Int. J. Nonlinear Anal. Appl. In Press, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2025.34637.5180

Nonexistence and multiplicity results for binonlocal Leray-Lions type problems

Zohreh Naghizadeh

Faculty of Sciences, Department of Mathematics, University of Science and Technology of Mazandaran, P.O. Box 48518-78195, Behshahr, Iran

(Communicated by Zakieh Avazzadeh)

Abstract

The Leray-Lions operators attract much attention because they are flexible enough to be specified for different elliptic operators. The goal of this paper is to obtain the existence of at least three distinct weak solutions for a Leray-Lions problem of $r(x)$-Kirchhoff type and a nonexistence result in the exponent constant case. The technique is constructed on variational methods.

Keywords: $r(x)$-Kirchhoff type problems, Leray-Lions type operators, Variational techniques
2020 MSC: 35D30, 35J35, 35J60

1 Introduction

The study of Leray-Lions type operator is a new subject for investigation, because they happen in some field, like as electrorheological fluids [12], image processing [14] and etc. Recently, some fourth order Leray-Lions type problems have been investigated. For instance, in [10] by using critical point theorem of [2], the authors ensured multiplicity of weak solutions for a nonlocal biharmonic system including Hardy potential and Leray-Lions operator. In [9], a multiplicity theorem for a fourth-order Leray Lions equation including indefinite weights, was established.

Relatively speaking, biharmonic $r(x)-$Kirchhoff type problems consisting of Leray-Lions operators have rarely been considered. In [13], by applying critical point theory and variational approach, some multiplicity results for a Leray-Lions $r(x)$-Kirchhoff type problem was obtained. Besides, the study of bi-nonlocal problems including Kirchhoff type operator together with the external force term with a nonlocal coefficient can better qualify multiple biological and physical systems (see [5]).

Here, we consider the following form of binonlocal $r(x)-$Kirchhoff type problems including Leray-Lions operator:

$$\begin{cases} M_1\Big(H_{r(x)}(u)\Big)\Delta\left(a(x,\Delta u)+|u|^{r(x)-2}u\right)=\lambda M_2\Big(K(u)\Big)f(x,u(x)) & \text{in } \Omega,\\ u=\Delta u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in which

$$H_{r(x)}(u)=\int_\Omega\left[A(x,\Delta u)+\frac{1}{r(x)}|u|^{r(x)}\right]dx,\quad K(u)=\int_\Omega F(x,u(x))dx, \tag{1.2}$$

and $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ represents a bounded domain with smooth boundary, $F(x,t)=\int_0^t f(x,\gamma)d\gamma$, $a:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ denotes a Carathéodory function obeying the subsequent assumptions:

Email address: z.naghizadeh@mazust.ac.ir (Zohreh Naghizadeh)

Received: July 2024 *Accepted:* February 2025

($\mathbf{A1}$) $a(x,0)=0,$ for $a.e.$ $x\in\overline{\Omega}$.

($\mathbf{A2}$) a verifies the growth condition

$$|a(x,t)|\leq c\left(|t|^{r(x)-1}+g(x)\right),\quad \forall t\in\mathbb{R},\ a.e.\ x\in\Omega,$$

in which $c>0$ denotes some constant, $g\in L^{\frac{r(x)}{r(x)-1}}(\Omega)$, denotes a nonnegative function and $r\in C(\overline{\Omega})$ denotes a Log-Holder continuous function obeying the relationship

$$1<r^{-}:=\inf_{x\in\Omega}r(x)\leq r(x)\leq r^{+}:=\sup_{x\in\Omega}r(x)<\frac{N}{2}. \tag{1.3}$$

($\mathbf{A3}$) For every $t,s\in\mathbb{R}$,

$$\Big(a(x,s)-a(x,t)\Big)(s-t)\geq 0 \text{ for a.e. } x\in\Omega.$$

($\mathbf{A4}$) There is $0<\bar{c}<3\min\{c,1\}$, obeying the following relationship

$$\bar{c}|t|^{r(x)}\leq\min\{r(x)A(x,t),a(x,t)t\}\ \forall t\in\mathbb{R},\ a.e.\ x\in\Omega,$$

in which $A:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ denotes the primitive function of a, in other words,

$$A(x,t)=\int_0^t a(x,\gamma)d\gamma,$$

the operators $\Delta\Big(a(x,\Delta u)\Big)$ represents the fourth order Leray-Lions. If we consider

$$a(x,t)=\varrho(x)|t|^{r(x)-2}t, \tag{1.4}$$

in which $r\in C_{+}(\overline{\Omega}), r^{+}<+\infty$, and select $\varrho\in L^{\infty}(\Omega)$ obeying the relationship

$$\exists\varrho_0>0\ ;\ \varrho(x)\geq\varrho_0>0,\ \text{for a.e. } x\in\Omega,$$

so, (1.4) satisfies conditions ($\mathbf{A1}$) $-$ ($\mathbf{A4}$) and we achieve the following operator

$$\varrho(.)\Delta\left(|\Delta u|^{r(.)-2}\Delta u\right).$$

Whenever $\varrho\equiv 1$, the upper operator becomes the well-known $r(x)-$biharmonic operator $\Delta^2_{r(.)}$, [8]. Now, we give the hypothesis concerning the functions M_1, M_2 and f:

