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Stanley's conjecture on the Cohen-Macaulay simplicial complexes of codimension 2

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Abstract

Let Δ be a simplicial complex on vertex set $\{x_1,\dots,x_n\}$. It is shown that if Δ is a Cohen-Macaulay simplicial complex of codimension 2, then Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$. As a consequence, we show that if Δ is a quasi-forest simplicial complex, then $\Delta^\vee$ is shellable.

Keywords: Stanley depth, Cohen-Macaulay, partitionable
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1 Introduction

A *simplicial complex* Δ over a set of vertices $V=\{x_1,\dots,x_n\}$ is a collection of subsets of V with the property that:

(a) $\{x_i\}\in\Delta$, for all i;

(b) if $F\in\Delta$, then all subsets of F are also in Δ (including the empty set).

An element of Δ is called a *face* of Δ and complement of a face F is $V\setminus F$ and it is denoted by F^c. Also, the complement of the simplicial complex $\Delta=\langle F_1,\dots,F_r\rangle$ is $\Delta^c=\langle F_1^c,\dots,F_r^c\rangle$. The *dimension* of a face F of Δ, $\dim F$, is $|F|-1$, where $|F|$ is the number of elements of F and $\dim\varnothing=-1$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. A *non-face* of Δ is a subset F of V with $F\notin\Delta$. We denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of Δ. The maximal faces of Δ under inclusion are called *facets* of Δ. The *dimension* of the simplicial complex Δ, $\dim\Delta$, is the maximum of dimensions of its facets. If all facets of Δ have the same dimension, then Δ is called *pure*.

Let $\mathcal{F}(\Delta)=\{F_1,\dots,F_q\}$ be the facet set of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ completely and we write $\Delta=\langle F_1,\dots,F_q\rangle$. A simplicial complex with only one facet is called a *simplex*. A simplicial complex Λ is called a *subcomplex* of Δ, if $\mathcal{F}(\Lambda)\subseteq\mathcal{F}(\Delta)$.

For $v\in V$, the subcomplex of Δ obtained by removing all faces $F\in\Delta$ with $v\in F$ is denoted by $\Delta\setminus v$. That is,

$$\Delta\setminus v=\langle F\in\Delta:\quad v\notin F\rangle.$$

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The *link* of a face $F \in \Delta$, denoted by $\text{link}_\Delta(F)$, is a simplicial complex on V with the faces $G \in \Delta$ such that $G \cap F = \varnothing$ and $G \cup F \in \Delta$. The link of a vertex $v \in V$ is simply denoted by $\text{link}_\Delta(v)$.

$$\text{link}_\Delta(v) = \{F \in \Delta : \quad v \notin F, \quad F \cup \{v\} \in \Delta\}.$$

Let Δ be a simplicial complex over n vertices $\{x_1, \ldots, x_n\}$. For $F \subseteq \{x_1, \ldots, x_n\}$, we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := R/I_\Delta$ the *Stanley-Reisner ring* of Δ.

One of the interesting problems in combinatorial commutative algebra is the Stanley's conjectures. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field K and M be a finitely generated $\mathbb{Z}^n$-graded R-module. Let $m \in M$ be a homogeneous element in M and $Z \subseteq \{x_1, \ldots, x_n\}$. We denote by $mK[Z]$ the K-subspace of M generated by all elements mf where f is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $mK[Z] \subseteq M$ is called a Stanley space of dimension $|Z|$ when $mK[Z]$ is a free $K[Z]$-module. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i].$$

Set $sdepth(\mathcal{D}) = \min\{|Z_i| : i = 1, \ldots, r\}$. The number

$$sdepth(M) = \max\{sdepth(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of M}\}$$

is called Stanley depth of M. The Stanley's conjectures are studied by many researchers. Let R be a $\mathbb{N}^n$-graded ring and M a $\mathbb{Z}^n$- graded R- module. Then Stanley [8] conjectured that

$$depth(M) \leq sdepth(M).$$

He also conjectured in [9] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi in [5] showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. In this paper, we show that if Δ is cohen macaulay simplicial complex of codimension 2, then Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$. Also, it is shown that every k-Cohen-Macaulay simplicial complexes of codimension 3 is partitionable.

2 Preliminaries

In this section we fix some notations and recall some definitions. For a monomial $u = x_1^{a_1} \ldots x_n^{a_n}$ in R, we denote the support of u by *supp*(u) and it is the set of those variables x_i that $a_i \neq 0$. Let m be another monomial in R. If for all $x_i \in supp(u)$, $x_i^{a_i} \nmid m$, then we set $[u, m] = 1$, otherwise we set $[u, m] \neq 1$.

