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Some results on the vanishing and coassociated prime ideals of the top generalized local cohomology module

Ali Fathi

Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, Iran

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Abstract

Let R be a commutative Noetherian ring, and let $\mathfrak{a}$ be a proper ideal of R. Let M be a non-zero finitely generated R-module with the finite projective dimension p, and let N be a non-zero finitely generated R-module of dimension d. Assume that c is the greatest non-negative integer with the property that $\mathrm{H}^i_{\mathfrak{a}}(M,N)$, the i-th generalized local cohomology module of M,N with respect to $\mathfrak{a}$, is non-zero. It is known that $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is zero for all $i > p+d$ and the top generalized local cohomology $\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)$ is Artinian. In this paper, we study the vanishing and attached primes of $\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)$. Also, we prove that if $\mathrm{H}^{p+r}_{\mathfrak{a}}(M,R/\mathfrak{p}) = 0$ for a fixed non-negative integer r and for all $\mathfrak{p}$ in $\mathrm{supp}_R(N)$, then $\mathrm{H}^{p+s}_{\mathfrak{a}}(M,N) = 0$ for all $s \geq r$ and so $c < p+r$. We deduce that if $p \leq c$, then

$$c = p + \min\{t \in \mathbb{N}_0 : \mathrm{H}^{p+t}_{\mathfrak{a}}(M,R/\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \mathrm{supp}_R(N)\} - 1$$

Also, we prove that, for each i with $p \leq i \leq c$, there exists $\mathfrak{p}_i$ in $\mathrm{supp}_R(N)$ such that $\mathrm{H}^i_{\mathfrak{a}}(M,R/\mathfrak{p}_i) \neq 0$.

Keywords: Generalized local cohomology module, cohomological dimension, coassociated prime ideal, attached prime ideal
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1 Introduction

Throughout this paper, let R be a commutative Noetherian ring with non-zero identity. Let $\mathfrak{a}$ be an ideal of R and N be an R-module. The i-th local cohomology module of N with respect to $\mathfrak{a}$ was defined by Grothendieck as follows:

$$\mathrm{H}^i_{\mathfrak{a}}(N) := \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, N);$$

see [3] for more details. For a pair of R-modules (M,N), the i-generalized local cohomology module of (M,N) with respect to $\mathfrak{a}$ was introduced by Herzog as follows:

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) := \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N);$$

see [2, 14] for more details. It is clear that $\mathrm{H}^i_{\mathfrak{a}}(R,N) = \mathrm{H}^i_{\mathfrak{a}}(N)$. The cohomological dimension of N with respect to $\mathfrak{a}$ and the cohomological dimension of (M,N) with respect to $\mathfrak{a}$ are defined, respectively, as follow:

$$\mathrm{cd}_{\mathfrak{a}}(N) := \sup\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(N) \neq 0\}$$

Email address: alif1387@gmail.com (Ali Fathi)

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and

$$\operatorname{cd}_{\mathfrak{a}}(M,N) := \sup\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,N) \neq 0\}.$$

Assume that N is finitely generated with finite dimension d. By [3, Theorem 6.1.2 and Exercise 7.1.7], $\operatorname{cd}_{\mathfrak{a}}(N) \leq d$ and $\mathrm{H}^d_{\mathfrak{a}}(N)$ is Artinian. When R is local, Dibaei and Yassemi proved in [4, Theorem A] that

$$\operatorname{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(N)) = \{\mathfrak{p} \in \operatorname{Ass}_R(N) : \operatorname{cd}_{\mathfrak{a}}(R/\mathfrak{p}) = d\}.$$

This equality also holds without the hypothesis that R is local (see [6, Theorem 2.5]). If M is finitely generated with finite projective dimension p, then $\operatorname{cd}_{\mathfrak{a}}(M,N) \leq p+d$ and $\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)$ is Artinian (see [2, Lemma 5.1] and [16, Theorem 2.9]). When R is local, as a generalization of the theorem of Dibaei and Yassemi, Gu and Chu show in [11, Theorem 2.3] that

$$\operatorname{Att}_R(\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)) = \{\mathfrak{p} \in \operatorname{Ass}_R(N) : \operatorname{cd}_{\mathfrak{a}}(M, R/\mathfrak{p}) = p+d\}.$$