($\mathbf{M}$) $M_1:\mathbb{R}^{+}\to\mathbb{R}^{+}$ and $M_2:\mathbb{R}^{+}\to\mathbb{R}^{+}$ denote continuous functions and there exist three constants $m_1,m_2,m_2'>0$ with $0<m_2\leq m_2'$ and two constants $\beta,\alpha>1$ obeying the relationship

$$M_1(t)\geq m_1t^{\alpha-1},\qquad m_2t^{\beta-1}\leq M_2(t)\leq m_2't^{\beta-1},\quad\forall t\geq 0.$$

$f:\Omega\times\mathbb{R}\to\mathbb{R}$ denotes a function described as

$$f(x,t)=\begin{cases}f_1(x,t) & |t|\geq 1,\\ f_2(x,t) & |t|<1,\end{cases} \tag{1.5}$$

and verifies the following assumption:

($\mathbf{F}$) There are $o_i\in C(\overline{\Omega})$ and $c_i>0, i=1,2$, obeying the relationship

$$|f_i(x,t)|\leq c_i|t|^{o_i(x)-1},\quad 1<o_1^{-}\leq o_1(x)\leq o_1^{+}<\frac{\alpha r^{-}}{\beta}<\frac{\alpha r^{+}}{\beta}<o_2^{-}\leq o_2(x)<r_2^{*}(x), \tag{1.6}$$

in which $r_2^{*}(x)$ will be defined by (2.2) with $m=2$.

The paper contains four sections as follows: Section 2 gives some background and notations related to the function space. The multiplicity and nonexistence theorems are presented in Section 3, whereas the proofs of this theorems are stated in subsections 4.1 and 4.2, respectively.

2 Background

We begin by recalling some essential notions of the generalized Sobolev spaces that will be applied in the next sections. For any $r \in C(\overline{\Omega})$ obeying the relationship (1.3), we describe the Lebesgue variable exponent space as

$$L^{r(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \ \text{ measurable} : \int_\Omega |u(x)|^{r(x)} \mathrm{d}x < +\infty \right\},$$

including the norm

$$|u|_{r(x)} := \inf \left\{ \nu > 0 : \int_\Omega \left| \frac{u(x)}{\nu} \right|^{r(x)} \mathrm{d}x \leq 1 \right\},$$

and the Holder-type inequality

$$\Big| \int_\Omega uv dx \Big| \leq \Big(\frac{1}{r^-} + \frac{1}{(\frac{r}{r-1})^-} \Big) |u|_{r(x)} |v|_{\frac{r(x)}{r(x)-1}}, \ \forall u \in L^{r(x)}(\Omega), v \in L^{\frac{r(x)}{r(x)-1}}(\Omega), \tag{2.1}$$

keeps true.

Proposition 2.1 (see [7, Proposition 2.7]). If $u \in L^{r(x)}(\Omega)$, then

$$\min \Big\{ |u|_{r(x)}^{r^-}, |u|_{r(x)}^{r^+} \Big\} \leq \int_\Omega |u(x)|^{r(x)} dx \leq \max \Big\{ |u|_{r(x)}^{r^-}, |u|_{r(x)}^{r^+} \Big\}.$$

For $m = 1, 2$, the variable exponent Sobolev space is described as

$$W^{m,r(x)}(\Omega) := \Big\{ u \in L^{r(x)}(\Omega) : \ D^\delta u \in L^{r(x)}(\Omega), |\delta| \leq m \Big\},$$

in which $\delta = (\delta_1, \ldots, \delta_N)$ denotes a multi-index, $|\delta| = \sum_{j=1}^N \delta_j$ and $D^\delta u = \frac{\partial^{|\delta|}}{\partial x_1^{\delta_1} \ldots \partial x_N^{\delta_N}} u$. The norm of this space is characterized by

$$\|u\|_{m,r(x)} := \inf \Big\{ \nu > 0 : \ \ B_{r(x)}(\frac{u}{\nu}) \leq 1 \Big\},$$

in which the modular $B_{r(x)} : W^{m,r(x)}(\Omega) \to \mathbb{R}$, is described as

$$B_{r(x)}(u) = \int_\Omega (|\Delta u(x)|^{r(x)} + |u|^{r(x)}) dx.$$

Let $W_0^{1,r(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\text{ in } W^{1,r(x)}(\Omega)}$. Our workspace Z is described as

$$Z := W^{2,r(x)}(\Omega) \cap W_0^{1,r(x)}(\Omega),$$

with

$$\|u\|_Z = \|u\|_{m,r(x)},$$

which displays separable and reflexive Banach space (see [4, 11]). Furthermore, the next embedding proposition take place.