For a monomial ideal $I \subset R$, we set $I^u = (m_i \in G(I) \ : \ [u, m_i] \neq 1)$ and $I_u = (m_i \in G(I) \ : \ [u, m_i] = 1)$. The concept of shedding monomial and k-decomposable monomial ideals was first introduced by Rahmati and Yassemi in [7].

Definition 2.1. Let I be a monomial ideal and $G(I) = \{m_1, \ldots, m_r\}$. The monomial $u = x_1^{a_1} \ldots x_n^{a_n}$ is called a shedding monomial of I if $I_u \neq 0$ and for each $m_i \in G(I_u)$ and each $x_l \in supp(u)$ there exists $m_j \in G(I^u)$ such that $\langle m_j : m_i \rangle = \langle x_l \rangle$.

Definition 2.2. Let I be a monomial ideal and $G(I) = \{m_1, \ldots, m_r\}$. Then I is a *k-decomposable ideal if* $r = 1$ *or else has a shedding monomial* u *with* $| \ supp(u) \ | \leq k + 1$ *such that the ideals* I^u *and* I_u *are* k*-decomposable. Note that since* $| \ G(I) \ |$ *is finite, the recursion procedure will stop.*

A 0-decomposable ideal is called *variable decomposable*. Also, a monomial ideal is decomposable if it is k-decomposable for some $k \geq 0$.

Definition 2.3. A simplicial complex Δ is recursively defined to be *vertex decomposable*, if it is either a simplex, or else has some vertex v so that

(a) Both $\Delta \setminus v$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable, and

(b) No face of $\mathrm{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.

A vertex v which satisfies in condition (b) is called a *shedding vertex*.

Definition 2.4. A simplicial complex Δ is *shellable*, if the facets of Δ can be ordered $F_1, \ldots, F_s$ such that, for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$.

A simplicial complex Δ is called disconnected, if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2. Otherwise Δ is connected. It is well-known that

vertex decomposable $\Longrightarrow$ shellable $\Longrightarrow$ Cohen-Macaulay

Definition 2.5. Given a simplicial complex Δ on V, we define $\Delta^\vee$, the *Alexander dual* of Δ, by

$$\Delta^\vee = \{V \setminus F : \quad F \notin \Delta\}.$$

It is known that for the complex Δ one has $I_{\Delta^\vee} = I(\Delta^c)$. Let $I \neq 0$ be a homogeneous ideal of R and $\mathbb{N}$ be the set of non-negative integers. For every $i \in \mathbb{N}$, one defines:

$$t_i^R(I) = \max\{j : \quad \beta_{i,j}^R(I) \neq 0\},$$

where $\beta_{i,j}^R(I)$ is the i,j-th graded Betti number of I as an R-module. The *Castelnuovo-Mumford regularity* of I is given by:

$$reg(I) = \sup\{t_i^R(I) - i : \quad i \in \mathbb{Z}\}.$$

We say that the ideal I has a *d-linear resolution*, if I is generated by homogeneous polynomials of degree d and $\beta_{i,j}^R(I) = 0$, for all $j \neq i + d$ and $i \geq 0$. For an ideal which has a d-linear resolution, the Castelnuovo-Mumford regularity would be d. If I is a graded ideal of R, then we write (I_d) for the ideal generated by all homogeneous polynomials of degree d belonging to I.

Definition 2.6. A graded ideal I is componentwise linear if (I_d) has a linear resolution for all d.

Also, we write $I_{[d]}$ for the ideal generated by the squarefree monomials of degree d belonging to I.

Definition 2.7. A graded S-module M is called *sequentially Cohen-Macaulay* (over K), if there exists a finite filtration of graded S-modules,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

The Alexander dual, allows us to make a bridge between (sequentially) Cohen-Macaulay ideals and (componetwise) linear ideals.

Definition 2.8 (Alexander duality). For a square-free monomial ideal $I = (M_1, \ldots, M_q) \subset R = K[x_1, \ldots, x_n]$, the *Alexander dual* of I, denoted by $I^\vee$, is defined to be:

$$I^\vee = P_{M_1} \cap \cdots \cap P_{M_q},$$

where P_{M_i} is prime ideal generated by $\{x_j : \quad x_j | M_i\}$.

Theorem 2.9 ([4, Proposition 8.2.20], [3, Theorem 3]). Let I be a square-free monomial ideal in $R = K[x_1, \ldots, x_n]$.

(i) The ideal I is componentwise linear ideal if and only if $R/I^\vee$ is sequentially Cohen-Macaulay.

(ii) The ideal I has a q-linear resolution if and only if $R/I^\vee$ is Cohen-Macaulay of dimension $n-q$.

