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Multiple nontrivial solutions to a class of fractional Schrödinger-Maxwell systems in Bessel potential space

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Abstract

This work deals with a class of fractional Schrödinger-Maxwell systems related to the distributional Riesz fractional gradient. First, we introduce the latter operator and investigate its appropriate functional framework. Then, we pose the given problem in that space. Applying variational methods combined with the Symmetric Mountain Pass critical point theorem, we obtain infinitely many nontrivial solutions for the system in Bessel potential space.

Keywords: Fractional Schrödinger-Maxwell systems, Bessel potential spaces, distributional Riesz fractional gradient, multiplicity of solutions
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1 Introduction

This paper is concerned with a class of fractional Schrödinger-Maxwell systems of the following form

$$\begin{cases} -D^\gamma.D^\gamma u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -D^\gamma.D^\gamma \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\gamma \in (0,1)$, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $D^\gamma.D^\gamma$ is the distributional Riesz fractional derivative that will be defined below, and we will show its consistency with the fractional Laplacian. This work is motivated by the large interest in the literature around nonlinear systems driven by fractional operators of elliptic type due to the fundamental role of such systems in analyzing of numerous complex phenomena, such as electrical circuits, diffusion, phase transitions, finance, and quantum mechanics. For more related results, we refer the interested reader to [7, 8, 12, 21, 22].

Another important reason for studying (1.1) is that in fractional cases the classical analysis is not available and elliptic PDEs cannot be treated pointwisely. Caffarelli and Silvestre in their celebrated work [13] introduced the reduction method to overcome these difficulties. Since then and with the aid of [17], there have been many excellent works that considered the existence of infinitely many solutions, nontrivial solutions, concentration of solutions, and ground state solutions to different classes of nonlinear fractional Schrödinger systems and nonlinear fractional

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Schrödinger-Poisson systems using different methods including variational tools and critical point theory, see for instance [2, 10, 11, 15, 18, 19, 20, 28, 31] and the references therein. Among them, we pay special attention in our recent work [11] where we studied the following system

$$\begin{cases}(-\Delta)^{\gamma}u+V(x)u+K(x)\phi u=\lambda f(x,u) & \text{in } \mathbb{R}^3,\\ (-\Delta)^{\beta}\phi=K(x)u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.2}$$

where $\gamma,\beta\in(0,1)$, $2\beta+4\gamma>3$, $\lambda\in\mathbb{R}_+$. By using the Fountain Theorem we get the existence of infinitely many solutions for any $\lambda>0$. When $\beta=\gamma$ and $K(x)\equiv 1$ in (1.2) and under different assumptions on f, Kim and Bae [20] showed the existence of infinitely many solutions. If $\beta=\gamma$ and $K(x)\equiv 1$ in (1.2), Jin and Yang [32] obtained three existence and multiplicity results of solutions under similar assumptions of this work. On the other hand, many authors have paid attention to the problems involving fractional (p,q)-Laplacian operator which is a natural generalization of fractional Laplace operator. The interest in nonlocal problems driven by fractional (p,q)-Laplacian is founded in their popularity in several fields of mathematical physics and biology, such as plasma physics, biophysics and strongly anisotropic materials. Interested readers can consult the nice works of Razani et al [5, 24, 26] for more details.

In recent years, great attention has been given to the search for a good concept of fractional derivatives operators. This considerable interest led many researchers to develop a variety of suitable definitions of fractional derivatives, see for instance [1]. In particular, an increasing number of authors have focused their attention on problems involving the so-called distributional Riesz fractional gradient in their works, see e.g. [6, 9, 19, 23, 28, 29]. Shieh and Spector in their pioneered research [28] have studied a new class of fractional PDEs related to the distributional Riesz fractional gradient, and they defined a functional space suitable to study fractional problems in which the latter operator is present. This space is introduced consistently with the well-known fractional Sobolev and Bessel potential spaces.

Furthermore, it is worth mentioning that the fundamental aspect of the distributional Riesz fractional gradient is, that the former (up to a constant) enjoys a unique combination of desirable translational and rotational invariance, and homogeneity properties of order γ as proved in [30] on fractional gradient analysis. This characterization establishes both from a mathematical and physical perspective the distributional Riesz fractional gradient, in some sense as the natural definition for a fractional differential object, and makes this object very interesting and deserves more attention in the literature. Another important aspect of such operator is that we have the convergence to its classical as the fractionality parameter when $\gamma\to 1$ (see [4, 6]), and this led not only fueling further developments in analysis but led also to new application areas and improving the existing ones.