Proposition 2.2 (see [6, Theorem 2.3]). If $h \in C(\overline{\Omega})$ obeying the relationship $1 < h^- \leq h^+ \leq \infty$ and $h(x) \leq r_m^*(x), \forall x \in \overline{\Omega}$, in which

$$r_m^*(x) = \begin{cases} \frac{Nr(x)}{N - mr(x)} & r(x) < \frac{N}{m}, \\ +\infty & r(x) \geq \frac{N}{m}. \end{cases} \tag{2.2}$$

Then the embedding $Z \hookrightarrow L^{h(x)}(\Omega)$ is continuous. If $h(x) < r_m^*(x)$ for each $x \in \overline{\Omega}$, the embedding becomes compact.

By Proposition 2.1, We arrive at the following Proposition.

Proposition 2.3. For every $u \in Z$, we get

(**i**) $\|u\|_Z < 1(> 1; = 1) \Leftrightarrow B_{r(x)}(u) < 1((> 1; = 1)$;

(**ii**) $\min\left\{\|u\|_Z^{r^-}, \|u\|_Z^{r^+}\right\} \le B_{r(x)}(u) \le \max\left\{\|u\|_Z^{r^-}, \|u\|_Z^{r^+}\right\}$.

Remark 2.4. From conditions (**A1**) − (**A4**), we conclude that the function $A(x,t)$ is C^1-Carathéodory and there is $\tilde{c} > 0$, obeying the following relation

$$\frac{\bar{c}}{r(x)}|t|^{r(x)} \le |A(x,t)| \le \tilde{c}\Big(|t|^{r(x)} + g(x)|t|\Big), \ \forall t \in \mathbf{R} \text{ and } a.e.\ x \in \Omega, \tag{2.3}$$

in which the constants $\bar{c}$, is as in condition (**A4**).

Note that if Z^* represents the dual space of Z, then a mapping $G : Z \to Z^*$ is of $(S+)$ type if $u_j \rightharpoonup u$ and $\limsup_{j\to\infty}\langle G(u_j), u_j - u\rangle \le 0$, imply $u_j \to u$ over Z. A functional $B : Z \to \mathbb{R}$ is sequentially weakly lower semicontinous if $u_j \rightharpoonup u$ over Z implies $B(u) \le \liminf_{j\to\infty} B(u_j)$.

Proposition 2.5 (see [3]). Suppose that (**A1**) − (**A4**) are fulfilled and the functional $H_{r(x)} : Z \to \mathbb{R}$ characterized by (1.2). Then we get

(**i**) $H_{r(x)} \in C^1(Z, \mathbb{R})$ with derivative denoted by

$$\langle H'_{r(x)}(u), \eta\rangle = \int_\Omega a(x, \Delta u)\Delta\eta dx + \int_\Omega |u|^{r(x)-2}u\eta dx, \tag{2.4}$$

for all $\eta \in Z$.

(**ii**) $H_{r(x)}$ is sequentially weakly lower semicontinuous.

(**iii**) $H'_{r(x)} : Z \to Z'$ denotes a mapping of (S_+) type .

Proposition 2.6. Suppose that (**F**) is fulfilled. Consider the functional $K : Z \to \mathbb{R}$ by (1.2). Then we get

(**i**) $K \in C^1(Z, \mathbb{R})$ together with derivative denoted by

$$\langle K'(u), \eta\rangle = \int_\Omega f(x, u)\eta dx, \tag{2.5}$$

for all $\eta \in Z$.

(**ii**) K is sequentially weakly continuous over Z, that is, $u_j \rightharpoonup u$ implies that $K(u_j) \to K(u)$.

Proof . By (**F**), the proof of statement (**i**) is immediate. Now let $\{u_j\}$ be any sequence with $u_j \rightharpoonup u$ over Z. By using (**F**) and (2.1), we arrive at

$$\begin{aligned} |K(u_j) - K(u)| &\le c_1 \int_\Omega |u + \mu_j(u_j - u)|^{o_1(x)-1}|u_j - u|dx + c_2 \int_\Omega |u + \mu_j(u_j - u)|^{o_2(x)-1}|u_j - u|dx \\ &\le 2c_1 \Big||u + \mu_j(u_j - u)|^{o_1(x)-1}\Big|_{\frac{o_1(x)}{o_1(x)-1}} |u_j - u|_{o_1(x)} \\ &\quad + 2c_2 \Big||u + \mu_j(u_j - u)|^{o_2(x)-1}\Big|_{\frac{o_2(x)}{o_2(x)-1}} |u_j - u|_{o_2(x)} \end{aligned} \tag{2.6}$$

in which for all $x \in \Omega$; $0 \le \mu_j(x) \le 1$. Besides, since $Z \hookrightarrow L^{o_i(x)}(\Omega)$ is compact for $i = 1, 2$, $u_j \to u$ in $L^{o_i(x)}(\Omega)$. So, it follows from (2.6) that (**ii**) holds true . □

Now, we will express the subsequent theorem that will be indispensable to prove the multiplicity result of this paper.