A monomial ideal $I \subset R = K[x_1,\dots,x_n]$ generated in a single degree is called polymatroidal if for any $u,v \in G(I)$ such that $\deg_{x_i}(u) > \deg_{x_i}(v)$ there an index j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in G(I)$. A squarefree polymatroidal ideal is called matroidal. Also, a monomial ideal I is called weakly polymatroidal if for every two monomials $u = x_1^{a_1}\dots x_n^{a_n} > v = x_1^{b_1}\dots x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1,\dots,a_{t-1} = b_{t-1}$ and $a_t > b_t$, there exists $j > t$ such that $x_t(v/x_j) \in I$. It is clear from the definition that a polymatroidal ideal is weakly polymatroidal.

The following results from [7] are crucial in this paper.

Theorem 2.10 ([7, Theorem 2.10]). Let Δ be a (not necessarily pure) d-dimensional simplicial complex on vertex set $\{x_1,\dots,x_n\}$. Then Δ is k-decomposable if and only if $I_{\Delta^\vee}$ is k-decomposable, where $k \le d$.

Proposition 2.11 ([7, Lemma 3.8]). If I is an squarefree monomial ideal generated in degree 2 which has a linear resolution, then after suitable renumbering of the variables, I is weakly polymatroidal.

Theorem 2.12 ([7, Theorem 3.5]). Let $I \subset R$ be a weakly polymatroidal ideal. Then I is 0-decomposable.

3 Partitionability of Cohen-Macaulay simplicial complexes of codimension 2

As the main result of this section, it is shown that if Δ is Cohen-Macaulay simplicial complex of codimension 2, then Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$. Stanley conjectured in [8] the upper bound for the depth of $K[\Delta]$ as the following:

$$depth(K[\Delta]) \le sdepth(K[\Delta]).$$

Also we recall another conjecture of Stanley. Let Δ be again a simplicial complex on $\{x_1,\dots,x_n\}$ with facets $G_1,\dots,G_t$. The complex Δ is called partitionable if there exists a partition $\Delta = \bigcup_{i=1}^{t}[F_i,G_i]$ where $F_i \subseteq G_i$ are suitable faces of Δ. Here the interval $[F_i,G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [9] and [10] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan and Yassemi [5] proved that for Cohen-Macaulay simplicial complex Δ on $\{x_1,\dots,x_n\}$ we have that $depth(K[\Delta]) \le sdepth(K[\Delta])$ if and only if Δ is partitionable.

Theorem 3.1. If Δ is a Cohen-Macaulay simplicial complex of codimension 2, then Δ is partitionable.

Proof . Since Δ is Cohen-Macaulay simplicial complex of codimension 2, by a result of Eagon and Reiner [3], $I_{\Delta^\vee}$ is a squarefree monomial ideal which has 2-linear resolution. Hence by Proposition 2.11 and Theorem 2.12, $I_{\Delta^\vee}$ is 0-decomposable. It follows from Theorem 2.10 that Δ is vertex decomposable. So Δ is partitionable. □

Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable, as a consequence of our result we obtain the following corollary.

Corollary 3.2. Let Δ be a Cohen-Macaulay simplicial complex of codimension 2 on vertex set $\{x_1,\dots,x_n\}$. Then Stanley's conjecture holds for $K[\Delta]$.

As an immediate consequence we have the following:

Corollary 3.3. Let Δ be a quasi-forest simplicial complex which is not a simplex. Then $\Delta^\vee$ is shellable and Stanley's conjecture holds for $k[\Delta^\vee]$.

Proof . It is proved in [11] that each quasi-forest is a flag complex. So I_Δ is generated by quadratic monomials and hence $ht(I_{\Delta^\vee}) = 2$. Since Δ is quasi-forest by [11, Corollary 5.5], we have $pd(K[\Delta^\vee]) = 2$. Therefore $\Delta^\vee$ is Cohen-Macaulay of codimension 2 and by Theorem 3.1, $\Delta^\vee$ is vertex decomposable and each vertex decomposable complex is shellable. □

Let Δ be a simplicial complex on the vertex set $\{x_1,\dots,x_n\}$ with $\dim\Delta < n-2$. Let F be an arbitrary face of $\Delta^\vee$ and x_0 a new vertex. A cone from x_0 over F, denoted by $co_{x_0}F$, is the simplex on the the vertex set $F \cup \{x_0\}$. Set $\Gamma = \Delta^\vee \cup co_{x_0}F$.

Lemma 3.4 ([1, Lemma1]). Let F be a face of Δ_1 such that F is not $\{x_1, \ldots, x_n\}$. If $\Delta_2 = \Delta_1 \cup co_{x_0}F$, then

$$pd(k[\Delta_2]) = \max\{pd(k[\Delta_1]) + 1, n- \mid F \mid\}.$$

Proposition 3.5. Let Δ be a quasi-forest simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$. Let F be an arbitrary face of $\Delta^\vee$ such that $\dim(\Delta^\vee) =\mid F \mid$ and x_0 a new vertex. Set $\Gamma = \Delta^\vee \cup co_{x_0}F$. Then Γ is Cohen-Macaulay.

