), & 0 < \beta, \gamma \leq 1, \\ \alpha_1 u(0) - \beta_1 (t^{(1-\gamma)} D_q u(0))\big|_{t=0} = \sigma_1 u(\eta_1), \quad \alpha_2 u(1) + \beta_2 D_q u(1) = \sigma_2 u(\eta_2), & \end{cases} \tag{1.1}$$

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where $t, q \in [0,1]$ and ${}^cD_q^\beta$ is the fractional Liouville-Caputo q-derivative. Moreover, the symbol $I_q^\xi(.)$ stands for the Riemann-Liouville integral for $\xi \in (0,1)$ and the functions f and g are two continuous functions. Finally, the parameters λ, p, k are real numbers. Similarly, we have $\alpha_\ell, \beta_\ell, \sigma_\ell \in \mathbb{R}$ and $\eta_\ell \in (0,1)$ for $\ell = 1,2$ ([5]). Furthermore, Abdeljawad *et al.*, considered (with proof) a novel discrete $q-$fractional version of the well-known Grönwall inequality: $({}_qC_a^\alpha f)(t) = T(t, f(t))$ and $f(a) = \gamma$ in a way that $\alpha \in (0,1]$, $a \in \mathbb{T}_q = \{q^n : n \in \mathbb{Z}\}$, t belongs to $\mathbb{T}_a = [0,\infty)_q = \{q^{-\ell}a : \ell = 0,1,2,\ldots\}$. Here, the notation ${}_qC_a^\alpha$ shows the Liouville-Caputo fractional difference of order α and a Lipschitz condition for the function $T(t,x)$ holds for all t and x [1]. Later in 2017, Zhou *et al.* provided the existence criteria for the solutions of p-Laplacian Langevin FDE

$$\begin{aligned} &D_{0^+}^\beta \phi_p[(D_{0^+}^\alpha + \lambda)x(t)] = f(t, x(t), D_{0^+}^\alpha x(t)), \\ &{}_qD_{0^+}^\beta \phi_p[(D_{0^+}^\alpha + \lambda)x(t)] = g(t, x(t), {}_qD_{0^+}^\alpha x(t)) \end{aligned} \tag{1.2}$$

under anti-periodic boundary conditions $x(0) = -x(1)$, ${}_qD_{0^+}^\alpha x(0) = -{}_qD_{0^+}^\alpha x(1)$, in the whole domain $0 \le t \le 1$. Here, $\phi_p(s) = |s|^{p-2}s$, with $p \in (1,2]$. Also, we have $0 < \alpha, \beta \le 1$, $0 \le \lambda$, $1 < \alpha + \beta < 2$, and $q \in (0,1)$ [31]. For more instance, see [7, 22, 23, 24].

In this work, some basic and fundamental results related to q-calculus are recalled in Sec. 2. Motivated by these achievements, in Sec. 3, we examine the positive solutions of FqDE in two consecutive segments

$${}^c\mathcal{D}_q^\sigma[k](t) = \begin{cases} f\left(t, k(t), {}^c\mathcal{D}_q^{\alpha_1}[k](t), \mathcal{I}_q^{\beta_1}[k](t)\right), & t \in [0, t_0), \\ g\left(t, k(t), {}^c\mathcal{D}_q^{\alpha_2}[k](t), \mathcal{I}_q^{\beta_2}[k](t)\right), & t \in [t_0, 1], \end{cases} \tag{1.3}$$

under simultaneous Dirichlet boundary conditions

$$k(0) = h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right), \qquad k(1) = h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right), \tag{1.4}$$

where $\mathcal{I}_q^\beta$ and ${}^c\mathcal{D}_q^\sigma$ stand for the Riemann-Liouville $q-$integral and the Liouville-Caputo fractional $q-$derivative of order β and $1 < \sigma \le 2$ respectively, $t \in \overline{J} = [0,1]$, $t_0 \in J = (0,1)$, $0 < \alpha_\ell < 1$ with $\ell = 1,2,3,4$, $\beta_\ell > 0$ with $\ell = 1,2,3,4$, and the functions f, g, h_1 and h_2 map $J \times \mathbb{R}^3$ to $\mathbb{R}$ with $f(t_0,\cdot,\cdot,\cdot) = g(t_0,\cdot,\cdot,\cdot)$. Finally in Sec. 4 we consider two illustrated examples associated to the obtained results for the above model problems are provided in detail.

2 Essential preliminaries

Throughout the context, we shall apply the notations of time scales calculus [8]. Let us assume that $\mathfrak{t}_0 \in \mathbb{R}$ and $q \in \mathbb{I}$. Next, we define the time scale $\mathbb{T}_{\mathfrak{t}_0} = \{0\} \cup \{\mathfrak{t} : \mathfrak{t} = \mathfrak{t}_0 q^n, \ \forall n \in \mathbb{N}\}$. However, for simplicity we sometimes drop the subscript $\mathfrak{t}_0$ and denote $\mathbb{T}_{\mathfrak{t}_0}$ by $\mathbb{T}$ if there is no confusion about $\mathfrak{t}_0$. For a given $s \in \mathbb{R}$, let us define the symbol $[s]_q = (1-q^s)/(1-q)$ [15]. The next aim is to define the notation $(y-z)_q^{(n)}$ for the q-factorial function. It is given by

$$(y-z)_q^{(n)} = \prod_{k=0}^{n-1}(y - zq^k), \qquad n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}, \, y, z \in \mathbb{R}, \tag{2.1}$$

and with $(y-z)_q^{(0)} = 1$, see ([2]). One can also show that $(y-z)_q^{(\sigma)} = y^\sigma \prod_{k=0}^\infty \frac{y - zq^k}{y - zq^{\sigma+k}}$, $\sigma \in \mathbb{R}$, $s \ne 0$. It should be stressed that for $z = 0$, we have obviously $y^{(\sigma)} = y^\sigma$. The next symbol is used for the q-Gamma function. It has the following definition $\Gamma_q(y) = (1-q)^{1-y}(1-q)_q^{(y-1)}$, $(y \in \mathbb{R}\backslash\{\cdots, -2, -1, 0\})$ [15]. To proceed, let us σ and ν be two positive numbers. Let a function $y : \mathbb{T} \to \mathbb{R}$ is given. We define the q-derivative of y in the form

$$\mathbb{D}_q[y](\mathfrak{t}) = \left(\tfrac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\right)_q y(\mathfrak{t}) = \tfrac{y(q\mathfrak{t}) - y(\mathfrak{t})}{\mathfrak{t}(1-q)}, \qquad \forall \mathfrak{t} \in \mathbb{T} \setminus \{0\}, \tag{2.2}$$

and for $\mathfrak{t} = 0$ we have $\mathbb{D}_q[y](0) = \lim_{\mathfrak{t}\to 0} \mathbb{D}_q[y](\mathfrak{t})$ [2]. One can also define the higher-order q-derivative of y recursivly through the relation $\mathbb{D}_q^n[y](\mathfrak{t}) = \mathbb{D}_q[\mathbb{D}_q^{n-1}[y]](\mathfrak{t})$ for all $n \ge 1$. Here, for $n = 0$ we get $\mathbb{D}_q^0[y](\mathfrak{t}) = y(\mathfrak{t})$ [2].

$$\mathbb{I}_q[y](\mathfrak{t}) = \int_0^{\mathfrak{t}} y(\xi)\, \mathrm{d}_q\xi = \mathfrak{t}(1-q)\sum_{k=0}^\infty q^k y(\mathfrak{t}q^k), \tag{2.3}$$

for $0 \le \mathfrak{t} \le b$ and under condition that the involved series is absolutely convergent, see [2]. From this we can conclude the next identity for s in $[0, b]$ as

$$\int_s^b y(\xi)\, \mathrm{d}_q\xi = \mathbb{I}_q[y](b) - \mathbb{I}_q[y](s) = (1-q)\sum_{k=0}^{\infty} q^k \left[by(bq^k) - sy(sq^k)\right],$$

based upon the existence of the series. Suppose that $y \in C([0, b])$. For $n = 0$, the integral operator $\mathbb{I}_q^n$ is defined as $\mathbb{I}_q^0[y](\mathfrak{t}) = y(\mathfrak{t})$ and for for $n \ge 1$ we have $\mathbb{I}_q^n[y](\mathfrak{t}) = \mathbb{I}_q[\mathbb{I}_q^{n-1}[y]](\mathfrak{t})$, see [2]. If the function y be continuous at $\mathfrak{t} = 0$ one can assert that $\mathbb{D}_q[\mathbb{I}_q[y]](\mathfrak{t}) = y(\mathfrak{t})$ and $\mathbb{I}_q[\mathbb{D}_q[y]](\mathfrak{t}) = y(\mathfrak{t}) - y(0)$ [2]. For the function y, the next is the definition of fractional Riemann–Liouville type q-integral in the form

$$\mathbb{I}_q^\sigma[y](\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)_q^{(\sigma-1)} \tfrac{y(\xi)}{\Gamma_q(\sigma)}\, \mathrm{d}_q\xi, \qquad \mathbb{I}_q^0[y](\mathfrak{t}) = y(\mathfrak{t}), \tag{2.4}$$

where $\sigma > 0$ and for all $\mathfrak{t} \in [0, 1]$ [6, 12]. Similarly for this function, the concept of Liouville-Caputo fractional q-derivative is given next as

$${}^{C}\mathbb{D}_q^\sigma[y](\mathfrak{t}) = \mathbb{I}_q^{[\sigma]-\sigma}\left[\mathbb{D}_q^{[\sigma]}[y]\right](\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)_q^{([\sigma]-\sigma-1)} \tfrac{\mathbb{D}_q^{[\sigma]}[y](\xi)}{\Gamma_q([\sigma]-\sigma)}\, \mathrm{d}_q\xi, \tag{2.5}$$

where $\sigma > 0$ and for all $\mathfrak{t} \in [0, 1]$ [12, 19]. For $\sigma, \nu \ge 0$, we can prove that $\mathbb{I}_q^\nu[\mathbb{I}_q^\sigma[y]](t) = \mathbb{I}_q^{\sigma+\nu}[y](t)$, and ${}^{C}\mathbb{D}_q^\sigma[\mathbb{I}_q^\sigma[y]](t) = y(t)$, see [12].