Proposition 2.7 (see [1, Theorem 2.1]). Assume that $J, I : Z \to \mathbb{R}$ denote two continuously Gâteaux differentiable functionals over reflexive and separable real Banach space Z. If $I(z) \geq 0$ for each $z \in Z$ and there is $z_0 \in Z$ together with $I(z_0) = J(z_0) = 0$ and there are $\eta_0 > 0, z_1 \in Z$ so that

(i) $\eta_0 < I(z_1)$;

(ii) $\sup_{I(z)<\eta_0} J(z) < \eta_0 \frac{J(z_1)}{I(z_1)}$. Moreover, put

$$\xi = \frac{\sigma\eta_0}{\eta_0 \frac{J(z_1)}{I(z_1)} - \sup_{I(z)<\eta_0} J(z)},$$

with $\sigma > 1$, and if $I - \lambda J$ denotes a sequentially weakly lower semicontinuous functional, verifies the (PS) condition and

(iii) $\lim_{\|z\|\to+\infty}(I(z) - \lambda J(z)) = +\infty$ for all $\lambda \in [0, \xi]$.

Then there is a number $\mu > 0$ and an open interval $L \subset [0, \xi]$, so that for all $\lambda \in L$, the equation

$$I'(z) - \lambda J'(z) = 0,$$

possesses at least three distinct solutions over Z whose norms are smaller than μ.

3 Main results

We consider two functionals $J, I : Z \to \mathbb{R}$ as

$$I(u) = \widehat{M_1}\Big(H_{r(x)}(u)\Big), \quad J(u) = \widehat{M_2}\Big(K(u)\Big) \quad \forall u \in Z, \tag{3.1}$$

in which $\widehat{M_i}(t) = \int_0^t M_i(\gamma)d\gamma$ for $i = 1, 2$ and $H_{r(x)}(u)$ and $K(u)$ are defined as (1.2). By Propositions 2.5 and 2.6, $J, I \in C^1(Z, \mathbb{R})$ and

$$\langle I'(u), \eta\rangle = M_1\Big(H_{r(x)}(u)\Big)\langle H'_{r(x)}(u), \eta\rangle, \qquad \langle J'(u), \eta\rangle = M_2\Big(K(u)\Big)\langle K'(u), \eta\rangle, \quad ,$$

for all $\eta, u \in Z$, in which $H'_{r(x)}$ and K' as defined in (2.4) and (2.5), respectively.

Any function $u \in Z$ is named a weak solution of problem (1.1), if the following relationship is verified:

$$\langle I'(u), \eta\rangle - \lambda\langle J'(u), \eta\rangle = 0, \quad \forall \eta \in Z. \tag{3.2}$$

The multiplicity result can be described by the following theorems.

Theorem 3.1. (Multiplicity result) Suppose that (**A1**) − (**A4**), (**M**) and (**F**) are fulfilled. Furthermore, there exist $\bar{t} > 0$ with $F(x, \bar{t}) > 0$ for all $x \in \Omega$. Then, there is an open interval $L \subset [0, \xi]$ and a number $\mu > 0$ so that for each $\lambda \in L$ problem (1.1) possesses at least three weak solutions whose norms are smaller than μ, in which ξ will given later one.

In the special case, when $r(x) \equiv r$ be a constant, problem (1.1) reduces to the following r-Kirchhoff type problem

$$\begin{cases} M_1\Big(H_r(u)\Big)\Delta\left(a(x, \Delta u) + |u|^{r-2}u\right) = \lambda M_2\Big(K(u)\Big)f(x, u(x)) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

and the nonexistence result are stated by the following theorem.

Theorem 3.2. (Nonexistence result in the exponent constant case) Suppose that (**A4**) is fulfilled. In case that $r(x) \equiv r$, the conditions (**M**) and (**F**) change as follow, respectively:

(**M′**) $M_1, M_2 : \mathbb{R}^+ \to \mathbb{R}^+$ denote two continuous functions and there are two numbers $m_1, m'_2 > 0$ and two constant $\beta, \alpha > 1$ verifying

$$M_1(t) \geq m_1 t^{\alpha-1}, \quad M_2(t) \leq m'_2 t^{\beta-1}.$$

(**F′**) There is $c_3 > 0$ so that

$$|f(x, t)| \leq c_3|t|^{\frac{\alpha r}{\beta}-1}, \quad 1 < \frac{\alpha r}{\beta} < r^*, \quad \forall t \in \mathbb{R} \text{ and } \forall \text{ } a.e. \text{ } x \in \Omega.$$

Then there is $\lambda_0 > 0$ so that, problem (3.3) hasn't any nontrivial weak solution over Z for any $\lambda < \lambda_0$.