Motivated by all the reasons just described, the aim of this paper is to prove the existence of infinitely many nontrivial solutions in Bessel potential space for (1.1).
In what follows, let $1<p<\infty$, and $0<\gamma<1$. Following [28], consider $u\in L^p\left(\mathbb{R}^N\right)$ such that $I_{1-\gamma}*u$ is well defined, we recall that the Riesz fractional partial derivatives $\dfrac{\partial u}{\partial x_j}$ of order γ can be characterized in distributionally by

$$\langle\frac{\partial^{\gamma}u}{\partial x_i^{\gamma}},\varphi\rangle=(-1)\langle(I_{1-\gamma}*u),\frac{\partial\varphi}{\partial x_j}\rangle=-\int_{\mathbb{R}^N}(I_{1-\gamma}*u)\frac{\partial\varphi}{\partial x_j}dx,$$

where I_γ denotes the Riesz potential of order γ

$$I_\gamma u\,(x):=C_{N,\gamma}\int_{\mathbb{R}^N}\frac{u\,(y)}{|x-y|^{N-\gamma}}dy,\quad\text{with}\quad C_{N,\gamma}:=\frac{\Gamma(\frac{N-\gamma}{2})}{\pi^{\frac{N}{2}}2^{\gamma}\Gamma(\frac{\gamma}{2})}.$$

Consequently, we define the distributional Riesz fractional gradient D^γ by

$$(D^\gamma u)_j=\frac{\partial^\gamma u}{\partial x_j^\gamma}=\frac{\partial}{\partial x_j}I_{1-\gamma}*u,\quad j=1,...,N.$$

As it was observed in [28], the fractional Laplacian for sufficiently regular function u can be written as

$$\begin{aligned}(-\Delta)^\gamma u &= C_{N,\gamma}P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|y|^{N+2\gamma}}dy\\ &= -\sum_{j=1}^{N}\frac{\partial^\gamma}{\partial x_j^\gamma}\frac{\partial^\gamma}{\partial x_j^\gamma}u\\ &= -D^\gamma.D^\gamma u,\end{aligned}\tag{1.3}$$

where $C_{N,\gamma}$ is a normalizing constant [17]. Among the nice properties of the distributional Riesz fractional gradient D^γ, we have an alternative representation of D^γ as it was shown in [28], can be given in terms of the classical gradient ∇

$$D^\gamma u = I_{1-\gamma} * \nabla u \quad \text{for } u \in C_0^\infty(\mathbb{R}^N).$$

Moreover, D^γ can be written for sufficiently regular functions u by [16, 23, 29]

$$D^\gamma u(x) := C_{N,\gamma} \int_{\mathbb{R}^N} [u(x) - u(y)] \frac{x-y}{|x-y|} \frac{1}{|x-y|^{N+\gamma}} dy,$$

Furthermore, for $u, w \in C_0^\infty(\mathbb{R}^N)$ equation (1.3) is to be understood in the following sense

$$\int_{\mathbb{R}^N} D^\gamma u . D^\gamma w dx = \int_{\mathbb{R}^N} (-\Delta)^\gamma u . w dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} u . (-\Delta)^{\frac{\gamma}{2}} w dx,$$

which is particularity useful for the variational formulation of PDEs involving fractional operator. We refer to [16, 23, 28, 29, 30] for more details about this fractional operator. The following assumptions on f and V will be needed throughout the paper:

(A_1) : The nonlinearity $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ fulfills the Carathéodory condition.

(A_2) : There exist constants $C_1 > 0$ and $s \in \mathbb{R}_0^+$ such that

$$|f(x,s)| \le C_1(|s| + |s|^{p-1}), \quad p \in \left(2\theta, 2_\gamma^*\right)$$

where $\theta \in (1, \dfrac{3}{3-2\gamma})$ and $2_\gamma^* = \frac{6}{3-2\gamma}$ is the critical Sobolev exponent.

(A_3) : For every $x \in \mathbb{R}^3$ and $s \in \mathbb{R}$, there exist $\mu > 2\theta$ and $\lambda > 0$ such that

$$\mu F(x,s) \le s f(x,s) + \lambda s^2,$$

where $F(x,s) = \int_0^s f(x,t) dt$.