Lemma 2.1 ([17]). Let $y \in AC^n[\mathfrak{t}_1, \mathfrak{t}_2]$. Then for $n-1 < \sigma \le n$, $n \in \mathbb{N}$ one has $\mathbb{I}^\sigma[{}^{C}\mathbb{D}_q^\sigma[y]](\mathfrak{t}) = y(\mathfrak{t}) + \sum_{i=0}^{n-1} c_i(\mathfrak{t}-\mathfrak{t}_1)^i$, $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$.

Lemma 2.2 ([17]). Let suppose that $\sigma \in (0, 1)$. Then for each $y \in AC[0, 1]$ we have $\mathbb{I}^\sigma[\mathbb{D}^\sigma[y]](\mathfrak{t}) = y(\mathfrak{t})$ for a.e. $\mathfrak{t} \in [0, 1]$. Here, we have $\mathbb{D}^\sigma[y](\mathfrak{t}) = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}} \int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)^{-\sigma} \frac{y(\xi)}{\Gamma(1-\sigma)}\, \mathrm{d}\xi$.

Theorem 2.3 ([26] Banach contraction principle). Let assume that the space $\mathcal{X}$ is a Banach space and let $A : \mathcal{X} \to \mathcal{X}$ be a contraction map. Then, there exists an $x \in \mathcal{X}$ such that $Ax = x$.

Theorem 2.4 ([26] Krasnoselskii's fixed point theorem). Consider a nonempty subset S of a Banach space $\mathcal{X}$ such that be a closed and convex and two maps A and B of S into $\mathcal{X}$ such that $A[k] + B[l] \in S$ for $k, l \in S$. Let suppose that B is a contraction map and let A is also compact and continuous map. Then, there exists a $k \in S$ such that $k = A[k] + B[k]$.

3 Main and basic results

The main aim of this section is to investigate the existence of the solutions for the FqDE (1.3)-(1.4) by considering the fixed point theorems. We consider the set $\mathcal{X} = C^1(\overline{J}, \mathbb{R})$ endowed with the norm $\|k\|_* = \sup_{t\in\overline{J}} |k(t)| + \sup_{t\in\overline{J}} |k'(t)|$.

Lemma 3.1. Assume that we have $v \in L^1(\overline{J}, \mathbb{R})$. Assume further that the FqDE ${}^{c}\mathcal{D}_q^\sigma[k](t) = v(t)$ under the conditions

$$k(0) = h_1\left(t_0, k(t_0), {}^{c}\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right), \qquad k(1) = h_2\left(t_0, k(t_0), {}^{c}\mathcal{D}_q^{\alpha_4}[x](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right), \tag{3.1}$$

is given. Then, the unique solution is obtained as

$$\begin{aligned} k(t) &= \mathcal{I}_q^\sigma[v](t) + h_1\left(t_0, k(t_0), {}^{c}\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) \\ &\quad + t\big[h_2\left(t_0, k(t_0), {}^{c}\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) - \mathcal{I}_q^\sigma[v](1) - h_1\left(t_0, k(t_0), {}^{c}\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\big]. \end{aligned} \tag{3.2}$$

Proof . We assume that $k(t)$ satisfies in the equation ${}^{c}\mathcal{D}_q^\sigma[k](t) = v(t)$. Lemma 2.2 implies that $k(t) = \mathcal{I}_q^\sigma[v](t) + c_0 + c_1 t$, where $c_0, c_1 \in \mathbb{R}$. Considering the boundary conditions, we conclude that

$$c_0 = h_1\left(t_0, k(t_0), {}^{c}\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right), \tag{3.3}$$

and

$$c_1 = h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) - \mathcal{I}_q^{\sigma}[v](1) - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right). \tag{3.4}$$

Clearly Eq. (3.2) satisfies on the boundary conditions

$$k(0) = h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) \qquad k(1) = h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), I_q^{\beta_4}[k](t_0)\right). \tag{3.5}$$

On the other hand, Lemmas 2.1 and 2.2 imply that

$${}^c\mathcal{D}_q^{\sigma}[k](t) = \mathcal{I}_q^{2-\sigma} k''(t) = \mathcal{I}_q^{2-\sigma}\left[\mathcal{I}_q^{-2+\sigma}[v]\right](t) = \mathcal{I}_q^{2-\sigma}\left[{}^c\mathcal{D}_q^{2-\sigma}[v]\right](t) = v(t). \tag{3.6}$$

Now, our proof is complete. □

Corollary 3.2. A given function $k \in \mathcal{X}$ is called a solution of FqDE (1.3)-(1.4) iff

$$\begin{aligned} k(t) &= \mathcal{I}_q^{\sigma} f\left(t, k(t), {}^c\mathcal{D}_q^{\alpha_1}[k](t), \mathcal{I}_q^{\beta_1}[k](t)\right) + h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) \\ &\quad + \Bigg[h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\ &\quad - \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\Bigg] t, \end{aligned} \tag{3.7}$$

whenever $0 \le t \le t_0$, and

$$\begin{aligned} k(t) &= \int_0^{t_0} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1})[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\ &\quad + \int_{t_0}^{t} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s \\ &\quad + h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) + \Bigg[h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) \\ &\quad - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\ &\quad - \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\beta_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\ &\quad - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\Bigg] t, \end{aligned} \tag{3.8}$$

whenever $t_0 \le t \le 1$.

Theorem 3.3. Let suppose that there exist $\ell \in (0, \sigma - 1)$ and $L_1, L_2 \in L^{1/\ell}(\overline{J}, (0, \infty))$ and $L_3, L_4 \in C(\overline{J}, (0, \infty))$ s.t

$$\begin{aligned} \left|f(t, k_1, k_2, k_3) - f(t, k_1', k_2', k_3')\right| &\le L_1(t) \sum_{i=1}^{3} |k_i - k_i'|, \\ \left|g(t, k_1, k_2, k_3) - g(t, k_1', k_2', k_3')\right| &\le L_2(t) \sum_{i=1}^{3} |k_i - k_i'|, \\ \left|h_1(t, k_1, k_2, k_3) - h_1(t, k_1', k_2', k_3')\right| &\le L_3(t) \sum_{i=1}^{3} |k_i - k_i'|, \\ \left|h_2(t, k_1, k_2, k_3) - h_2(t, k_1', k_2', k_3')\right| &\le L_4(t) \sum_{i=1}^{3} |k_i - k_i'|, \end{aligned} \tag{3.9}$$

for all $t \in \overline{J}$ and k_ℓ, k_ℓ', with $\ell = 1, 2, 3$. Then FqDE (1.3)-(1.4) has a unique solution if

$$\begin{aligned} \Lambda_1 &= \tfrac{3\|L_1\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)}\left[1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right] + \tfrac{3\|L_2\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)}\left[1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right] \\ &\quad + 3\|L_3\|\left[1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right] + 2\|L_4\|\left[1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right] \\ &\quad + \tfrac{\|L_1\|_{1/\ell}\eta_2}{\Gamma_q(\sigma-1)}\left[1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right] + \tfrac{\|L_2\|_{1/\ell}\eta_2}{\Gamma_q(\sigma-1)}\left[1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right] < 1, \end{aligned} \tag{3.10}$$

where $\eta_1 = \left(\frac{1-\ell}{\sigma-\ell}\right)^{1-\ell}$, $\eta_2 = \left(\frac{1-\ell}{\sigma-\ell+1}\right)^{1-\ell}$, and $\|L\|_p = \left(\int_0^1 |L(s)|^p \,\mathrm{d}s\right)^{1/p}$, for each $L \in L^p(\overline{J}, \mathbb{R})$.