4 Proof the of the main results

This section is devoted to proving Theorem 3.1 and Theorem 3.2, respectively.

4.1 Proof of Theorem 3.1

We will use Proposition 2.7 to prove Theorem 3.1. So, it is necessary to check all conditions of Proposition 2.7.

Lemma 4.1. The functional $I-\lambda J$ is sequentially weakly lower semicontinuous over Z for each $\lambda>0$.

Proof . Assume that $\{u_j\}$ is a sequence that $u_j \rightharpoonup u$ over Z. By (ii) in Proposition 2.5, we get

$$\liminf_{j\to\infty} H_{r(x)}(u_j) \geq H_{r(x)}(u).$$

Besides, the function $t\to \hat{M}_1(t)$ is monotone. So, we deduce that

$$\liminf_{j\to\infty} I(u_j)=\liminf_{j\to\infty}\widehat{M_1}\Big(H_{r(x)}(u_j)\Big)\geq \widehat{M_1}\Big(\liminf_{j\to\infty} H_{r(x)}(u_j)\Big)\geq \widehat{M_1}\Big(H_{r(x)}(u)\Big)=I(u).$$

So, I is sequentially weakly lower semicontinuous over Z. By (ii) in Proposition 2.6, we arrive at

$$\lim_{j\to\infty} K(u_j)=K(u).$$

Besides, since the function $t\to \hat{M}_2(t)$ is continuous, we arrive at

$$\lim_{j\to\infty} J(u_j)=\lim_{j\to\infty}\widehat{M_2}\Big(K(u_j)\Big)=\widehat{M_2}\Big(\lim_{j\to\infty} K(u_j)\Big)=\widehat{M_2}\Big(K(u)\Big)=J(u).$$

Hence, J is sequentially weakly continuous and hence $I-\lambda J$ is sequentially weakly lower semicontinuous. □

Lemma 4.2. $I-\lambda J$ represents a coercive functional, that is, $\lim_{\|u\|\to+\infty}[I(u)-\lambda J(u)]=+\infty$.

Proof . Let $u\in Z$ with $\|u\|_Z>1$. Put

$$\Omega_1=\{x\in\Omega;\ |u(x)|\geq 1\},\quad \Omega_2=\{x\in\Omega;\ |u(x)|<1\}.$$

By (**M**), (**F**) and (2.3), we arrive at

$$(I-\lambda J)(u)\geq \frac{m_1(\min\{1,\bar{c}\})^\alpha}{\alpha(r^+)^\alpha}\Big(B_{r(x)}(u)\Big)^\alpha-\lambda\frac{m_2'}{\beta}\Big(\frac{c_1}{o_1^-}\int_{\Omega_1}|u|^{o_1(x)}dx+\frac{c_2}{o_2^-}\int_{\Omega_2}|u|^{o_2(x)}dx\Big)^\beta.$$

By (1.6), we infer that

$$\begin{aligned}(I-\lambda J)(u)&\geq \frac{m_1(\min\{1,\bar{c}\})^\alpha}{\alpha(r^+)^\alpha}\Big(B_{r(x)}(u)\Big)^\alpha-\lambda\frac{m_2'}{\beta}\Big(\frac{c_1}{o_1^-}\int_{\Omega_1}|u|^{o_1(x)}dx+\frac{c_2}{o_2^-}\int_{\Omega_2}|u|^{o_1(x)}dx\Big)^\beta\\&\geq \frac{m_1(\min\{1,\bar{c}\})^\alpha}{\alpha(r^+)^\alpha}\Big(B_{r(x)}(u)\Big)^\alpha-\lambda\frac{m_2'}{\beta o_1^-}\Big(\max\{c_1,c_2\}\Big)^\beta\Big(\int\limits_\Omega|u(x)|^{o_1(x)}dx\Big)^\beta.\end{aligned}$$

Now, since the embedding $Z\hookrightarrow L^{o_1(x)}(\Omega)$ is continuous, we infer that

$$\exists\ e_1>0;\quad e_1|u|_{o_1(x)}\leq\|u\|_Z.$$

Since $\|u\|_Z>1$, by Proposition 2.1, we arrive at

$$\int_\Omega|u|^{o_1(x)}dx\leq\max\Big\{\frac{1}{e_1^{o_1^+}},\frac{1}{e_1^{o_1^-}}\Big\}\|u\|_Z^{o_1^+}.$$