(A_4) : $\lim_{|s|\to\infty} \frac{F(x,s)}{|s|^{2\theta}} = \infty$ uniformly for a.e. $x \in \mathbb{R}^3$.

(V) : $V \in C(\mathbb{R}^3, \mathbb{R}), V_0 := \inf_{x\in\mathbb{R}^3} V(x) > 0$, where V_0 is a constant, and for every $M > 0$ meas $\left\{x \in \mathbb{R}^3 : \ V(x) \le M\right\} < \infty$.

Our main result is as follows:

Theorem 1.1. Assume that system (1.1) satisfies the assumptions (A_1)-(A_4) and (V), then (1.1) has infinitely many nontrivial solutions.

The rest of the paper is organized as follows. In section 2, we present some auxiliary results that will be used in this paper. The proof of Theorem 1.1 is provided in section 3. In section 4, we give a discussion about our result.

Remark 1.2. (i) In this paper, we do not impose the Ambrosetti-Rabinowitz's 4-superlinearity condition:

$$\exists \mu > 4 \ \text{ such that } \quad 0 < \mu F(x,s) \le s f(x,s) \ \text{ for all } x \in \mathbb{R}^3, \tag{1.4}$$

which was first introduced by Ambrosetti and Rabinowitz in [3]. This condition is important to ensure that the corresponding functional I has the Mountain Pass geometry and to guarantee that the (PS), or (C) sequence of I is bounded.

(ii) Hypothesis (A_3), is weaker than the (1.4)-condition.

2 Preliminaries and Variational Settings

In this section, we state some preliminary results that will be needed later. In the sequel, let $\gamma \in (0,1)$. The fractional Sobolev space $W^{\gamma,2}(\mathbb{R}^N)$ is defined by

$$W^{\gamma,2}(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\gamma}} dxdy < \infty \right\}.$$

endowed with the norm

$$\|u\|_{W^{\gamma,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\gamma}} dxdy + \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}}.$$

The fractional Sobolev space $W^{\gamma,2}(\mathbb{R}^N)$ is simply denoted by $H^\gamma(\mathbb{R}^N)$. Furthermore, for $u \in C_0^\infty(\mathbb{R}^N)$, we can thus define the vector space of fractional differentiable functions $S^{\gamma,2}(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ naturally endowed with the norm

$$\|u\|^2_{S^{\gamma,2}(\mathbb{R}^N)} = \|u\|^2_{L^2(\mathbb{R}^N)} + \|D^\gamma u\|^2_{L^2(\mathbb{R}^N)}. \tag{2.1}$$

According to the Theorem 1.7 in [28], it is exactly the Bessel potential space $L^{\gamma,2}(\mathbb{R}^N)$ defined for $\gamma \in \mathbb{R}_+$ as

$$L^{\gamma,2}(\mathbb{R}^N) := G_\gamma(L^2(\mathbb{R}^N)) = \{G_\gamma * f : f \in L^2(\mathbb{R}^N)\},$$

where the Bessel potential G_γ is defined by (see [27, 28])

$$G_\gamma(x) := \frac{1}{(4\pi)^{\frac{\gamma}{2}}\Gamma(\frac{\gamma}{2})} \int_0^{+\infty} e^{\frac{-\pi|x|^2}{t}} e^{\frac{-t}{4\pi}} t^{\frac{\gamma-N}{2}-1} dt.$$

The norm of this space is $\|u\|_{L^{\gamma,2}(\mathbb{R}^N)} = \|f\|_{L^2(\mathbb{R}^N)}$ if $u = G_\gamma * f$. The following theorem summarize the main properties of this space (see [28]).

Theorem 2.1.
1. If $\gamma \in (0,1)$, then $H^\gamma(\mathbb{R}^N) = W^{\gamma,2}(\mathbb{R}^N) = L^{\gamma,2}(\mathbb{R}^N) = S^{\gamma,2}(\mathbb{R}^N)$ with the norm given by (2.1).
2. If $\gamma \geq 0$ and $2 \leq q \leq 2^*_\gamma$, then $L^{\gamma,2}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$, and the embedding is locally compact if $2 \leq q < 2^*_\gamma$,

Remark 2.2. According to the Theorem 2.1, the Bessel potential space $L^{\gamma,2}(\mathbb{R}^N)$ is topologically undistinguishable from the well known fractional Sobolev space $H^\gamma(\mathbb{R}^N)$.