Proof . Define the operator $\mathcal{T}$ on $\mathcal{X}$ by

$$\begin{aligned}\mathcal{T}[k](t) &= \mathcal{I}_q^{\sigma} f\left(t, k(t), {}^c\mathcal{D}_q^{\alpha_1}[k](t), \mathcal{I}_q^{\beta_1}[k](t)\right) + h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\\ &\quad + \Bigg[h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s\\ &\quad - \int_{t_0}^1 \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2} k(s), \mathcal{I}_q^{\beta_2} k(s)\right) \mathrm{d}_q s - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\Bigg] t, \qquad (3.11)\end{aligned}$$

for $0 \le t \le t_0$, and

$$\begin{aligned}\mathcal{T}[k](t) &= \int_0^{t_0} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\alpha)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s\\ &\quad + \int_{t_0}^{t} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s\\ &\quad + h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) + \Bigg[h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right)\\ &\quad - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s\\ &\quad - \int_{t_0}^1 \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\Bigg] t, \qquad (3.12)\end{aligned}$$

for $t_0 \le t \le 1$. Clearly, FqDE (1.3)-(1.4) admits a solution iff the relation $\mathcal{T}[k] = k$ has a fixed point. Let $k, l \in \mathcal{X}$. We then for $0 \le t \le t_0$ have

$$\begin{aligned}|\mathcal{T}[k](t) - \mathcal{T}[l](t)| &= \Big|\mathcal{I}_q^{\sigma} f\left(t, k(t), {}^c\mathcal{D}_q^{\alpha_1}[k](t), \mathcal{I}_q^{\beta_1}[k](t)\right) + h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\\ &\quad + \big[h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s\\ &\quad - \int_{t_0}^1 \tfrac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s\\ &\quad -h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\big] t - \mathcal{I}_q^{\alpha} f\left(t, l(t), {}^c\mathcal{D}_q^{\alpha_1}[l](t), \mathcal{I}_q^{\beta_1} l(t)\right)\\ &\quad - h_1\left(t_0, l(t_0), {}^c\mathcal{D}_q^{\alpha_3}[l](t_0), \mathcal{I}_q^{\beta_3}[l](t_0)\right) - \big[h_2\left(t_0, l(t_0), {}^c\mathcal{D}_q^{\alpha_4}[l](t_0), \mathcal{I}_q^{\beta_4}[l](t_0)\right)\\ &\quad - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_1}[l](s), \mathcal{I}_q^{\beta_1}[l](s)\right) \mathrm{d}_q s\\ &\quad - \int_{t_0}^1 \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_2}[l](s), \mathcal{I}_q^{\beta_2}[l](s)\right) \mathrm{d}_q s \; -h_1\left(t_0, l(t_0), {}^c\mathcal{D}_q^{\alpha_3}[l](t_0), \mathcal{I}_q^{\beta_3}[l](t_0)\right)\big] t\Big|\\ &\le \mathcal{I}_q^{\sigma}\big|f\left(t, k(t), {}^c\mathcal{D}_q^{\alpha_1}[k](t), \mathcal{I}_q^{\beta_1}[k](t)\right) - f\left(t, l(t), {}^c\mathcal{D}_q^{\alpha_1}[l](t), \mathcal{I}_q^{\beta_1}[l](t)\right)\big|\\ &\quad + 2\big|h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) - h_1\left(t_0, l(t_0), {}^c\mathcal{D}_q^{\alpha_3}[l](t_0), \mathcal{I}_q^{\beta_3}[l](t_0)\right)\big|\\ &\quad + \big|h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) - h_2\left(t_0, l(t_0), {}^c\mathcal{D}_q^{\alpha_4}[l](t_0), \mathcal{I}_q^{\beta_4}[l](t_0)\right)\big|\\ &\quad + \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \big|f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) - f\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_1}[l](s), \mathcal{I}_q^{\beta_1}[l](s)\right)\big| \mathrm{d}_q s\\ &\quad + \int_{t_0}^1 \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \big|g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) - g\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_2}[l](s), \mathcal{I}_q^{\beta_2}[l](s)\right)\big| \mathrm{d}_q s\\ &\le \mathcal{I}_q^{\sigma} L_1(t)\left(|k(t) - l(t)| + |{}^c\mathcal{D}_q^{\alpha_1}[k](t) - {}^c\mathcal{D}_q^{\alpha_1}[l](s)| + |\mathcal{I}_q^{\beta_1}[k](t) - \mathcal{I}_q^{\beta_1}[l](t)|\right)\\ &\quad + 2L_3(t_0)\left(|k(t_0) - l(t_0)| + |{}^c\mathcal{D}_q^{\alpha_3}[k](t_0) - {}^c\mathcal{D}_q^{\alpha_3}[l](t_0)| + |\mathcal{I}_q^{\beta_3}[k](t_0) - \mathcal{I}^{\beta_3}[l](t_0)|\right)\\ &\quad + L_4(t_0)\left(|k(t_0) - l(t_0)| + |{}^c\mathcal{D}_q^{\alpha_4}[k](t_0) - {}^c\mathcal{D}_q^{\alpha_4}[l](t_0)| + |\mathcal{I}_q^{\beta_4}[k](t_0) - \mathcal{I}_q^{\beta_4}[l](t_0)|\right)\\ &\quad + \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} L_1(s)\left(|k(s) - l(s)|\right.\end{aligned}$$

$$
\begin{aligned}
&+|{}^c\mathcal{D}_q^{\alpha_1}[k](s) - {}^c\mathcal{D}_q^{\alpha_1}[l](s)| + |\mathcal{I}_q^{\beta_1}[k](s) - \mathcal{I}_q^{\beta_1}[l](s)|\big) \, \mathrm{d}_q s \\
&+ \int_{t_0}^1 \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} L_2(s) \left(|k(s) - l(s)| + |{}^c\mathcal{D}_q^{\alpha_2}[k](s) - {}^c\mathcal{D}_q^{\alpha_2}[l](s)| + |\mathcal{I}_q^{\beta_1}[k](s) - \mathcal{I}_q^{\beta_2}[l](s)|\right) \mathrm{d}_q s \\
\le\, & \mathcal{I}_q^{\sigma} L_1(t) \left(|k(t) - l(t)| + \mathcal{I}_q^{1-\alpha_1}|k'(s) - l'(w)| + \mathcal{I}_q^{\beta_1}|k(s) - l(s)|\right) \\
&+ 2L_3(t_0) \left(|k(t_0) - l(t_0)| + \mathcal{I}_q^{1-\alpha_3}|k'(t_0) - l'(t_0)| + \mathcal{I}_q^{\beta_3}|k(t_0) - l(t_0)|\right) \\
&+ L_4(t_0) \left(|k(t_0) - l(t_0)| + \mathcal{I}_q^{1-\alpha_4}|k'(t_0) - l'(t_0)| + \mathcal{I}_q^{\beta_4}|k(t_0) - l(t_0)|\right) \\
&+ \int_0^{t_0} (1-qs)^{(\sigma-1)} \tfrac{L_1(s)}{\Gamma_q(\sigma)} \Big(|k(s) - l(s)| + \mathcal{I}_q^{1-\alpha_1}|k'(s) - l'(s)| + \mathcal{I}_q^{\beta_1}|k(s) - l(s)|\Big) \\
&+ \int_{t_0}^1 (1-qs)^{(\sigma-1)} \tfrac{L_2(s)}{\Gamma_q(\sigma)} \Bigg(|k(s) - l(s)| + \int_0^s (s-u)^{-\alpha_2} \tfrac{|k'(u) - l'(u)|}{\Gamma_q(1-\alpha_2)} \, \mathrm{d}u \\
&+ \int_0^s (s-u)^{\beta_2 - 1} \tfrac{|k(u) - l(u)|}{\Gamma_q(\beta_2)} \, \mathrm{d}u\Bigg) \, \mathrm{d}s \\
\le\, & \mathcal{I}_q^{\sigma} L_1(t) \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) \|k - l\|_* + 2L_3(t_0) \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right) \|k - l\|_* \\
&+ L_4(t_0) \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right) \|k - l\|_* \\
&+ \int_0^{t_0} \tfrac{(1-s)^{\sigma-1}}{\Gamma_q(\sigma)} L_1(s) \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) \|k - l\|_* ds \\
&+ \int_{t_0}^1 \tfrac{(1-s)^{\sigma-1}}{\Gamma_q(\sigma)} L_2(s) \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) \|k - l\|_* \, \mathrm{d}s \\
\le\, & \tfrac{\|k-l\|_*}{\Gamma_q(\sigma)} \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) \left[\int_0^t \left((t-s)^{\sigma-1}\right)^{1/1-k} \mathrm{d}s\right]^{1-k} \left[\int_0^t (L_1(s))^{1/k} \, \mathrm{d}s\right]^{\ell} \\
&+ \left[2\|L_3\| \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right) + \|L_4\| \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right)\right] \|k - l\|_* \\
&+ \tfrac{\|k-l\|_*}{\Gamma_q(\sigma)} \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) \left[\int_0^{t_0} \left((1-s)^{\sigma-1}\right)^{1/1-\ell} \mathrm{d}s\right]^{1-\ell} \left[\int_0^{t_0} (L_1(s))^{1/\ell} \, \mathrm{d}s\right]^{\ell} \\
&+ \tfrac{\|k-l\|_*}{\Gamma_q(\sigma)} \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) \left[\int_{t_0}^1 \left((1-s)^{\sigma-1}\right)^{1/1-\ell} \mathrm{d}s\right]^{1-\ell} \left[\int_{t_0}^1 (L_2(s))^{1/\ell} \, \mathrm{d}s\right]^{\ell} \\
\le\, & \Bigg[\tfrac{2\|L_1\|_{1/\ell}}{\Gamma_q(\sigma)} \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) \left[\tfrac{1-\ell}{\sigma-\ell}\right]^{1-\ell} + \tfrac{\|L_2\|_{1/\ell}}{\Gamma_q(\sigma)} \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) \left[\tfrac{1-\ell}{\sigma-\ell}\right]^{1-\ell} \\
&+ 2\|L_3\| \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right) + \|L_4\| \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right)\Bigg] \|k - l\|_*, \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
|(\mathcal{T}[k])'(t) - (\mathcal{T}[l])'(t)| = \Big|& \mathcal{I}_q^{\sigma-1} f\left(t, k(t), {}^c\mathcal{D}_q^{\alpha_1}[k](t), \mathcal{I}_q^{\beta_1}[k](t)\right) + h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}^{\beta_4}[k](t_0)\right) \\
&- \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\
&- \int_{t_0}^1 \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{(\alpha_2)}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s \\
&- h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) - \mathcal{I}_q^{\sigma-1} f\left(t, l(t), {}^c\mathcal{D}_q^{\alpha_1} l(t), \mathcal{I}_q^{\beta_1}[l](t)\right) \\
&- h_2\left(t_0, l(t_0), {}^c\mathcal{D}_q^{\alpha_4}[l](t_0), \mathcal{I}_q^{\beta_4}[l](t_0)\right) + \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_1}[l](s), \mathcal{I}_q^{\beta_1}[l](s)\right) \mathrm{d}_q s \\
&+ \int_{t_0}^1 \tfrac{(1-qs)^{\sigma-1}}{\Gamma_q(\sigma)} g\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_2}[l](s), \mathcal{I}_q^{\beta_2}[l](s)\right) \mathrm{d}_q s + h_1\left(t_0, l(t_0), {}^c\mathcal{D}_q^{\alpha_3}[l](t_0), \mathcal{I}_q^{\beta_3}[l](t_0)\right)\Big| \\
\le \Bigg[& \tfrac{\|L_1\|_{1/\ell}\eta_2}{\Gamma_q(\sigma-1)} \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) + \tfrac{\|L_1\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)} \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) \\
&+ \tfrac{\|L_2\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)} \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) + \|L_3\| \left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right)
\end{aligned}
$$