So, by using (ii) in proposition 2.3, we infer that

$$(I-\lambda J)(u)\geq \frac{m_1(\min\{1,\bar{c}\})^\alpha}{\alpha(r^+)^\alpha}\|u\|_Z^{\alpha r^-}-\lambda\frac{m_2'}{\beta o_1^-}\Big(\max\{c_1,c_2\}\Big)^\beta\Big(\max\Big\{\frac{1}{e_1^{o_1^+}},\frac{1}{e_1^{o_1^-}}\Big\}\Big)^\beta\|u\|_Z^{\beta o_1^+}.$$

By (1.6), we have $\alpha r^->\beta o_1^+$ and so $I-\lambda J$ is coercive. □

Lemma 4.3. For each $\lambda > 0$, $I - \lambda J$ verifies the (PS) condition, that is, each sequence $\{u_j\}$ obeying the following condition

$$\Big|(I-\lambda J)(u_j)\Big| \le c, \quad (I'-\lambda J')(u_j) \to 0 \text{ over } \ Z^* \text{ as } j\to\infty, \tag{4.1}$$

admits a convergent subsequence in Z.

Proof . Let $\{u_j\}$ be a (PS) sequence for $I - \lambda J$. By Lemma 4.2, $I - \lambda J$ is coercive on Z, so by the first relation in (4.1), the sequence $\{u_j\}$ is bounded over Z. Via the reflexivity of Z, there is a subsequence indicated by $\{u_j\}$ and some $u \in Z$ with $u_j \rightharpoonup u$. We will prove that $\{u_j\} \to u$. Indeed, since $u_j \rightharpoonup u$ over Z, we deduce that

$$M_1(H_{r(x)}(u_j))\langle H'_{r(x)}(u_j),(u_j-u)\rangle - \lambda M_2(K(u_j))\langle K'(u_j),(u_j-u)\rangle \to 0. \tag{4.2}$$

By (**F**) and (1.6) , we infer that

$$\begin{aligned}\Big|\langle K'(u_j), u_j-u\rangle\Big| &\le \int_\Omega |f(x,u_j)||u_j-u|dx\\ &\le \max\{c_1,c_2\}\int_\Omega |u_j|^{o_1(x)-1}|u_j-u|dx.\end{aligned}$$

Since the embedding $Z \hookrightarrow L^{o_1(x)}$ is compact, by (2.1), we deduce that

$$\begin{aligned}\Big|\langle K'(u_j), u_j-u\rangle\Big| &\le \max\{c_1,c_2\}\int_\Omega |u_j|^{o_1(x)-1}|u_j-u|dx\\ &\le 2\max\{c_1,c_2\} \ \Big||u_j|^{o_1(x)-1}\Big|_{\frac{o_1(x)}{o_1(x)-1}}|u_j-u|_{o_1(x)} \to 0.\end{aligned} \tag{4.3}$$

Combining (4.3) with the continuity of the function M_2 and (ii) in Proposition 2.6, we arrive at

$$M_2\Big(K(u_j)\Big)\langle K'(u_j),(u_j-u)\rangle \to 0. \tag{4.4}$$

Hence by (4.2) and (4.4), we arrive at

$$M_1(H_{r(x)}(u_j))\langle H'_{r(x)}(u_j),(u_j-u)\rangle \to 0.$$

Besides, by (ii) in Proposition 2.3 and (**M**), we arrive at

$$\begin{aligned}\Big|M_1\Big(H_{r(x)}(u_j)\Big)\langle H'_{r(x)}(u_j),(u_j-u)\rangle\Big| &\ge \Big|\frac{m_1(\min\{1,\bar c\})^{\alpha-1}}{(r^+)^{\alpha-1}}\Big(B_{r(x)}(u)\Big)^{\alpha-1}\Big| \times \Big|\langle H'_{r(x)}(u_j),(u_j-u)\rangle\Big|\\ &\ge \Big|\frac{m_1(\min\{1,\bar c\})^{\alpha-1}}{(r^+)^{\alpha-1}}\min\Big\{\|u_j\|_Z^{(\alpha-1)r^-},\|u_j\|_Z^{(\alpha-1)r^+}\Big\}\Big|\\ &\quad\times\Big|\langle H'_{r(x)}(u_j),(u_j-u)\rangle\Big| \ge 0.\end{aligned}$$

If $\|u_j\|_Z \to 0$ then $u_j \to 0$ over Z. Otherwise, $\|u_j\|_Z$ is bounded in Z. So

$$\lim_{j\to\infty}\langle H'_{r(x)}(u_j),(u_j-u)\rangle = 0.$$

By (iii) in Proposition 2.5, we obtain $u_j \to u$ over Z and this ends the proof. □