The homogeneous Sobolev space $D^{\gamma,2}(\mathbb{R}^N)$ for $\gamma \in (0,1)$, is defined by

$$D^{\gamma,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_\gamma}(\mathbb{R}^N) : D^\gamma u \in L^2(\mathbb{R}^N) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{\gamma,2}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |D^\gamma u|^2 dx)^{\frac{1}{2}}.$$

Now, we introduce our working space E as follows

$$E = \left\{ u \in L^{\gamma,2}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} |D^\gamma u|^2 + V(x)|u|^2) dx < \infty \right\},$$

which is a Hilbert space equipped with the norm

$$\|u\|_E = \int_{\mathbb{R}^N} (|D^\gamma u|^2 + V(x)|u|^2)^{\frac{1}{2}} dx.$$

We notice that the two above definitions of $D^{\gamma,2}(\mathbb{R}^N)$ and E coincide with the usual definitions of them in terms of the Gagliardo seminorm, and the assumption (V) implies that $\|u\|_E = \|u\|_{L^{\gamma,2}(\mathbb{R}^N)}$. The following embedding properties are necessary.

Lemma 2.3. E is compactly embedded in $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*_\gamma)$, and continuously embedded in $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*_\gamma]$.

Lemma 2.4. ([17]) For any $\gamma \in (0, \frac{3}{2})$, $D^{\gamma,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2^*_\gamma}(\mathbb{R}^N)$, i.e there exists $C_\gamma > 0$ such that

$$\left(\int_{\mathbb{R}^N} |u|^{2^*_\gamma} dx \right)^{\frac{2}{2^*_\gamma}} \leq C_\gamma \int_{\mathbb{R}^N} |D^\gamma u|^2 dx, \ u \in D^{\gamma,2}(\mathbb{R}^N).$$

The existence of a nontrivial solution to a linear fractional PDEs with variable coefficients is established by the following theorem.

Theorem 2.5. ([28]) Let $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded open set. Assume that $h \in L^2(\Omega)$, such that $I_{1-\gamma} * u$ is well defined and $A : \mathbb{R}^N \longrightarrow \mathbb{R}^{N \times N}$ with coefficients bounded and measurable such that

$$c_*|y|^2 \leq A(x)y.y \ \text{ and } \ A(x)y.y \leq c^*|y|^2$$

For all $x, y \in \mathbb{R}^N$ and some $c_*, c^* > 0$. Then, there exists a unique $u \in L^{\gamma,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} A(x) D^\gamma u . D^\gamma w dx = \int_\Omega h w dx$$

for every $w \in L^{\gamma,2}(\mathbb{R}^N)$.

From now on, we restrict the work space in dimension $N = 3$.

2.1 A reduced problem

For any $u \in E$ and $w \in D^{\gamma,2}(\mathbb{R}^3)$ we have from Hölder inequality, Lemma 2.3 and Lemma 2.4

$$\begin{aligned} \int_{\mathbb{R}^3} u^2 w dx &\leq \|u\|^2_{L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)} \|w\|_{L^{2^*_\gamma}(\mathbb{R}^3)} \\ &\leq C \|u\|^2_E \|w\|_{D^{\gamma,2}(\mathbb{R}^3)}. \end{aligned} \tag{2.2}$$

For any $u \in E$, the Lax-Milgram Theorem implies that there exists a unique $\phi^\gamma_u \in D^{\gamma,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} D^\gamma \phi^\gamma_u . D^\gamma w dx = \int_{\mathbb{R}^3} u^2 w dx \ \ \forall w \in D^{\gamma,2}(\mathbb{R}^3). \tag{2.3}$$

i.e. ϕ^γ_u is a weak solution of $-D^\gamma . D^\gamma \phi^\gamma_u = u^2$. Moreover ϕ^γ_u is given by

$$\phi^\gamma_u(x) = c_\gamma \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2\gamma}} dy, \tag{2.4}$$

which is the Riesz potential (see [27]), where

$$c_\gamma = \pi^{-\frac{3}{2}} 2^{-2\gamma} \frac{\Gamma\left(\frac{3-2\gamma}{2}\right)}{\Gamma(\gamma)}.$$