$$+\|L_4\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_4)}+\tfrac{1}{\Gamma_q(1+\beta_4)}\right)\Big]\|k-l\|_*, \tag{3.14}$$

where $\eta_1=\left(\frac{1-\ell}{\sigma-\ell}\right)^{1-\ell}$ and $\eta_2=\left(\frac{1-\ell}{\sigma-\ell+1}\right)^{1-\ell}$. Similarly, for $t_0\le t\le 1$ we get

$$\begin{aligned}
|\mathcal{T}[k](t)-\mathcal{T}[l](t)| = \Big|&\int_0^{t_0}\tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_1}[k](s),\mathcal{I}_q^{\beta_1}[k](s)\right)\,\mathrm{d}_qs\\
&+\int_{t_0}^{t}\tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_2}[k](s),\mathcal{I}_q^{\beta_2}[k](s)\right)\,\mathrm{d}_qs\\
&+h_1\left(t_0,k(t_0),{}^c\mathcal{D}_q^{\alpha_3}[k](t_0),\mathcal{I}_q^{\beta_3}[k](t_0)\right)+\big[h_2\left(t_0,k(t_0),{}^c\mathcal{D}_q^{\alpha_4}[k](t_0),\mathcal{I}_q^{\beta_4}[k](t_0)\right)\\
&-\int_0^{t_0}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_1}[k](s),\mathcal{I}_q^{\beta_1}[k](s)\right)\,\mathrm{d}_qs\\
&-\int_{t_0}^{1}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_2}[k](s),\mathcal{I}_q^{\beta_2}[k](s)\right)\,\mathrm{d}_qs\\
&-h_1\left(t_0,k(t_0),{}^c\mathcal{D}_q^{\alpha_3}[k](t_0),\mathcal{I}_q^{\beta_3}x)(t_0)\right)\big]\,t-\int_0^{t_0}\tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_1}[l](s),\mathcal{I}_q^{\beta_1}[l](s)\right)\,\mathrm{d}_qs\\
&-\int_{t_0}^{t}\tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_2}[l](s),\mathcal{I}_q^{\beta_2}[l](s)\right)\,\mathrm{d}_qs\\
&-h_1\left(t_0,l(t_0),{}^c\mathcal{D}_q^{\alpha_3}[l](t_0),\mathcal{I}_q^{\beta_3}[l](t_0)\right)-\big[h_2\big(t_0,l(t_0),{}^c\mathcal{D}^{\alpha_4}[l](t_0),\mathcal{I}_q^{\beta_4}l[y](t_0)\big)\\
&-\int_0^{t_0}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_1}[l](s),\mathcal{I}_q^{\beta_1}[l](s)\right)\,\mathrm{d}_qs\\
&-\int_{t_0}^{1}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_2}[l](s),\mathcal{I}_q^{\beta_2}[l](s)\right)\,\mathrm{d}_qs\ -h_1\left(t_0,l(t_0),{}^c\mathcal{D}_q^{\alpha_3}[l](t_0),\mathcal{I}_q^{\beta_3}[l](t_0)\right)\big]\,t\Big|\\
\le\Big[&\tfrac{2\|L_1\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right)+\tfrac{2\|L_2\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_2)}+\tfrac{1}{\Gamma_q(1+\beta_2)}\right)\\
&+2\|L_3\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_3)}+\tfrac{1}{\Gamma_q(1+\beta_3)}\right)+\|L_4\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_4)}+\tfrac{1}{\Gamma_q(1+\beta_4)}\right)\Big]\|k-l\|_*,
\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
|(\mathcal{T}[k])'(t)-(\mathcal{T}[l])'(t)| = \Big|&\int_0^{t_0}\tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)}f\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_1}[k](s),\mathcal{I}_q^{\beta_1}[k](s)\right)\,\mathrm{d}_qs\\
&+\int_{t_0}^{t}\tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)}g\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_g}[k](s),\mathcal{I}_q^{\beta_g}[k](s)\right)\,\mathrm{d}_qs+h_2\left(t_0,k(t_0),{}^c\mathcal{D}_q^{\alpha_4}[k](t_0),\mathcal{I}_q^{\beta_4}[k](t_0)\right)\\
&-\int_0^{t_0}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_1}[k](s),\mathcal{I}_q^{\beta_1}[k](s)\right)\,\mathrm{d}_qs\\
&-\int_{t_0}^{1}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_2}[k](s),\mathcal{I}_q^{\beta_2}[k](s)\right)\,\mathrm{d}_qs-h_1\left(t_0,k(t_0),{}^c\mathcal{D}_q^{\alpha_3}[k](t_0),\mathcal{I}_q^{\beta_3}[k](t_0)\right)\\
&-\int_0^{t_0}\tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)}f\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_1}[l](s),\mathcal{I}_q^{\beta_1}[l](s)\right)\,\mathrm{d}_qs\\
&-\int_{t_0}^{t}\tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)}g\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_g}[l](s),\mathcal{I}_q^{\beta_g}[l](s)\right)\,\mathrm{d}_qs-h_2\left(t_0,l(t_0),{}^c\mathcal{D}_q^{\alpha_4}[l](t_0),\mathcal{I}_q^{\beta_4}[l](t_0)\right)\\
&+\int_0^{t_0}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_1}[l](s),\mathcal{I}_q^{\beta_1}[l](s)\right)\,\mathrm{d}_qs\\
&+\int_{t_0}^{1}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_2}[l](s),\mathcal{I}_q^{\beta_2}[l](s)\right)\,\mathrm{d}_qs+h_1\left(t_0,l(t_0),{}^c\mathcal{D}_q^{\alpha_3}[l](t_0),\mathcal{I}_q^{\beta_3}[l](t_0)\right)\Big|\\
\le\Big[&\tfrac{\|L_1\|_{1/k}k_2}{\Gamma_q(\sigma-1)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right)+\tfrac{\|L_2\|_{1/\ell}\eta_2}{\Gamma_q(\sigma-1)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_2)}+\tfrac{1}{\Gamma_q(1+\beta_2)}\right)\\
&+\tfrac{\|L_1\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right)+\tfrac{\|L_2\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_2)}+\tfrac{1}{\Gamma_q(1+\beta_2)}\right)\\
&+\|L_3\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_3)}+\tfrac{1}{\Gamma_q(1+\beta_3)}\right)+\|L_4\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_4)}+\tfrac{1}{\Gamma_q(1+\beta_4)}\right)\Big]\|k-l\|_*,
\end{aligned}\tag{3.16}$$

where $\eta_1 = \left(\frac{1-\ell}{\sigma-\ell}\right)^{1-\ell}$ and $\eta_2 = \left(\frac{1-\ell}{\sigma-\ell+1}\right)^{1-\ell}$. By utilizing relations (3.13), (3.14), (3.15), and (3.16) we have

$$\begin{aligned}\|\mathcal{T}[k]-\mathcal{T}[l]\|_* &= \|\mathcal{T}[k]-\mathcal{T}[l]\| + \|(\mathcal{T}[k])'-(\mathcal{T}[l])'\| \\ &\leq \Bigg[\frac{3\|L_1\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)}\left(1+\frac{1}{\Gamma_q(2-\alpha_1)}+\frac{1}{\Gamma_q(1+\beta_1)}\right)+\frac{3\|L_2\|_{1/\ell}\eta_1}{\Gamma_q(\sigma)}\left(1+\frac{1}{\Gamma_q(2-\alpha_2)}+\frac{1}{\Gamma_q(1+\beta_2)}\right) \\ &\quad + 3\|L_3\|\left(1+\frac{1}{\Gamma_q(2-\alpha_3)}+\frac{1}{\Gamma_q(1+\beta_3)}\right)+2\|L_4\|\left(1+\frac{1}{\Gamma_q(2-\alpha_4)}+\frac{1}{\Gamma_q(1+\beta_4)}\right) \\ &\quad + \frac{\|L_1\|_{1/\ell}\eta_2}{\Gamma_q(\sigma-1)}\left(1+\frac{1}{\Gamma_q(2-\alpha_1)}+\frac{1}{\Gamma_q(1+\beta_1)}\right) \\ &\quad + \frac{\|L_2\|_{1/\ell}\eta_2}{\Gamma_q(\sigma-1)}\left(1+\frac{1}{\Gamma_q(2-\alpha_2)}+\frac{1}{\Gamma_q(1+\beta_2)}\right)\Bigg]\|k-l\|_* = \Lambda_1\|k-l\|_*. \end{aligned} \tag{3.17}$$