In [10, Theorem 5.3], the author of this paper, Tehranian and Zakeri proved this equality in the case when R is not necessarily local. Also, it is shown in [10, Theorem 5.6] that

$$\operatorname{Att}_R(\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)) = \operatorname{Supp}_R(\operatorname{Ext}^p_R(M,R)) \cap \operatorname{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(N)) \qquad (\dagger)$$

whenever $R/\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(N))$ is a complete semilocal ring. This equality allows us to compute the set of attached prime ideals of the top generalized local cohomology module $\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)$ from the set of attached prime ideals of the top local cohomology module $\mathrm{H}^d_{\mathfrak{a}}(N)$.

Now we assume that $N \neq \mathfrak{a}N$ and set $c := \operatorname{cd}_{\mathfrak{a}}(N)$. For all $i > p+c$, $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$; see [13, Proposition 2.8]. Since $c \leq d$, $p+c$ yields a sharper upper bound for $\operatorname{cd}_{\mathfrak{a}}(M,N)$. Note that $\mathrm{H}^c_{\mathfrak{a}}(N)$ and $\mathrm{H}^{p+c}_{\mathfrak{a}}(M,N)$ are not necessarily Artinian. In Theorem 3.1, using the set of coassociated prime ideals of $\mathrm{H}^c_{\mathfrak{a}}(N)$, we compute the set of coassociated prime ideals of $\mathrm{H}^{p+c}_{\mathfrak{a}}(M,N)$. More precisely, we have

$$\operatorname{Coass}_R\left(\mathrm{H}^{p+c}_{\mathfrak{a}}(M,N)\right) = \{\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Coass}_R\left(\mathrm{H}^c_{\mathfrak{a}}(N)\right) : \operatorname{proj\,dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = p\}.$$

As a consequence of this equality, Corollary 3.2 shows that the equality (†) holds even if $R/\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(N))$ is not a complete semilocal ring, and so there is the following equality

$$\operatorname{Att}_R\left(\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)\right) = \{\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N) : \operatorname{proj\,dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = p, \operatorname{cd}_{\mathfrak{a}}(R/\mathfrak{p}) = d\}. \qquad (\ddagger)$$

In particular, if R is local and M is Cohen-Macaulay, then

$$\operatorname{Att}_R(\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)) = \{\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N) : \operatorname{cd}_{\mathfrak{a}}(R/\mathfrak{p}) = d\}$$

(see Corollary 3.3). Also, as an application of the equality (‡), the Lichtenbaum-Hartshorne vanishing theorem is extended for generalized local cohomology modules in Theorem 3.4. More precisely, when R is a local ring, the top generalized local cohomology module $\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N)$ is zero if and only if for all $\mathfrak{P} \in \operatorname{Supp}_{\widehat{R}}(\widehat{M}) \cap \operatorname{Ass}_{\widehat{R}}(\widehat{N})$ satisfying $\dim_{\widehat{R}}(\widehat{R}/\mathfrak{P}) = d$ and $\operatorname{proj\,dim}_{\widehat{R}_{\mathfrak{P}}}(\widehat{M}_{\mathfrak{P}}) = p$, we have $\dim_{\widehat{R}}(\widehat{R}/(\mathfrak{a}\widehat{R}+\mathfrak{P})) > 0$.