Taking $w = \phi^\gamma_u$ in (2.2) and (2.3), we derive

$$\|\phi^\gamma_u\|_{D^{\gamma,2}(\mathbb{R}^3)} \leq C \|u\|^2_E. \tag{2.5}$$

Substituting ϕ^γ_u in (1.1), it leads to the equivalent form

$$-D^\gamma . D^\gamma u + V(x) u + \phi^\gamma_u u = f(x, u), \quad x \in \mathbb{R}^3, \tag{2.6}$$

whose corresponding functional $I : E \to \mathbb{R}$ is given as follows

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|D^\gamma u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^\gamma u^2 dx - \int_{\mathbb{R}^3} F(x,u)\,dx. \tag{2.7}$$

Moreover, if we take $w = \phi_u^\gamma$ in (2.2) and (2.3) again, and by (2.5) we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_u^\gamma u^2 dx &\le C\|u\|_E^2 \|\phi_u^\gamma\|_{D^{\gamma,2}(\mathbb{R}^3)}\\ &\le C\|u\|_E^4.\end{aligned} \tag{2.8}$$

Clearly, I is well defined in E and $I \in C^1(E,\mathbb{R})$. Moreover, its derivative is

$$< I'(u), w >= \int_{\mathbb{R}^3} (D^\gamma u.D^\gamma w + V(x)uw + \phi_u^\gamma uw - f(x,u)w)dx, \;\; w \in E. \tag{2.9}$$

Thus, we have the following result:

Theorem 2.6. the pair $(u,\phi) \in E \times D^{\gamma,2}(\mathbb{R}^3)$ is a weak nontrivial solution of (1.1) if and only if $u \in E$ is a critical point of functional I, where $\phi = \phi_u^\gamma$.

Since we do not suppose (1.4), the verification of $(\mathrm{PS})_c$-condition at level $c \in \mathbb{R}$ becomes complicated, thus we introduce the Cerami condition, which was established by Cerami [14]. Assuming that $I \in C^1(E,\mathbb{R})$.

Definition 2.7. The functional I satisfies the Cerami condition at level $c \in \mathbb{R}$, denoted by $(\mathrm{C})_c$-condition, if any sequence $\{u_n\} \subset E$ satisfying

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|(1 + \|u_n\|_E) \to 0 \;\text{ as } n \to \infty.$$

has a convergence subsequence.

Choosing $\{e_i\}$ an orthonormal basis of E and define $X_i = \mathbb{R}e_i$,

$$Y_k = \oplus_{i=1}^k X_i \quad Z_k = \overline{\oplus_{i=k}^\infty X_i} \quad k \in \mathbb{Z}.$$

Clearly, $E = Y_k \oplus Z_k$. To prove our result, we need the following theorem [25] under the $(\mathrm{C})_c$-condition.

Theorem 2.8. (Symmetric Mountain Pass Theorem) Let $E = Y_k \oplus Z_k$ be an infinite dimensional Banach space where Y is finite dimensional, let $I \in C^1(E,\mathbb{R})$ be even, satisfies the $(\mathrm{C})_c$-condition and $I(0) = 0$, if

(i) there exist constants $\rho, \delta > 0$ satisfying $I\left|_{\partial B_\rho \bigcap Z}\right. = \inf_{u \in Z, \|u\| = \rho} I(u) \ge \delta$;

(ii) for every finite dimensional subspace $\tilde{E} \subset E$, there is a constant $C = C(\tilde{E}) > 0$ such that $\max_{u \in \tilde{E}, \|u\| \ge C} I(u) < 0$,

then, I has an unbounded sequence of critical points.

3 Proof of Main Result

Before the proof of theorem, the following Lemma plays an important role in obtaining the existence of weak nontrivial solution for (1.1).

Lemma 3.1. Let $\gamma \in (0,1)$. The functional I satisfies $(\mathrm{C})_c$-condition on E, if (A_1),(A_3)-(A_4) and (V) hold.