Thus $\mathcal{T}$ is a contraction mapping due to the fact that $\Lambda_1 < 1$. Therefore, by using the Banach contraction principle we conclude that $\mathcal{T}$ has a unique fixed point. This fixed point is the unique solution of the model problem (1.3)-(1.4) by using Corollary 3.2. □

Corollary 3.4. Assume that there exist L_1, L_2, L_3 and $L_4 \in \mathbb{R}^+$ such that

$$\begin{aligned} \left|f(t,k_1,k_2,k_3)-f(t,k_1',k_2',k_3')\right| &\leq L_1(t)\sum_{i=1}^{3}|k_i-k_i'|, \\ \left|g(t,k_1,k_2,k_3)-g(t,k_1',k_2',k_3')\right| &\leq L_2(t)\sum_{i=1}^{3}|k_i-k_i'|, \\ \left|h_1(t,k_1,k_2,k_3)-h_1(t,k_1',k_2',k_3')\right| &\leq L_3(t)\sum_{i=1}^{3}|k_i-k_i'|, \\ \left|h_2(t,k_1,k_2,k_3)-h_2(t,k_1',k_2',k_3')\right| &\leq L_4(t)\sum_{i=1}^{3}|k_i-k_i'|, \end{aligned} \tag{3.18}$$

for all $t \in \overline{J}$ and k_ℓ, k_ℓ' with $\ell = 1,2,3$. Then the FqDE (1.3)-(1.4) has a unique solution whenever

$$\begin{aligned}&\frac{3L_1}{\Gamma_q(\sigma+1)}\left(1+\frac{1}{\Gamma_q(2-\alpha_1)}+\frac{1}{\Gamma_q(1+\beta_1)}\right)+\frac{3L_2}{\Gamma_q(\sigma+1)}\left(1+\frac{1}{\Gamma_q(2-\alpha_2)}+\frac{1}{\Gamma_q(1+\beta_2)}\right)+3L_3\left(1+\frac{1}{\Gamma_q(2-\alpha_3)}+\frac{1}{\Gamma_q(1+\beta_3)}\right) \\ &\quad + 2L_4\left(1+\frac{1}{\Gamma_q(2-\alpha_4)}+\frac{1}{\Gamma_q(1+\beta_4)}\right)+\frac{L_1}{\Gamma_q(\sigma)}\left(1+\frac{1}{\Gamma_q(2-\alpha_1)}+\frac{1}{\Gamma_q(1+\beta_1)}\right)+\frac{L_2}{\Gamma_q(\sigma)}\left(1+\frac{1}{\Gamma_q(2-\alpha_2)}+\frac{1}{\Gamma_q(1+\beta_2)}\right) < 1.\end{aligned}$$

By the aid of the Krasnoselskii's fixed-point theorem, we state our next existence result.

Theorem 3.5. Suppose that there exist L_1, L_2, μ_1 and $\mu_2 \in C(\overline{J}, [0,\infty))$ and two nondecreasing self-functions ψ_1 and ψ_2 defined on $\mathbb{R}^+$ s.t $|f(t,k_1,k_2,k_3)-f(t,k_1',k_2',k_3')|$ and $|g(t,k_1,k_2,k_3)-g(t,k_1',k_2',k_3')|$, are less than or equal to $L_1(t)\sum_{i=1}^{3}|k_i-k_i'|$, $L_2(t)\sum_{i=1}^{3}|k_i-k_i'|$ respectively, and $|h_1(t,k_1,k_2,k_3)| \leq \mu_1(t)\psi_1\sum_{i=1}^{3}|k_i|$, and $|h_2(t,k_1,k_2,k_3)| \leq \mu_2(t)\psi_1\sum_{i=1}^{3}|k_i|$, for each $t \in \overline{J}$ and k_ℓ, k_ℓ' with $\ell = 1,2,3$. If

$$\Lambda_2 = \left[\frac{\|L_1\|}{\Gamma_q(\sigma)}\left(1+\frac{1}{\Gamma_q(2-\alpha_1)}+\frac{1}{\Gamma_q(1+\beta_1)}\right)+\frac{\|L_2\|}{\Gamma_q(\sigma)}\left(1+\frac{1}{\Gamma_q(2-\alpha_2)}+\frac{1}{\Gamma_q(1+\beta_2)}\right)\right]\left(\frac{1}{\sigma}+1\right) < 1, \tag{3.19}$$

then, the model problem (1.3) admits a solution on $\overline{J}$.

Proof . Consider $S = \{k \in \mathcal{X} : \|k\| \leq r\}$, where

$$\begin{aligned}&3\|\mu_1\|\psi_1\left(\left(1+\frac{1}{\Gamma_q(2-\alpha_3)}+\frac{1}{\Gamma_q(1+\beta_3)}\right)r\right)+2\|\mu_2\|\psi_2\left(\left(1+\frac{1}{\Gamma_q(2-\alpha_4)}+\frac{1}{\Gamma_q(1+\beta_4)}\right)r\right) \\ &\quad + \frac{r}{\Gamma_q(\sigma)}\left(\frac{2}{\sigma}+\sigma+1\right)\left[\|L_1\|\left(1+\frac{1}{\Gamma_q(2-\alpha_1)}+\frac{1}{\Gamma_q(1+\beta_1)}\right)+F_0\right] \\ &\quad + \frac{r}{\Gamma_q(\sigma)}\left(\frac{2}{\sigma}+\sigma+1\right)\left[\|L_2\|\left(1+\frac{1}{\Gamma_q(2-\alpha_2)}+\frac{1}{\Gamma_q(1+\beta_2)}\right)+G_0\right] \leq r.\end{aligned} \tag{3.20}$$

Clearly the set S is an nonempty subset of the Banach space $\mathcal{X}$, closed and convex. Now, we define two operators A and B on S by

$$\begin{aligned} A[k](t) &= h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) + \Big[h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) \\ &\quad - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\ &\quad - \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\Big] t, \end{aligned} \tag{3.21}$$

for each $0 \le t \le 1$, and $B[k](t) = \mathcal{I}_q^{\sigma} f)\left(t, k(t), {}^c\mathcal{D}_q^{\alpha_1}[k](t), \mathcal{I}_q^{\beta_1}[k](t)\right)$, whenever $0 \le t \le t_0$, and

$$\begin{aligned} B[k](t) &= \int_0^{t_0} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\ &\quad + \int_{t_0}^{t} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s, \end{aligned} \tag{3.22}$$

whenever $t_0 \le t \le 1$. Let $k, l \in S$. On the interval $0 \le t \le t_0$ we get

$$\begin{aligned} |A[k](t) + B[l](t)| &= \big|h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) + \big[h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) \\ &\quad - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\ &\quad - \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s \\ &\quad - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right)\big] t + \mathcal{I}_q^{\sigma} f\left(t, l(t), {}^c\mathcal{D}_q^{\alpha_1}[l](t), \mathcal{I}_q^{\beta_1}[l](t)\right)\big| \\ &\le 2\mu_1(t_0)\psi_1\left(|k(t_0)| + |{}^c\mathcal{D}_q^{\alpha_3}[k](t_0)| + |\mathcal{I}_q^{\beta_3}[k](t_0)|\right) + \mu_2(t_0)\psi_2\left(|k(t_0)| + |{}^c\mathcal{D}_q^{\alpha_4}[k](t_0)| + |\mathcal{I}_q^{\beta_4}[k](t_0)|\right) \\ &\quad + \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \left(L_1(s)|k(s) + {}^c\mathcal{D}_q^{\alpha_1}[k](s) + \mathcal{I}_q^{\beta_1}[k](s)| + F_0\right) \mathrm{d}_q s \\ &\quad + \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \left(L_2(s)|k(s) + {}^c\mathcal{D}_q^{\alpha_2}[k](s) + \mathcal{I}_q^{\beta_2}[k](s)| + G_0\right) \mathrm{d}_q s \\ &\quad + \int_0^{t} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \left(L_1(s)|l(s) + {}^c\mathcal{D}_q^{\alpha_1}[l](s) + \mathcal{I}_q^{\beta_1}[l](s)| + F_0\right) \mathrm{d}_q s \\ &\le 2\|\mu_1\|\psi_1\left(\left[1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right] r\right) + \|\mu_2\|\psi_2\left(\left[1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right] r\right) \\ &\quad + \tfrac{r}{\Gamma_q(\sigma+1)}\left[2\|L_1\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) + 2F_0 + \|L_2\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) + G_0\right], \end{aligned} \tag{3.23}$$

$$\begin{aligned} |(A[k])'(t) + (B[l])'(t)| &= \Big|h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) \\ &\quad - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\ &\quad - \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) \\ &\quad + I_q^{\sigma-1} f\left(t, l(t), {}^c\mathcal{D}_q^{\alpha_1}[l](t), \mathcal{I}_q^{\beta_1}[l](t)\right)\Big| \\ &\le \|\mu_2\|\psi_2\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right) r\right) + \|\mu_1\|\psi_1\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right) r\right) \\ &\quad + \tfrac{r(\sigma+1)}{\Gamma_q(\sigma)}\left[\|L_1\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) + F_0\right] \\ &\quad + \tfrac{r}{\Gamma_q(\sigma+1)}\left[\|L_2\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) + G_0\right]. \end{aligned} \tag{3.24}$$