In [12, Proposition 2.3], Hartshorne proved that, for a non-negative integer r, if $\mathrm{H}^r_{\mathfrak{a}}(N) = 0$ for all R-modules N, then the corresponding condition holds for all $s \geq r$. Dibaei and Yassemi proved in [5] that, for a non-negative integer r and a finitely generated R-module N, if $\mathrm{H}^r_{\mathfrak{a}}(R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$, then $\mathrm{H}^s_{\mathfrak{a}}(N) = 0$ for all $s \geq r$. We extend this result for generalized local cohomology modules. Assume that M, N are two non-zero finitely generated R-modules and $p := \operatorname{proj\,dim}_R(M) < \infty$. In Theorem 3.8, we prove that if $\mathrm{H}^{p+r}_{\mathfrak{a}}(M, R/\mathfrak{p}) = 0$ for a fixed non-negative integer r and for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$, then $\mathrm{H}^{p+s}_{\mathfrak{a}}(M,N) = 0$ for all $s \geq r$ and so $\operatorname{cd}_{\mathfrak{a}}(M,N) < p+r$. Then we deduce in Corollary 3.10 that if $p \leq \operatorname{cd}_{\mathfrak{a}}(M,N)$, then

$$\operatorname{cd}_{\mathfrak{a}}(M,N) = p + \min\{t \in \mathbb{N}_0 : \mathrm{H}^{p+t}_{\mathfrak{a}}(M, R/\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Supp}_R N\} - 1.$$

Also, we prove in Corollary 3.9, for each i with $p \leq i \leq \operatorname{cd}_{\mathfrak{a}}(M,N)$, there exists $\mathfrak{p}_i \in \operatorname{Supp}_R(N)$ such that $\mathrm{H}^i_{\mathfrak{a}}(M, R/\mathfrak{p}_i) \neq 0$.

2 Preliminaries

Let M be an R-module. We denote the localization of M at $\mathfrak{p}$ by $M_{\mathfrak{p}}$, and the set of all prime ideals $\mathfrak{p}$ of R such that $M_{\mathfrak{p}}$ is nonzero is called the support of M and denoted by $\mathrm{Supp}_R(M)$. The annihilator of M in R, denoted by $\mathrm{Ann}_R(M)$, is defined to be the set $\{r \in R : rx = 0 \text{ for all } x \in M\}$. If $\mathfrak{p} := \mathrm{Ann}_R(Rx)$ is a prime ideal of R for some $x \in M$, then $\mathfrak{p}$ is called an associated prime ideal of M, and we denote the set of all associated prime ideals of M by $\mathrm{Ass}_R(M)$. We will denote the set of all positive integers (respectively, non-negative integers) by $\mathbb{N}$ (respectively, $\mathbb{N}_0$).

The concepts of attached prime ideal and secondary representation as the duals of the concepts of associated prime ideal and primary decomposition were introduced by Macdonald in [15]. An R-module M is said to be secondary if $M \neq 0$ and, for each $r \in R$, the endomorphism $\mu_r : M \to M$ defined by $\mu_r(x) = rx$ (for $x \in M$) is either surjective or nilpotent. If M is secondary, then $\mathfrak{p} := \sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal and we say that M is $\mathfrak{p}$-secondary. A prime ideal $\mathfrak{p}$ is called an attached prime ideal of M if M has a $\mathfrak{p}$-secondary quotient. We denote the set of all attached prime ideals of M by $\mathrm{Att}_R(M)$. If M can be written as a finite sum of its secondary submodules, then we say that M has a secondary representation. Such a secondary representation

$$M = M_1 + \cdots + M_t \quad \text{with } \mathfrak{p}_i := \sqrt{\mathrm{Ann}_R(M_i)} \text{ for } i = 1, \ldots, t$$

of M is said to be minimal when none of the modules M_i $(1 \leq i \leq t)$ is redundant (that is, $M_i \nsubseteq M_1 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_t$) and the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are distinct. Since the sum of two $\mathfrak{p}$-secondary submodules of M is again $\mathfrak{p}$-secondary, so if M has a secondary representation, then it has a minimal one. When the above secondary representation is minimal, then $\mathrm{Att}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$, and hence t and the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ are independent of the choice of minimal secondary representation of M. Artinian modules have secondary representation. We refer the readers to [15] for more details.

In [18], Yassemi has introduced the coassociated prime ideal as a dual of associated prime ideal. In Yassemi's definition, we do not need to assume that the module has a secondary representation, and note that if a module has a secondary representation, then its sets of coassociated prime ideals and attached prime ideals are same (see [18, Theorem 1.14]).