Proof . Let $\{u_n\} \subset E$ be a $(\mathrm{C})_c$ sequence of I. This implies that

$$c = I(u_n) + o_n(1) \;\text{ and }\; \langle I'(u_n), u_n\rangle = o_n(1), \tag{3.1}$$

where $o_n(1) \to 0$ as $n \to \infty$. Seeking a contradiction. We assume that $\|u_n\|_E \to \infty$ as $n \to \infty$. Define $\{v_n\} \subset E$ such that $v_n = \frac{u_n}{\|u_n\|_E}$, then clearly $\|v_n\|_E = 1$. Hence, there exists a subsequence $\{v_n\}$ such that $v_n \rightharpoonup v$ in E as $n \to \infty$. From Lemma 2.3 we get, for $2 \le p < 2^*_\gamma$

$$v_n \to v \text{ a.e. in } \mathbb{R}^3 \quad \text{and} \quad v_n \to v \quad \text{in} \quad L^p\left(\mathbb{R}^3\right) \quad \text{as} \quad n \to \infty. \tag{3.2}$$

There are two possible cases. First, we consider the case $v(x) = 0$. From (A_3) we have

$$\begin{aligned} c+1 &\ge I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_E^2 + \left(\frac{1}{4} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \phi_{u_n}^\gamma u_n^2 dx + \int_{\mathbb{R}^3}\left(\frac{f(x,u_n)\,u_n}{\mu} - F(x,u_n)\right)dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_E^2 - \frac{\lambda}{\mu}\int_{\mathbb{R}^3}|u_n|^2 dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_E^2 - \frac{\lambda}{\mu}\|v_n\|_{L^2(\mathbb{R}^3)}^2\|u_n\|_E^2, \end{aligned}$$

which implies

$$\frac{c+1}{\|u_n\|_E^2} \ge \frac{1}{2} - \frac{1}{\mu} - \frac{\lambda}{\mu}\|v_n\|_{L^2(\mathbb{R}^3)}^2.$$

Since $\|u_n\|_E \to \infty$ as $n \to \infty$, then

$$\|v_n\|_{L^2(\mathbb{R}^3)}^2 \ge \frac{\mu-2}{2\lambda},$$

which shows that $v(x) \neq 0$, then we arrive at contradiction. In the second case $v(x) \neq 0$ in $\mathbb{R}^3$, we set $\Sigma = \left\{x \in \mathbb{R}^3 : v(x) \neq 0\right\}$, thus meas$(\Sigma) > 0$. By (3.1), we obtain

$$c = I(u_n) + o_n(1) = \frac{1}{2}\|u_n\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}^\gamma u_n^2 dx - \int_{\mathbb{R}^3} F(x,u_n)dx. \tag{3.3}$$

Since $\|u_n\|_E \to \infty$ as $n \to \infty$, we assert that

$$\int_{\mathbb{R}^3} F(x,u_n)dx \ge \frac{1}{2}\|u_n\|_E^2 - c + o_n(1) \to \infty \quad \text{as } n \to \infty. \tag{3.4}$$

Thus, combining (2.8) and (3.3), we obtain

$$\int_{\mathbb{R}^3} F(x,u_n)dx + c - o_n(1) = \frac{1}{2}\|u_n\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}^\gamma u_n^2 dx \le \frac{3}{4}\|u_n\|_E^4. \tag{3.5}$$

Moreover, it follows from (A_4) that there exists $s_0 > 1$ such that $F(x,s) > |s|^{2\theta}$, for all $|s| > s_0$, $x \in \mathbb{R}^3$. Since f satisfies the Carathéodory condition and by means of (A_2), we infer that there exists a positive number K such that $|F(x,s)| < K$, for all $(x,s) \in \mathbb{R}^3 \times [-s_0, s_0]$. Then, we can choose $K_0 \in \mathbb{R}$ such that $F(x,s) \ge K_0$, for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$, and thus

$$\frac{F(x,u_n) - K_0}{\|u_n\|_E^{2\theta}} \ge 0. \tag{3.6}$$

It follows from (A_4) that

$$\lim_{n\to\infty}\frac{F(x,u_n)}{\|u_n\|_E^{2\theta}} = \lim_{n\to\infty}\frac{F(x,u_n)}{|u_n|^{2\theta}}.|v_n|^{2\theta} = \infty, \tag{3.7}$$

for all $x \in \Sigma$. Thus, we see that meas$(\Sigma) = 0$. Indeed, if meas$(\Sigma) \neq 0$, then it follows from (3.4)-(3.7) and Fatou's Lemma that