Proof . Let Δ be a quasi-forest simplicial complex on vertex set $\{x_1, \ldots, x_n\}$. Then by Corollary 3.3, $\Delta^\vee$ is Cohen-Macaulay. So $ht(I_{\Delta^\vee}) = pd(k[\Delta^\vee])$. On the other hand,

$$ht(I_\Gamma) = ht(I_{\Delta^\vee}) + 1.$$

By 3.4, we obtain

$$pd(k[\Gamma]) = \max\{pd(k[\Delta^\vee]) + 1, n- \mid F \mid\},$$

Auslander-Buchbaum formula implies that

$$pd(k[\Delta^\vee]) = n - depth(k[\Delta^\vee]).$$

Since $\Delta^\vee$ is Cohen-Macaulay, we have

$$\begin{aligned} pd(k[\Delta^\vee]) &= n - \dim(k[\Delta^\vee]) \\ &= n - (\dim \Delta^\vee + 1) \\ &= n- \mid F \mid -1. \end{aligned}$$

Therefore

$$\begin{aligned} pd(k[\Gamma]) &= n- \mid F \mid \\ &= n - \dim \Delta^\vee \\ &= n - (\dim(k[\Delta^\vee]) - 1) \\ &= n - \dim(k[\Delta^\vee]) + 1 \\ &= ht(I_{\Delta^\vee}) + 1 \\ &= ht(I_\Gamma). \end{aligned}$$

This shows that Γ is Cohen-Macaulay. □

As one of our main results, we prove that every k-Cohen-Macaulay simplicial complexes of codimension 3 is partitionable. But before that the following lemmas are needed.

Lemma 3.6 ([6, Lemma 2.3]). Let Δ be a simplicial complex with vertex set V. Let $W \subseteq V$ and let σ be a face in Δ. If $W \cap \sigma = \varnothing$, then $\text{link}_{\Delta \setminus W}\{\sigma\} = \text{link}_\Delta\{\sigma\} \setminus W$.

Definition 3.7. Let K be a field. A simplicial complex Δ with vertex set V is called k-Cohen-Macaulay of dimension r over K if for any subset W of V (including $\varnothing$), $\Delta \setminus W$ is Cohen-Macaulay of dimension r over K.

Lemma 3.8. Let Δ be a simplicial complex with vertex set V. Then the following conditions are equivalent:

(i) Δ is k-Cohen-Macaulay;

(ii) for all $\sigma \in \Delta$, $\text{link}_\Delta\{\sigma\}$ is k-Cohen-Macaulay ;

Proof . By lemma 3.6, for any subset W of V, we have $\text{link}_{\Delta \setminus W}\{\sigma\} = \text{link}_\Delta\{\sigma\} \setminus W$. Since $\Delta \setminus W$ is Cohen-Macaulay so $\text{link}_\Delta\{\sigma\} \setminus W$ is Cohen-Macaulay. Therefore $\text{link}_\Delta\{\sigma\}$ is k-Cohen-Macaulay. □

Now, we are ready that prove one of the main result of this section.

Theorem 3.9. Let Δ be a k-Cohen-Macaulay simplicial complex of codimension 3 on vertex set $\{x_1, \ldots, x_n\}$. Then Δ is partitionable.

Proof . We prove the theorem by induction on the number of vertices $\{x_1, \ldots, x_n\}$ of Δ. If $n = 0$, then $\Delta = \emptyset$ which is a vertex decomposable. Now Let $n \geq 1$ and $x_d \in \{x_1, \ldots, x_n\}$ be a vertex of Δ. Then the simplicial complex $\text{link}_\Delta\{x_d\}$ is a complex on vertices $\{x_1, \ldots, \widehat{x_d}, \ldots, x_n\}$ of dimension $n-1$. By lemma 3.8, $\text{link}_\Delta\{x_d\}$ is again k-Cohen-Macaulay of codimension 3. Therefore by induction hypothesis $\text{link}_\Delta\{x_d\}$ is partitionable.

On the other hand, since Δ is k-Cohen-Macaulay, for each $x_d \in \Delta$, $\Delta \setminus \{x_d\}$ is Cohen-Macaulay of codimension 2 and by Theorem 3.1, $\Delta \setminus \{x_d\}$ is partitionable. It is easy to see that no face of $\text{link}_\Delta\{x_d\}$ is a facet of $\Delta \setminus \{x_d\}$. Therefore any vertex x_d is a shedding vertex. So Δ is vertex decomposable and each vertex decomposable complex is partitionable. □

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