Definition 2.1. We say that an R-module M is cocyclic when M is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R, where $E(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$.

Definition 2.2. We say that a prime ideal $\mathfrak{p}$ of R is a coassociated prime ideal of an R-module M when there exists a cocyclic homomorphic image L of M such that $\mathfrak{p} = \mathrm{Ann}_R(L)$. We denote by $\mathrm{Coass}_R(M)$ the set of all coassociated prime ideals of M.

3 Main Results

The following theorem computes the set of coassociated prime ideals of the top generalized local cohomology module by using the set of coassociated prime ideals of the top local cohomology module.

Theorem 3.1 ([9]). Let $\mathfrak{a}$ be an ideal of R and M be a non-zero finitely generated R-module with finite projective dimension p. Let N be an R-module such that $N \neq \mathfrak{a}N$ and $c := \mathrm{cd}_{\mathfrak{a}}(N)$. Then

$$\mathrm{Coass}_R\left(\mathrm{H}^{p+c}_{\mathfrak{a}}(M, N)\right) = \{\mathfrak{p} \in \mathrm{Supp}_R(M) \cap \mathrm{Coass}_R\left(\mathrm{H}^c_{\mathfrak{a}}(N)\right) : \mathrm{proj\,dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = p\}.$$

Now, let the notations and assumptions be as in Corollary 3.2. The author of this paper, Tehranian and Zakeri proved in [10, Theorem 5.6] that

$$\mathrm{Att}_R(\mathrm{H}^{p+d}_{\mathfrak{a}}(M, N)) = \mathrm{Supp}_R(\mathrm{Ext}^p_R(M, R)) \cap \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(N))$$

whenever $B := R/\mathrm{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(N))$ is a complete semilocal ring. Corollary 3.2 shows that the above equality holds without the hypothesis that B is a complete semilocal ring.

Corollary 3.2 ([9]). Let $\mathfrak{a}$ be an ideal of R, and let M and N be two non-zero finitely generated R-modules such that $p := \mathrm{proj\,dim}_R(M) < \infty$ and $d := \dim_R(N) < \infty$. Then $\mathrm{H}^{p+d}_{\mathfrak{a}}(M, N)$ is Artinian and

$$\mathrm{Att}_R(\mathrm{H}^{p+d}_{\mathfrak{a}}(M, N)) = \{\mathfrak{p} \in \mathrm{Supp}_R(M) \cap \mathrm{Ass}_R(N) : \mathrm{proj\,dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = p, \mathrm{cd}_{\mathfrak{a}}(R/\mathfrak{p}) = d\}.$$

Corollary 3.3 ([9]). Let R be a local ring and $\mathfrak{a}$ be an ideal of R. Let M and N be two non-zero finitely generated R-modules such that M is Cohen-Macaulay, $p := \operatorname{proj\,dim}_R(M) < \infty$ and $d := \dim_R(N)$. Then we have

$$\operatorname{Att}_R \mathrm{H}^{p+d}_{\mathfrak{a}}(M,N) = \{\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N) : \operatorname{cd}_{\mathfrak{a}}(R/\mathfrak{p}) = d\}.$$

Theorem 3.4 ([9]). (The Lichtenbaum-Hartshorne vanishing theorem for generalized local cohomology modules). Let $(R,\mathfrak{m})$ be a local ring and $\mathfrak{a}$ be a proper ideal of R. Let M and N be two non-zero finitely generated R-modules such that $p := \operatorname{proj\,dim}_R(M) < \infty$ and $d := \dim_R(N)$. Then the following statements are equivalent:

(i) $\mathrm{H}^{p+d}_{\mathfrak{a}}(M,N) = 0$;

(ii) for each $\mathfrak{P} \in \operatorname{Supp}_{\widehat{R}}(\widehat{M}) \cap \operatorname{Ass}_{\widehat{R}}(\widehat{N})$ satisfying $\operatorname{proj\,dim}_{\widehat{R}_{\mathfrak{P}}}(\widehat{M}_{\mathfrak{P}}) = p$ and $\dim_{\widehat{R}}(\widehat{R}/\mathfrak{P}) = d$, we have $\dim_{\widehat{R}}(\widehat{R}/(\mathfrak{a}\widehat{R}+\mathfrak{P})) > 0$.