Similarly, on the interval $t_0 \leq t \leq 1$ one gets

$$\begin{aligned}
|A[k](t)+B[l](t)| &= \Big| h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) + \Big[h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) \\
&\quad - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\
&\quad - \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) \Big] t \\
&\quad + \int_0^{t_0} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_1}[l](s), \mathcal{I}_q^{\beta_1}[l](s)\right) \mathrm{d}_q s \\
&\quad + \int_{t_0}^{t} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_2}[l](s), \mathcal{I}_q^{\beta_2}[l](s)\right) \mathrm{d}_q s \Big| \\
&\leq 2\mu_1(t_0)\psi_1\left(|k(t_0)| + |{}^c\mathcal{D}_q^{\alpha_3}[k](t_0)| + |\mathcal{I}_q^{\beta_3}[k](t_0)|\right) + \mu_2(t_0)\psi_2\left(|k(t_0)| + |{}^c\mathcal{D}_q^{\alpha_4}[k](t_0)| + |\mathcal{I}_q^{\beta_4}[k](t_0)|\right) \\
&\quad + \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \left(L_1(s)|k(s) + {}^c\mathcal{D}_q^{\alpha_1}[k](s) + \mathcal{I}_q^{\beta_1}[k](s)| + F_0\right) \mathrm{d}_q s \\
&\quad + \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \left(L_2(s)|k(s) + {}^c\mathcal{D}_q^{\alpha_2}[k](s) + \mathcal{I}_q^{\beta_2}[k](s)| + G_0\right) \mathrm{d}_q s \\
&\quad + \int_0^{t_0} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \left(L_1(s)|l(s) + {}^c\mathcal{D}_q^{\alpha_1}[l](s) + \mathcal{I}_q^{\beta_1}[l](s)| + F_0\right) \mathrm{d}_q s \\
&\quad + \int_{t_0}^{t} \tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} \left(L_2(s)|l(s) + {}^c\mathcal{D}_q^{\alpha_2}[l](s) + \mathcal{I}_q^{\beta_2}[l](s)| + G_0\right) \mathrm{d}_q s \\
&\leq 2\|\mu_1\|\psi_1\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right) r\right) + \|\mu_2\|\psi_2\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right) r\right) \\
&\quad + \tfrac{2r}{\Gamma_q(\sigma+1)} \left[\|L_1\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) + F_0 + \|L_2\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) + G_0\right], \qquad (3.25)
\end{aligned}$$

$$\begin{aligned}
|(A[k])'(t) + (B[l])'(t)| &= \Big| h_2\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_4}[k](t_0), \mathcal{I}_q^{\beta_4}[k](t_0)\right) \\
&\quad - \int_0^{t_0} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} f\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_1}[k](s), \mathcal{I}_q^{\beta_1}[k](s)\right) \mathrm{d}_q s \\
&\quad - \int_{t_0}^{1} \tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)} g\left(s, k(s), {}^c\mathcal{D}_q^{\alpha_2}[k](s), \mathcal{I}_q^{\beta_2}[k](s)\right) \mathrm{d}_q s - h_1\left(t_0, k(t_0), {}^c\mathcal{D}_q^{\alpha_3}[k](t_0), \mathcal{I}_q^{\beta_3}[k](t_0)\right) \\
&\quad + \int_0^{t_0} \tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)} f\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_1}[l](s), \mathcal{I}_q^{\beta_1}[l](s)\right) \mathrm{d}_q s \\
&\quad + \int_{t_0}^{t} \tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)} g\left(s, l(s), {}^c\mathcal{D}_q^{\alpha_2}[l](s), \mathcal{I}_q^{\beta_2}[l](s)\right) \mathrm{d}_q s \Big| \\
&\leq \|\mu_2\|\psi_2\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right) r\right) + \|\mu_1\|\psi_1\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right) r\right) \\
&\quad + \tfrac{r(\sigma+1)}{\Gamma_q(\sigma)} \left[\|L_1\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) + F_0\right] \\
&\quad + \tfrac{r(\sigma+1)}{\Gamma_q(\sigma)} \left[\|L_2\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) + G_0\right], \qquad (3.26)
\end{aligned}$$

where $F_0 = \sup_{t \in \overline{J}} |f(t, 0, 0, 0)|$ and $G_0 = \sup_{t \in \overline{J}} |g(t, 0, 0, 0)|$. Thus

$$\begin{aligned}
\|A[k] + B[l]\|_* &= \|A[k] + B[l]\| + \|(A[k])' + (B[l])'\| \leq 2\|\mu_1\|\psi_1\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right) r\right) \\
&\quad + \|\mu_2\|\psi_2\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right) r\right) + \tfrac{2r}{\Gamma_q(\sigma+1)} \Big[\|L_1\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) + F_0 \\
&\quad + \|L_2\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) + G_0\Big] + \|\mu_2\|\psi_2\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_4)} + \tfrac{1}{\Gamma_q(1+\beta_4)}\right) r\right) \\
&\quad + \|\mu_1\|\psi_1\left(\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_3)} + \tfrac{1}{\Gamma_q(1+\beta_3)}\right) r\right) + \tfrac{r(\sigma+1)}{\Gamma_q(\sigma)} \left[\|L_1\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_1)} + \tfrac{1}{\Gamma_q(1+\beta_1)}\right) + F_0\right] \\
&\quad + \tfrac{r(\sigma+1)}{\Gamma_q(\sigma)} \left[\|L_2\|\left(1 + \tfrac{1}{\Gamma_q(2-\alpha_2)} + \tfrac{1}{\Gamma_q(1+\beta_2)}\right) + G_0\right]
\end{aligned}$$

$$
\begin{aligned}
&= 3\|\mu_1\|\psi_1\left(\left(1+\tfrac{1}{\Gamma_q(2-\alpha_3)}+\tfrac{1}{\Gamma_q(1+\beta_3)}\right)r\right)+2\|\mu_2\|\psi_2\left(\left(1+\tfrac{1}{\Gamma_q(2-\alpha_4)}+\tfrac{1}{\Gamma_q(1+\beta_4)}\right)r\right)\\
&+\tfrac{r}{\Gamma_q(\sigma)}\left(\tfrac{2}{\sigma}+\sigma+1\right)\left[\|L_1\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right)+F_0\right]\\
&+\tfrac{r}{\Gamma_q(\sigma)}\left(\tfrac{2}{\sigma}+\sigma+1\right)\left[\|L_2\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_2)}+\tfrac{1}{\Gamma_q(1+\beta_2)}\right)+G_0\right]\le r.
\end{aligned}
\tag{3.27}
$$

Hence for each k and $l\in S$, $A[k]+B[l]\in S$. For each $k\in S$, we have

$$
\begin{aligned}
\|A[k]\|_*\le\ & 3\|\mu_1\|\psi_1\left(\left(1+\tfrac{1}{\Gamma_q(2-\alpha_3)}+\tfrac{1}{\Gamma_q(1+\beta_3)}\right)r\right)+2\|\mu_2\|\psi_2\left(\left(1+\tfrac{1}{\Gamma_q(2-\alpha_4)}+\tfrac{1}{\Gamma_q(1+\beta_4)}\right)r\right)\\
&+\tfrac{2r}{\Gamma_q(\sigma+1)}\left[\|L_1\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right)+F_0+\|L_2\|\left(1+\tfrac{1}{\Gamma_q(2-\alpha_2)}+\tfrac{1}{\Gamma_q(1+\beta_2)}\right)+G_0\right].
\end{aligned}
\tag{3.28}
$$

Thus, we conclude the uniformly boundedness of the operator A on S. For any $k\in S$ and $t<\tau\in\overline{J}$, we also have

$$
\begin{aligned}
|A[k](\tau)-A[k](t)|=&(\tau-t)\Bigg[h_2\left(t_0,k(t_0),{}^c\mathcal{D}_q^{\alpha_4}[k](t_0),\mathcal{I}_q^{\beta_4}[k](t_0)\right)\\
&-\int_0^{t_0}\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_1}[k](s),\mathcal{I}_q^{\beta_1}[k](s)\right)\,\mathrm{d}_qs\\
&-\int_{t_0}^1\tfrac{(1-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_2}[k](s),\mathcal{I}_q^{\beta_2}[k](s)\right)\,\mathrm{d}_qs-h_1\left(t_0,k(t_0),{}^c\mathcal{D}_q^{\alpha_3}[k](t_0),\mathcal{I}_q^{\beta_3}[k](t_0)\right)\Bigg],
\end{aligned}
\tag{3.29}
$$

which is not dependent on k and approaches to 0 as $t\to\tau$. Indeed, the operator A is equicontinuous. Consequently, by invoking the Arzelá-Ascoli theorem we asserted that the operator A is compact on S. We now consider two elements k and l belonging to S. Then, we get