$$\begin{aligned}
\frac{3}{4} =& \liminf_{n\to\infty} \frac{\frac{3}{4}\int_{\mathbb{R}^3} F(x,u_n)dx}{\int_{\mathbb{R}^3} F(x,u_n)dx + c} \\
\geq& \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{\frac{3}{4}F(x,u_n)}{\frac{3}{4}\left\|u_n\right\|_E^4} dx \\
\geq& \liminf_{n\to\infty} \int_{\Sigma} \frac{F(x,u_n)}{\left\|u_n\right\|_E^4} dx - \liminf_{n\to\infty} \int_{\Sigma} \frac{K_0}{\left\|u_n\right\|_E^4} dx \\
\geq& \liminf_{n\to\infty} \int_{\Sigma} \frac{F(x,u_n) - K_0}{\left\|u_n\right\|_E^4} dx. \\
\geq& \int_{\Sigma} \liminf_{n\to\infty} \frac{F(x,u_n)}{\left\|u_n\right\|_E^4} dx - \int_{\Sigma} \limsup_{n\to\infty} \frac{K_0}{\left\|u_n\right\|_E^4} dx = \infty,
\end{aligned} \tag{3.8}$$

which is a contradiction, then $v(x) = 0$ a.e $x \in \mathbb{R}^3$. Thus, $\{u_n\}$ is bounded in E. Up to a subsequence, we can assume that $u_n \rightharpoonup u$ in E, from Lemma 2.3 we conclude $u_n \to u$ in $L^p\left(\mathbb{R}^3\right)$, for all $2 \leq p < 2_\gamma^*$. Clearly, we have

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \text{ and } \|u_n - u\|_{L^2(\mathbb{R}^3)}^2 \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Combining the Hölder inequality, Lemma 2.4 and (2.5), we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} \phi_{u_n}^\gamma u_n (u_n - u)\, dx\right| &\leq \left\|\phi_{u_n}^\gamma\right\|_{L^{2_\gamma^*}(\mathbb{R}^3)} \left\|u_n\right\|_{L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)} \left\|u_n - u\right\|_{L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)} \\
&\leq C\left\|\phi_{u_n}^\gamma\right\|_{D^{\gamma,2}(\mathbb{R}^3)} \left\|u_n\right\|_{L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)} \left\|u_n - u\right\|_{L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)} \\
&\leq C\left\|u_n\right\|_E^3 \left\|u_n - u\right\|_{L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)}.
\end{aligned}$$

Similarly, we derive that

$$\left|\int_{\mathbb{R}^3} \phi_u^\gamma u (u_n - u)\, dx\right| \leq C\left\|u\right\|_E^3 \left\|u_n - u\right\|_{L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)}.$$

We have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \left(\phi_{u_n}^\gamma u_n - \phi_u^\gamma u\right)(u_n - u)\, dx\right| \\
&\leq \left|\int_{\mathbb{R}^3} \phi_{u_n}^\gamma u_n (u_n - u)\, dx\right| + \left|\int_{\mathbb{R}^3} \phi_u^\gamma u (u_n - u)\, dx\right| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.10}$$

According to (A_1) and the Hölder inequality, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \left(f(x,u_n) - f(x,u)\right)(u_n - u)\, dx\right| \\
&\leq C_1 \int_{\mathbb{R}^3} (|u_n| + |u|)\,|u_n - u|\, dx + C_1 \int_{\mathbb{R}^3} \left(|u_n|^{p-1} + |u|^{p-1}\right)|u_n - u|\, dx \\
&\leq C_1 \left(\|u_n\|_{L^2(\mathbb{R}^3)} + \|u\|_{L^2(\mathbb{R}^3)}\right)\|u_n - u\|_{L^2(\mathbb{R}^3)} + C_1\left(\|u_n\|_{L^p(\mathbb{R}^3)}^{p-1} + \|u\|_{L^p(\mathbb{R}^3)}^{p-1}\right)\|u_n - u\|_{L^p(\mathbb{R}^3)} \\
&\leq C\left(\|u_n\|_E + \|u\|_E\right)\|u_n - u\|_{L^2(\mathbb{R}^3)} + C\left(\|u_n\|_E^{p-1} + \|u\|_E^{p-1}\right)\|u_n - u\|_{L^p(\mathbb{R}^3)} \to 0
\end{aligned}$$

as $n \to \infty$. Thus

$$\|u_n - u\|_E^2 = \langle I'(u_n) - I'(u), u_n - u\rangle - \int\limits_{\mathbb{R}^3} (V(x)u_n(u_n-u) - V(x)u(u_n-u))\,dx$$
$$- \int\limits_{\mathbb{R}^3} \left(\phi_{u_n}^{\gamma} u_n - \phi_u^{\gamma} u\right)(u_n - u)\,dx + \int\limits_{\mathbb{R}^3} (f(x,u_n) - f(x,u))(u_n-u)dx \to 0$$