Remark 3.5. Let $(R,\mathfrak{m})$ be a local ring. Let M and N be two non-zero finitely generated R-modules such that $p := \operatorname{proj\,dim}_R(M) < \infty$ and $d := \dim_R(N)$. By Grothendieck's vanishing and non-vanishing theorems [3, Theorems 6.1.2 and 6.1.4], we have $\operatorname{cd}_{\mathfrak{m}}(N) = \dim_R(N)$. The exact value of $\operatorname{cd}_{\mathfrak{m}}(M,N)$ is unknown under the above assumptions. However, if in addition R is Cohen-Macaulay, then Divaani-Aazar and Hajikarimi in [7, Theorem 3.5] proved that

$$\operatorname{cd}_{\mathfrak{m}}(M,N) = \dim_R(R) - \operatorname{grade}_R(\operatorname{Ann}_R(N), M).$$

By [2, Lemma 1.5], $p+d$ is an upper bound for $\operatorname{cd}_{\mathfrak{m}}(M,N)$. If we set $\mathfrak{a} := \mathfrak{m}$ in Theorem 3.4, then it is not true to say that since $\dim_{\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R}+\mathfrak{P}) = 0$ for each prime ideal $\mathfrak{P}$ of $\widehat{R}$, $\mathrm{H}^{p+d}_{\mathfrak{m}}(M,N)$ is non-zero and so $\operatorname{cd}_{\mathfrak{m}}(M,N) = p+d$. The following example shows that $p+d$ can be a strict upper bound for $\operatorname{cd}_{\mathfrak{m}}(M,N)$. In fact, if there does not exist a prime ideal $\mathfrak{P}$ in $\operatorname{Supp}_{\widehat{R}}(\widehat{M}) \cap \operatorname{Ass}_{\widehat{R}}(\widehat{N})$ satisfying $\dim_{\widehat{R}}(\widehat{R}/\mathfrak{P}) = d$ and $\operatorname{proj\,dim}_{\widehat{R}_{\mathfrak{P}}}(\widehat{M}_{\mathfrak{P}}) = p$, then the statement (ii) in Theorem 3.4 is true and hence $\mathrm{H}^{p+d}_{\mathfrak{m}}(M,N) = 0$.

Example 3.6. Let K be a field and $R := K[[x,y]]$ be the ring of formal power series over K in indeterminates x,y. Then R is a complete regular local ring of dimension 2 with maximal ideal $\mathfrak{m} := (x,y)$. We set $M := R/(x^2,xy)$. It follows from $\operatorname{Ass}_R(M) = \{(x),(x,y)\}$ that $\operatorname{depth}_R(M) = 0$ and $\dim_R(M) = \dim_R(R/(x)) = 1$. Since R is regular, M has finite projective dimension and so the Auslander-Buchsbaum formula implies that $\operatorname{proj\,dim}_R(M) = 2$. Therefore $\operatorname{proj\,dim}_R(M) + \dim_R(R) = 4$. Now since $\operatorname{Ass}_R(R) = \{0\}$, $\operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(R) = \emptyset$ and so, by Theorem 3.4 or Corollary 3.2, we obtain $\mathrm{H}^4_{\mathfrak{m}}(M,R) = 0$. Hence

$$\operatorname{cd}_{\mathfrak{m}}(M,R) < \operatorname{proj\,dim}_R(M) + \dim_R(R).$$

Furthermore, since R is Cohen-Macaulay and M has a finite projective dimension, the Divaani-Azar–Hajikarimi formula implies that

$$\operatorname{cd}_{\mathfrak{m}}(M,R) = \dim_R(R) - \operatorname{grade}_R(\operatorname{Ann}_R(R), M) = 2 - 0 = 2.$$