$$
\begin{aligned}
|B[k](t)-B[l](t)|&=\left|\mathcal{I}_q^{\sigma}f\left(t,k(t),{}^c\mathcal{D}_q^{\alpha_1}[k](t),\mathcal{I}_q^{\beta_1}[k](t)\right)-\mathcal{I}_q^{\sigma}f\left(t,l(t),{}^c\mathcal{D}_q^{\alpha_1}[l](t),\mathcal{I}_q^{\beta_1}[l](t)\right)\right|\\
&\le\tfrac{\|L_1\|}{\Gamma_q(\sigma+1)}\left[1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right]\|k-l\|_*,\\
|(B[k])'(t)-(B[l])'(t)|&=\left|\mathcal{I}_q^{\sigma-1}f\left(t,k(t),{}^c\mathcal{D}_q^{\alpha_1}[k](t),\mathcal{I}_q^{\beta_1}[k](t)\right)-\mathcal{I}_q^{\sigma-1}f\left(t,l(t),{}^c\mathcal{D}_q^{\alpha_1}[l](t),\mathcal{I}_q^{\beta_1}[l](t)\right)\right|\\
&\le\tfrac{\|L_1\|}{\Gamma_q(\sigma)}\left[1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right]\|k-l\|_*,
\end{aligned}
\tag{3.30}
$$

whenever $0\le t\le t_0$. Also, we have

$$
\begin{aligned}
|B[k](t)-B[l](t)|=&\Bigg|\int_0^{t_0}\tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_1}[k](s),\mathcal{I}_q^{\beta_1}[k](s)\right)\,\mathrm{d}_qs\\
&+\int_{t_0}^{t}\tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_2}[k](s),\mathcal{I}_q^{\beta_2}[k](s)\right)\,\mathrm{d}_qs\\
&-\int_0^{t_0}\tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}f\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_1}[l](s),\mathcal{I}_q^{\beta_1}[l](s)\right)\,\mathrm{d}_qs\\
&-\int_{t_0}^{t}\tfrac{(t-qs)^{(\sigma-1)}}{\Gamma_q(\sigma)}g\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_2}[l](s),\mathcal{I}_q^{\beta_2}[l](s)\right)\,\mathrm{d}_qs\Bigg|\\
&\le\|x-y\|_*\left[\tfrac{\|L_1\|}{\Gamma_q(\sigma+1)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right)+\tfrac{\|L_2\|}{\Gamma_q(\sigma+1)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_2)}+\tfrac{1}{\Gamma_q(1+\beta_2)}\right)\right],
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
|(B[k])'(t)-(B[l])'(t)|=&\Bigg|\tfrac{1}{\Gamma_q(\sigma-1)}\int_0^{t_0}(t-qs)^{(\sigma-2)}f\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_1}[k](s),\mathcal{I}_q^{\beta_1}[k](s)\right)\,\mathrm{d}_qs\\
&+\int_{t_0}^{t}\tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)}g\left(s,k(s),{}^c\mathcal{D}_q^{\alpha_2}[k](s),\mathcal{I}_q^{\beta_2}[k](s)\right)\,\mathrm{d}_qs\\
&-\int_0^{t_0}\tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)}f\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_1}[l](s),\mathcal{I}_q^{\beta_1}[l](s)\right)\,\mathrm{d}_qs\\
&-\int_{t_0}^{t}\tfrac{(t-qs)^{(\sigma-2)}}{\Gamma_q(\sigma-1)}g\left(s,l(s),{}^c\mathcal{D}_q^{\alpha_2}[l](s),\mathcal{I}_q^{\beta_2}[l](s)\right)\,\mathrm{d}_qs\Bigg|\\
\le&\|k-l\|_*\left[\tfrac{\|L_1\|}{\Gamma_q(\sigma)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_1)}+\tfrac{1}{\Gamma_q(1+\beta_1)}\right)+\tfrac{\|L_2\|}{\Gamma_q(\sigma)}\left(1+\tfrac{1}{\Gamma_q(2-\alpha_2)}+\tfrac{1}{\Gamma_q(1+\beta_2)}\right)\right],
\end{aligned}
\tag{3.32}
$$

whenever $t_0 \leq t \leq 1$. Therefore,

$$\|B[k]-B[l]\|_* \leq \Big[\frac{\|L_1\|}{\Gamma_q(\sigma)}\left(1+\frac{1}{\Gamma_q(2-\alpha_1)}+\frac{1}{\Gamma_q(1+\beta_1)}\right)+\frac{\|L_2\|}{\Gamma_q(\sigma)}\left(1+\frac{1}{\Gamma_q(2-\alpha_2)}+\frac{1}{\Gamma_q(1+\beta_2)}\right)\Big]\left(\frac{1}{\sigma}+1\right)\|k-l\|_* \leq \Lambda_2\|k-l\|_*, \tag{3.33}$$

since $\Lambda_2 < 1$. Indeed, operator B is contraction. In the other words, all conditions of Theorem 2.3 are fulfilled. This indicates that there exists a k belonging to S such that $A[k]+B[k]=k$. So, we conclude that equation (1.3) has a solution, which is in $\overline{J}$. This finishes the validation. □

4 Applications

At present, we provide two examples for illustrating our main results which their numerical results are presented using the required algorithms. In this way, some computational results are carried out to check the feasibility of the theoretical findings for FqDE (1.3)-(1.4).

Example 4.1. As the first test example, let us pay attention to the boundary value differential $q-$fractional problem in the form

$$^{c}\mathcal{D}_q^{3/2}[k](t) = \begin{cases} \frac{1}{100}\left(t^2+\frac{1}{2}t-\frac{1}{2}\right)\left[k(t)+\tan^{-1}\left({}^{c}\mathcal{D}_q^{1/3}[k](t)\right)+\sin\left(\mathcal{I}_q^{\sqrt{2}}[k](t)\right)\right], & 0 \leq t \leq \frac{3}{7}, \\ \frac{1}{100}\left(t^2+\frac{(\sqrt{2}-1)}{2}t-\frac{\sqrt{2}}{4}\right)\left[\frac{|k(t)|}{1+|k(t)|}+\frac{\left|{}^{c}\mathcal{D}_q^{1/4}[k](t)+\mathcal{I}_q^{\sqrt{3}}[k](t)\right|}{1+\left|{}^{c}\mathcal{D}_q^{1/4}[k](t)+\mathcal{I}_q^{\sqrt{3}}[k](t)\right|}\right], & \frac{3}{7} \leq t \leq 1. \end{cases} \tag{4.1}$$

The prescribed boundary conditions are

$$k(0) = \frac{e^{3/7}}{100}\left[\frac{\left|k\left(\frac{3}{7}\right)+{}^{c}\mathcal{D}_q^{1/5}[k]\left(\frac{3}{7}\right)+\mathcal{I}_q^{\sqrt{5}}[k]\left(\frac{3}{7}\right)\right|}{1+\left|k\left(\frac{3}{7}\right)+{}^{c}\mathcal{D}_q^{1/5}[k]\left(\frac{3}{7}\right)+\mathcal{I}_q^{\sqrt{5}}[k]\left(\frac{3}{7}\right)\right|}\right], \tag{4.2}$$

and

$$k(1) = \frac{1}{100}\sin\left(\frac{3}{7}\right)\left[\cos\left(k\left(\frac{3}{7}\right)\right)+\sin\left(\left({}^{c}\mathcal{D}_q^{1/6}k\right)\left(\frac{3}{7}\right)\right)+\tan^{-1}\left(\left(\mathcal{I}_q^{\sqrt{6}}k\right)\left(\frac{3}{7}\right)\right)\right]. \tag{4.3}$$

Here, $\sigma=\frac{3}{2}$, $\alpha_1=\frac{1}{3}$, $\alpha_2=\frac{1}{4}$, $\alpha_3=\frac{1}{5}$, $\alpha_4=\frac{1}{6}$, $\beta_1=\sqrt{2}$, $\beta_2=\sqrt{3}$, $\beta_3=\sqrt{5}$, $\beta_4=\sqrt{6}$, $t_0=\frac{1}{2}$,

$$\begin{aligned} f(t,k_1,k_2,k_3) &= \frac{1}{100}\left[t^2+\frac{1}{2}t-\frac{1}{2}\right]\left(k_1+\tan^{-1}(k_2)+\sin(k_3)\right), \\ g(t,k_1,k_2,k_3) &= \frac{1}{100}\left[t^2+\frac{(\sqrt{2}-1)}{2}t-\frac{\sqrt{2}}{4}\right]\left(\frac{|k_1|}{1+|k_1|}+\frac{|k_2+k_3|}{1+|k_2+k_3|}\right), \end{aligned} \tag{4.4}$$

and $h_1(t,k_1,k_2,k_3) = \frac{e^t}{100}\left(\frac{|k_1+k_2+k_3|}{1+|k_1+k_2+k_3|}\right)$, $h_2(t,k_1,k_2,k_3) = \frac{1}{100}\sin(t)\left[\cos(k_1)+\sin(k_2)+\tan^{-1}(k_3)\right]$. Clearly

$$\begin{aligned} |f(t,k_1,k_2,k_3)-f(t,k_1',k_2',k_3')| &\leq \frac{1}{100}(|k_1-k_1'|+|k_2-k_2'|+|k_3-k_3'|), \\ |g(t,k_1,k_2,k_3)-g(t,k_1',k_2',k_3')| &\leq \frac{2+\sqrt{2}}{400}(|k_1-k_1'|+|k_2-k_2'|+|k_3-k_3'|), \\ |h_1(t,k_1,k_2,k_3)-h_1(t,k_1',k_2',k_3')| &\leq \frac{1}{100}e(|k_1-k_1'|+|k_2-k_2'|+|k_3-k_3'|), \end{aligned} \tag{4.5}$$

and $|h_2(t,k_1,k_2,k_3)-h_2(t,k_1',k_2',k_3')| \leq \frac{1}{100}\sin(1)(|k_1-k_1'|+|k_2-k_2'|+|k_3-k_3'|)$, for $t \in \overline{J}$ and $k_1,k_1',k_2,k_2',k_3,k_3' \in \mathbb{R}$. Hence, $L_1=\frac{1}{100}$, $L_2=\frac{2+\sqrt{2}}{400}$, $L_3=\frac{1}{100}e$, $L_4=\frac{1}{100}$, and by using Eq. (3.10), we obtain $\Lambda_1 \approx 0.35919$, 0.32314, 0.30295 for $q=\frac{1}{10}, \frac{1}{2}, \frac{8}{9}$, respectively. These results show in Tables 1 such that they emphasize with underline. Hence, all conditions of Corollary 3.4 are hold. This indicates that the differential $q-$fractional equation (4.1) has an unique solution under the Dirichlet boundary conditions (4.2) and (4.3), here the unique solution is in $\overline{J}$. We also note that, L_1, L_2, L_3, and L_4 are maximum of functions f, g, h_1, and h_2, respectively.