Here, we are prepare to prove Theorem 3.1. Obviously,

$$I(u) \ge 0 \ \ \forall u \in Z, \quad I(0) = J(0) = 0 \tag{4.5}$$

Given $\bar t$ as in Theorem 3.1. If $\tilde\Omega \subset \Omega$ denotes a sufficiently large compact subset and $\bar u \in C_0^\infty(\Omega)$, so that $0 \le \bar u(x) \le \bar t$ over $\Omega\backslash\tilde\Omega$, $\bar u(x) = \bar t$ over $\tilde\Omega$. Put

$$\Omega_3 = \{x\in\Omega;\ |\bar u(x)| \ge 1\}, \quad \Omega_4 = \{x\in\Omega;\ |\bar u(x)| < 1\}.$$

Then by (**M**) and (**F**), we arrive at

$$\begin{aligned} J(\bar{u}) &\geq \frac{m_2}{\beta}(K(\bar{u}))^{\beta} \geq \frac{m_2}{\beta}\Big(\int_{\tilde{\Omega}} F(x,\bar{t})dx - \frac{c_1}{o_1^-}\int_{(\Omega\backslash\tilde{\Omega})\cap\Omega_3} |\bar{u}|^{o_1(x)}dx - \frac{c_2}{o_2^-}\int_{(\Omega\backslash\tilde{\Omega})\cap\Omega_4} |\bar{u}|^{o_2(x)}dx\Big)^{\beta} \\ &\geq \frac{m_2}{\beta}\Big(\int_{\tilde{\Omega}} F(x,\bar{t})dx - \frac{c_1}{o_1^-}\max\Big\{|\bar{t}|^{o_1^+}, |\bar{t}|^{o_1^-}\Big\}\Big)\Big|(\Omega\backslash\tilde{\Omega})\cap\Omega_3\Big| - \frac{c_2}{o_2^-}\max\Big\{|\bar{t}|^{o_2^+}, |\bar{t}|^{o_2^-}\Big\}\Big)\Big|(\Omega\backslash\tilde{\Omega})\cap\Omega_4\Big|\Big)^{\beta} \\ &> 0, \end{aligned}$$

while $|\Omega\backslash\tilde{\Omega}|$ is small enough. Besides, we have

$$I(\bar{u}) = \widehat{M_1}\Big(H_{r(x)}(\bar{u})\Big) \geq \frac{m_1(\min\{1,\bar{c}\})^{\alpha}}{\alpha(r^+)^{\alpha}} \min\Big\{|\bar{t}|^{\alpha r^+}, |\bar{t}|^{\alpha r^-}\Big\}|\tilde{\Omega}|^{\alpha} > 0.$$

Hence

$$0 < \frac{J(\bar{u})}{I(\bar{u})}. \tag{4.6}$$

Now, choose $0 < \eta_0 < \min\Big\{\frac{m_1(\min\{1,\bar{c}\})^{\alpha}}{\alpha(r^+)^{\alpha}}, I(\bar{u})\Big\}$. So, (i) in Proposition 2.7 is achieved with $z_1 = \bar{u}$. Whenever $I(u) < \eta_0$, we arrive at

$$\frac{m_1(\min\{1,\bar{c}\})^{\alpha}}{\alpha(r^+)^{\alpha}}\Big(B_{r(x)}(u)\Big)^{\alpha} \leq I(u) < \eta_0 < \frac{m_1(\min\{1,\bar{c}\})^{\alpha}}{\alpha(r^+)^{\alpha}}. \tag{4.7}$$

So

$$B_{r(x)}(u) < 1,$$

and by (i) in Proposition 2.3, $\|u\|_Z < 1$ which implies by (4.7) that

$$\|u\|_Z^{r^+} = \min\{\|u\|_Z^{r^+}, \|u\|_Z^{r^-}\} \leq B_{r(x)}(u) \leq \Big(\frac{\eta_0\alpha(r^+)^{\alpha}}{m_1(\min\{1,\bar{c}\})^{\alpha}}\Big)^{\frac{1}{\alpha}} = (\eta_0 c_4)^{\frac{1}{\alpha}}.$$

So, we get

$$\|u\|_Z \leq (\eta_0 c_4)^{\frac{1}{\alpha r^+}}. \tag{4.8}$$

By (**F**) and (1.6), we arrive at

$$\begin{aligned} J(u) &\leq \frac{m_2'}{\beta}(K(u))^{\beta} \leq \frac{m_2'}{\beta}\Big(\frac{c_1}{o_1^-}\int_{\Omega_1}|u|^{o_1(x)}dx + \frac{c_2}{o_2^-}\int_{\Omega_2}|u|^{o_2(x)}dx\Big)^{\beta} \\ &\leq \frac{m_2'}{\beta}\Big(\frac{c_1}{o_1^-}\int_{\Omega_1}|u|^{o_2(x)}dx + \frac{c_2}{o_2^-}\int_{\Omega_2}|u|^{o_2(x)}dx\Big)^{\beta} \\ &\leq \frac{m_2'}{\beta o_1^-}\Big(\max\{c_1,c_2\}\Big)^{\beta}\Big(\int_{\Omega}|u|^{o_2(x)}dx\Big)^{\beta}, \end{aligned}$$