In [12, Proposition 2.3], Hartshorne proved that, for a non-negative integer r, if $\mathrm{H}^r_{\mathfrak{a}}(N) = 0$ for all R-modules N, then the corresponding condition holds for all $s \geq r$. Dibaei and Yassemi proved in [5] that, for a non-negative integer r and a finitely generated R-module N, if $\mathrm{H}^r_{\mathfrak{a}}(R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$, then $\mathrm{H}^s_{\mathfrak{a}}(N) = 0$ for all $s \geq r$. In the following theorem we extend this result for generalized local cohomology modules.

Lemma 3.7. Let M,N be two non-zero R-modules such that N is finitely generated. If $\mathrm{H}^t_{\mathfrak{a}}(M,R/\mathfrak{p}) = 0$ for some non-negative integer t and for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$, then $\mathrm{H}^t_{\mathfrak{a}}(M,N) = 0$.

Proof . There is a finite filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = N$$

of submodules of N such that, for each $1 \leq i \leq n$, $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Supp}_R(N)$. For each $1 \leq i \leq n$, the exact sequence

$$0 \to N_{i-1} \to N_i \to R/\mathfrak{p}_i \to 0,$$

induces the exact sequence

$$\mathrm{H}^t_{\mathfrak{a}}(M,N_{i-1}) \to \mathrm{H}^t_{\mathfrak{a}}(M,N_i) \to \mathrm{H}^t_{\mathfrak{a}}(M,R/\mathfrak{p}_i) = 0.$$

It follows, by induction on i, that $\mathrm{H}^t_{\mathfrak{a}}(M,N_i) = 0$ for all $i = 1,\ldots,n$. In particular, $\mathrm{H}^t_{\mathfrak{a}}(M,N) = 0$. □

Theorem 3.8. Let M and N be two non-zero finitely generated R-modules such that $p := \operatorname{proj\,dim}_R M < \infty$. Let $r \in \mathbb{N}_0$ be such that $\mathrm{H}^{p+r}_{\mathfrak{a}}(M, R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$. Then $\mathrm{H}^{p+s}_{\mathfrak{a}}(M, N) = 0$ for all $s \geq r$; in other words, $\operatorname{cd}_{\mathfrak{a}}(M, N) < p + r$.

Proof . In view of Lemma 3.7, it is sufficient for us to prove that $\mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$ and all s with $s \geq r$. We argue by induction on s for $s \geq r$. When $s = r$, there is nothing to prove. Now assume, inductively, that $s > r$ and the result has been proved for smaller values of s. Assume that $\mathfrak{p} \in \operatorname{Supp}_R(N)$. If $\mathfrak{a} \subseteq \mathfrak{p}$, then $\Gamma_{\mathfrak{a}}(R/\mathfrak{p}) = R/\mathfrak{p}$ and so, by [19, Lemma 1.1] (see [8, Corollary 2.3] for the generalization of this lemma), we have

$$\mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p}) \cong \operatorname{Ext}^{p+s}_R(M, R/\mathfrak{p}) = 0$$

because $\operatorname{proj\,dim}_R(M) < p + s$. Now suppose that $\mathfrak{a} \nsubseteq \mathfrak{p}$ and $x \in \mathfrak{a} \setminus \mathfrak{p}$. For each $t \in \mathbb{N}$, the following exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x^t} R/\mathfrak{p} \to R/(\mathfrak{p} + Rx^t) \to 0$$

induces the exact sequence

$$\mathrm{H}^{p+s-1}_{\mathfrak{a}}(M, R/(\mathfrak{p} + Rx^t)) \to \mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p}) \xrightarrow{x^t} \mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p}).$$