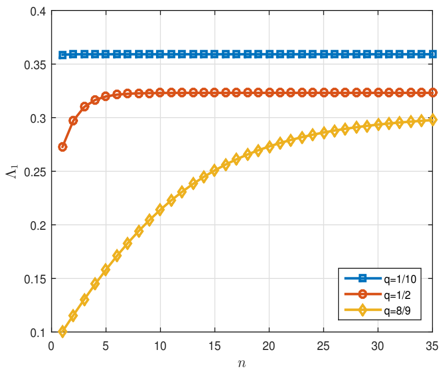

Figure 1: Numerical evaluations of Λ_1 for various $q = \frac{1}{10}, \frac{1}{2}, \frac{8}{9}$ in Example 4.1

Table 1: Numerical evaluations of Λ_1 for various $q = \frac{1}{10}$, $\frac{1}{2}$, and $\frac{8}{9}$ in Example 4.1.

	Λ_1		
n	$q=\frac{1}{10}$	$q=\frac{1}{2}$	$q=\frac{8}{9}$
1	0.35874	0.27282	0.10052
2	0.35915	0.29729	0.11569
3	0.35919	0.31004	0.13043
4	0.35919	0.31655	0.14462
⋮	⋮	⋮	⋮
12	0.35919	0.32312	0.23091
13	0.35919	0.32313	0.23821
14	0.35919	0.32314	0.24485
15	0.35919	0.32314	0.25087
⋮	⋮	⋮	⋮
75	0.35919	0.32314	0.30293
76	0.35919	0.32314	0.30293
77	0.35919	0.32314	0.30294
78	0.35919	0.32314	0.30294
79	0.35919	0.32314	0.30294
80	0.35919	0.32314	0.30295
81	0.35919	0.32314	0.30295
82	0.35919	0.32314	0.30295

Example 4.2. The second test example devoted to the following boundary value differential $q-$fractional problem

$$ {}^{c}\mathcal{D}_q^{4/3}[k](t) = \begin{cases} \frac{\ln\left(t+\frac{7}{8}\right)}{3t+\pi^2+3}\left[\frac{\left|k(t)+{}^{c}\mathcal{D}_q^{1/6}[k](t)+\mathcal{I}_q^{4/5}[k](t)\right|}{1+\left|k(t)+{}^{c}\mathcal{D}_q^{1/6}[k](t)+\mathcal{I}_q^{4/5}[k](t)\right|}\right], & 0 \le t \le \frac{2}{5}, \\ \frac{1}{e^3+1}\left[t-\frac{2}{6}\right]^2\left[k(t)+\cos\left({}^{c}\mathcal{D}_q^{3/7}[k](t)\right)+\sin\left(\mathcal{I}_q^{3/4}[k](t)\right)\right], & \frac{2}{5} \le t \le 1, \end{cases} \tag{4.6} $$

subjected to the boundary conditions

$$ k(0) = e^{2/5}\left[k\left(\tfrac{2}{5}\right) + {}^{c}\mathcal{D}_q^{3/8}[k]\left(\tfrac{2}{5}\right) + \mathcal{I}_q^{9/7}[k]\left(\tfrac{2}{5}\right)\right] \tag{4.7} $$

and $k(1) = \sin\left(\frac{2}{5}\right)\left[k\left(\frac{2}{5}\right) + {}^{c}\mathcal{D}_q^{5/8}[k]\left(\frac{2}{5}\right) + \mathcal{I}_q^{10/7}[k]\left(\frac{2}{5}\right)\right]^{1/2}$. Here, $\sigma = \frac{4}{3}$, $\alpha_1 = \frac{1}{6}$, $\alpha_2 = \frac{3}{7}$, $\alpha_3 = \frac{3}{8}$, $\alpha_4 = \frac{5}{8}$, $\beta_1 = \frac{4}{5}$, $\beta_2 = \frac{3}{4}$, $\beta_3 = \frac{9}{7}$, $\beta_4 = \frac{10}{7}$, $t_0 = \frac{2}{5}$,

$$ f(t, k+1, k_2, k_3) = \frac{\ln\left(t+\frac{7}{8}\right)}{3t+\pi^2+3}\left[\frac{|k_1+k_2+k_3|}{1+|k_1+k_2+k_3|}\right], \qquad g(t, k_1, k_2, k_3) = \frac{1}{e^3+1}\left(t-\tfrac{2}{5}\right)^2(k_1+k_2+k_3), \tag{4.8} $$

and $h_1(t, k_1, k_2, k_3) = e^t\,(k_1+k_2+k_3)$, $h_2(t, k_1, k_2, k_3) = \sin(t)\,(k_1+k_2+k_3)^{1/2}$.

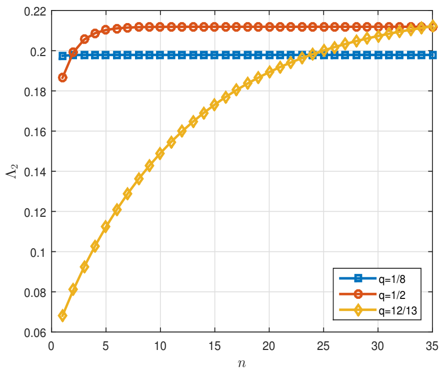

Figure 2: Numerical evaluations of Λ_2 for various $q = \frac{1}{8}, \frac{1}{2}, \frac{12}{13}$ in Example 4.2

Clearly

$$\begin{aligned} |f(t,k_1,k_2,k_3) - f(t,k_1',k_2',k_3')| &\le \tfrac{\ln\left(t+\frac{7}{8}\right)}{3t+\pi^2+3} \left(|k_1-k_1'| + |k_2-k_2'| + |k_3-k_3'|\right), \\ |g(t,k_1,k_2,k_3) - g(t,k_1',k_2',k_3')| &\le \tfrac{1}{e^3+1} \left(t-\tfrac{2}{5}\right)^2 \left(|k_1-k_1'| + |k_2-k_2'| + |k_3-k_3'|\right), \end{aligned} \tag{4.9}$$

and $|h_1(t,k_1,k_2,k_3)| \le e^t(|k_1|+|k_2|+|k_3|)$, $|h_2(t,k_1,k_2,k_3)| \le \sin(t)(|k_1|+|k_2|+|k_3|)^{1/2}$, for all $t \in \overline{J}$ and $k_1, k_2, k_3, k_1', k_2'$ and $k_3' \in \mathbb{R}$. Choose $L_1(t) = \frac{1}{3t+\pi^2+3}\ln(t+\frac{7}{8})$, $L_2(t) = \frac{1}{e^3+1}(t-\frac{2}{5})^2$, $\mu_1(t) = e^t$, $\mu_2(t) = \sin(t)$, $\psi_1(t) = t$, and $\psi_2(t) = t^{1/2}$. Eq. (3.19) yields $\Lambda_2 \approx 0.19790$, 0.21190 and $0.0.22194$ for $q = \frac{1}{8}, \frac{1}{2}$ and $\frac{12}{13}$, respectively. These

Table 2: Numerical evaluations of Λ_2 for various $q = \frac{1}{8}$, $\frac{1}{2}$, and $\frac{12}{13}$ in Example 4.2.

	Λ_2		
n	$q=\frac{1}{8}$	$q=\frac{1}{2}$	$q=\frac{12}{13}$
1	0.19745	0.18656	0.06811
2	0.19785	0.19924	0.081
3	0.1979	0.20557	0.0925
4	0.1979	0.20874	0.10289
5	0.1979	0.21032	0.11235
⋮	⋮	⋮	⋮
11	0.1979	0.21188	0.15468
12	0.1979	0.21189	0.1599
13	0.1979	0.2119	0.1647
14	0.1979	0.2119	0.16913
⋮	⋮	⋮	⋮
112	0.1979	0.2119	0.22193
113	0.1979	0.2119	0.22193
114	0.1979	0.2119	0.22193
115	0.1979	0.2119	0.22194
116	0.1979	0.2119	0.22194
117	0.1979	0.2119	0.22194

values show in Tables 2 such that they emphasize with underline. Consequently, all the assumptions of Theorem 3.5 hold. This implies that the given fractional $q-$differential equation (4.6) admits at least one solution under the given Dirichlet boundary conditions.

5 Conclusion

In this paper, we first gave some properties of the fractional q–derivative and integral, and then using the proposed properties we have established the existence of solutions for the single and multi-dimensional fractional neutral functional q–differential equation (1.3) with Dirichlet boundary conditions (1.4) on a time scale. By numerical evaluations we confirmed our theoretical finding for the underlying model problem. Compared to existing published outcomes in the literature, this results of the current work are new form point of theoretical and numerical computational point of views.

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