Now, by the continuous embedding $Z \hookrightarrow L^{o_2(x)}(\Omega)$, we infer that

$$\exists\ e_2 > 0; \quad e_2|u|_{o_2(x)} \leq \|u\|_Z.$$

Since $\|u\|_Z < 1$, by Proposition 2.1, we have

$$\int_{\Omega}|u|^{o_2(x)}dx \leq \max\Big\{\frac{1}{e_2^{o_2^+}}, \frac{1}{e_2^{o_2^-}}\Big\}\|u\|_Z^{o_2^-}.$$

By (4.8), we arrive at

$$\begin{aligned} J(u) &\leq \frac{m_2'}{\beta o_1^-}\Big(\max\{c_1,c_2\}\Big)^{\beta}\Big(\max\Big\{\frac{1}{e_2^{o_2^+}}, \frac{1}{e_2^{o_2^-}}\Big\}\Big)^{\beta}\|u\|_Z^{\beta o_2^-} \\ &\leq \frac{m_2'}{\beta o_1^-}\Big(\max\{c_1,c_2\}\Big)^{\beta}\Big(\max\Big\{\frac{1}{e_2^{o_2^+}}, \frac{1}{e_2^{o_2^-}}\Big\}\Big)^{\beta}(\eta_0 c_4)^{\frac{\beta o_2^-}{\alpha r^+}} = c_5\eta_0^{\frac{\beta o_2^-}{\alpha r^+}}. \end{aligned}$$

Note that by (1.6), we obtain $\alpha r^+ < \beta o_2^-$. Therefore, relation (4.6) permits us to select η_0 small enough so that

$$\frac{\sup_{I(u)<\eta_0} J(u)}{\eta_0} < \frac{J(\bar{u})}{I(\bar{u})}, \tag{4.9}$$

and (ii) in Proposition 2.7 is achieved. Lemmas 4.1-4.3 and relations (4.5),(4.6) and (4.9) permits us to apply Proposition 2.7 with $z_0 = 0, z_1 = \bar{u}$. Thus there exists a number $\mu > 0$ and an open interval $L \subset [0, \xi]$, in which

$$\xi = \frac{\sigma \eta_0}{\eta_0 \frac{J(\hat{u})}{I(\hat{u})} - \sup_{I(u)<\eta_0} J(u)},$$

with $\sigma > 1$, so that for all $\lambda \in L$ problem (1.1) possesses at least three distinct solutions over Z whose norms are smaller than μ.

4.2 Proof of Theorem 3.2

Note that we consider the exponent constant case $r(x) \equiv r$ be a constant. So, our working space reduce to the space $Z = W^{2,r} \cap W_0^{1,r}(\Omega)$ with

$$\|u\|_{Z_0} = \Big(B_r(u)\Big)^{\frac{1}{r}} = \Big(\int_\Omega (|\Delta u|^r + |u|^r)dx\Big)^{\frac{1}{r}}.$$

Since $1 < \frac{\alpha r}{\beta} < r^*$, the embedding $Z \hookrightarrow L^{\frac{\alpha r}{\beta}}(\Omega)$ is continuous. Let $e_3 > 0$ be the best Sobolev constants for that embedding, that is,

$$e_3 = \inf_{0 \neq u \in Z} \frac{\|u\|_Z}{|u|_{\frac{\alpha r}{\beta}}}.$$

Now, if u denotes any weak solution for the problem (1.1). By $(\mathbf{M'})$, $(\mathbf{A4})$ and $(\mathbf{F'})$, there is a number $c_6 > 0$ so that

$$\frac{m_1(\min\{1, \bar{c}\})^\alpha}{(r)^{\alpha-1}} \|u\|_{Z_0}^{\alpha r} \leq M_1\Big(H_{r(x)}(u)\Big)\langle H'_{r(x)}(u), u\rangle = \lambda M_2\Big(K(u)\Big)\langle K'(u), u\rangle \leq \lambda c_6 m_2' |u|_{\frac{\alpha r}{\beta}}^{\frac{\alpha r}{\beta}}.$$

By taking $\lambda_0 = \frac{m_1(\min\{1,\bar{c}\})^\alpha}{(r)^{\alpha-1}} \frac{e_3^{\alpha r}}{m_2' c_6}$, Theorem 3.2 is proved.

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