By the induction hypothesis, $\mathrm{H}^{p+s-1}_{\mathfrak{a}}(M, R/\mathfrak{q}) = 0$ for all $\mathfrak{q} \in \operatorname{Supp}_R(N)$. Therefore Lemma 3.7 implies that $\mathrm{H}^{p+s-1}_{\mathfrak{a}}(M, R/(\mathfrak{p} + Rx^t)) = 0$ for all $t \in \mathbb{N}$ (we note that $\operatorname{Supp}_R(R/(\mathfrak{p} + Rx^t)) \subseteq \operatorname{Supp}_R(N)$). Hence the multiplication map $\mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p}) \xrightarrow{x^t} \mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p})$ is a monomorphism for all $t \in \mathbb{N}$. On the other hand, since $\mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p})$ is $\mathfrak{a}$-torsion, each element of $\mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p})$ is annihilated by some power of x. Therefore $\mathrm{H}^{p+s}_{\mathfrak{a}}(M, R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$, as required. □

Corollary 3.9. Let $\mathfrak{a}$ be an ideal of R and M, N be two non-zero finitely generated R-modules with $\operatorname{proj\,dim}_R(M) \leq \operatorname{cd}_{\mathfrak{a}}(M, N) < \infty$. Then for each integer i with $\operatorname{proj\,dim}_R(M) \leq i \leq \operatorname{cd}_{\mathfrak{a}}(M, N)$, there exists $\mathfrak{p}_i \in \operatorname{Supp}_R(N)$ such that $\mathrm{H}^i_{\mathfrak{a}}(M, R/\mathfrak{p}_i) \neq 0$.

Proof . It is an immediate consequence of Theorem 3.8. □

Corollary 3.10. Let $\mathfrak{a}$ be an ideal of R and let M, N be two non-zero finitely generated R-modules such that $\operatorname{proj\,dim}_R(M) < \infty$. Then $\operatorname{cd}_{\mathfrak{a}}(M, N) < \infty$ and if, in addition, $\operatorname{proj\,dim}_R(M) \leq \operatorname{cd}_{\mathfrak{a}}(M, N)$, then

$$\operatorname{cd}_{\mathfrak{a}}(M, N) = \operatorname{proj\,dim}_R(M) + \min\{t \in \mathbb{N}_0 : \mathrm{H}^{\operatorname{proj\,dim}_R(M)+t}_{\mathfrak{a}}(M, R/\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Supp}_R N\} - 1.$$

Proof . By [17, Theorem 2.5], we have

$$\operatorname{cd}_{\mathfrak{a}}(M, N) \leq \operatorname{proj\,dim}_R(M) + \operatorname{ara}(\mathfrak{a})$$

and so $\operatorname{cd}_{\mathfrak{a}}(M, N) < \infty$. Also, [1, Theorem B] implies that $\operatorname{cd}_{\mathfrak{a}}(M, R/\mathfrak{p}) \leq \operatorname{cd}_{\mathfrak{a}}(M, N) < \infty$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$. Therefore the set

$$\Sigma := \{t \in \mathbb{N}_0 : \mathrm{H}^{\operatorname{proj\,dim}_R(M)+t}_{\mathfrak{a}}(M, R/\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Supp}_R N\}$$

is not empty (for example $\operatorname{cd}_{\mathfrak{a}}(M, N) + 1 \in \Sigma$) and so there exists $t \in \mathbb{N}_0$ such that $t = \min \Sigma$. Now Theorem 3.8 implies that

$$\operatorname{cd}_{\mathfrak{a}}(M, N) < \operatorname{proj\,dim}_R(M) + t.$$

On the other hand, the minimality of t yields $\mathrm{H}^{\operatorname{proj\,dim}_R(M)+t-1}_{\mathfrak{a}}(M, R/\mathfrak{p}) \neq 0$ for some $\mathfrak{p} \in \operatorname{Supp}_R(N)$. Hence, in view of [1, Theorem B], we have

$$\operatorname{cd}_{\mathfrak{a}}(M, N) \geq \operatorname{cd}_{\mathfrak{a}}(M, R/\mathfrak{p}) \geq \operatorname{proj\,dim}_R(M) + t - 1.$$

The above inequalities prove the claimed equality. □

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