as $n \to \infty$. Therefore, $\{u_n\}$ converges strongly in E. □

Lemma 3.2. Suppose that (A_1),(A_4) and (V) are satisfied. Then, for each finite dimensional subspace $\tilde{E} \subset E$, we have

$$I(u) \to -\infty, \qquad \|u\|_E \to \infty, \quad u \in \tilde{E}.$$

Proof . Arguing indirectly, suppose that there exists $M > 0$ for some $\{u_n\} \subset \tilde{E}$ and all $n \in \mathbb{N}$, such that $I(u_n) \geq -M$ with $\|u_n\|_E \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|_E}$, then $\|v_n\|_E = 1$, up to subsequence we may suppose that $v_n \rightharpoonup v$ in E. Since $\dim(\tilde{E}) < \infty$, then $v_n \to v \in \tilde{E}$ in E, $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^3$ and so $\|v\|_E = 1$. Set $\Sigma = \left\{x \in \mathbb{R}^3 : v(x) \neq 0\right\}$. Thus meas$(\Sigma) > 0$, and we have $|u_n(x)| \to \infty$ for a.e. $x \in \Sigma$. From (2.8) we get

$$\lim_{n\to\infty} \frac{4\int\limits_{\mathbb{R}^3} F(x,u_n)}{\|u_n\|_E^4} dx = \lim_{n\to\infty} \int\limits_{\Sigma} \frac{2\|u_n\|_E^2 + \int\limits_{\mathbb{R}^3} \phi_{u_n}^{\gamma} u_n^2 dx - 4I(u_n)}{\|u_n\|_E^4} dx \leq C, \tag{3.11}$$

for $x \in \Sigma$. Thus, since we have $|u_n(x)| \to \infty$, by similar argument in (3.6)-(3.8) and from (A_4), for large n we get

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^3} \frac{4F(x,u_n)}{\|u_n\|_E^{2\theta}} dx \geq \lim_{n\to\infty} \int\limits_{\Sigma} \frac{4F(x,u_n)}{\|u_n\|_E^{2\theta}} dx = \infty. \tag{3.12}$$

which is a contradiction with (3.11). □

Corollary 3.3. Under assumptions (A_1)-(A_4) and (V), for every finite dimensional subspace $\tilde{E} \subset E$, there exists a constant $C = C(\tilde{E}) > 0$ such that

$$I(u) \leq 0 \quad \text{for all} \ \ u \in \tilde{E} \quad \text{with } \|u\|_E \geq C.$$

Lemma 3.4. For $2 \leq p < 2_\gamma^*$, we have that

$$\Gamma_k := \sup_{u \in Z_k, \|u\|=1} \|u\|_{L^p(\mathbb{R}^3)} \to 0 \quad \text{as} \ \ k \to \infty$$

Proof . Since the embedding from E into L^p is compact, then we can prove Lemma 3.4 by a similar way as Lemma 2.10 in [15]. □

By Lemma 3.4, we can choose an integer $m \geq 1$ such that

$$\|u\|_{L^2}^2 \leq \frac{1}{2C_1} \|u\|_E^2, \quad \|u\|_{L^p}^p \leq \frac{p}{4C_1} \|u\|_E^p \quad \forall u \in Z_m. \tag{3.13}$$

Lemma 3.5. Suppose that (A_1),(A_4) and (V) are satisfied, there exist constants $\rho, \delta > 0$ satisfying $I\left|_{\partial B_\rho \bigcap Z_m}\right. \geq \delta > 0$.

Proof . From (A_1) and (3.13), for $u \in Z_m$, choosing $\rho := \|u\|_E = \frac{1}{2}$, we derive

$$\begin{aligned}
I(u) &= \frac{1}{2}\int\limits_{\mathbb{R}^3} \left(|D^\gamma u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int\limits_{\mathbb{R}^3} \phi_u^\gamma u^2 dx - \int\limits_{\mathbb{R}^3} F(x,u)\,dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{C_1}{2}\|u\|_{L^2(\mathbb{R}^3)}^2 - \frac{C_1}{p}\|u\|_{L^p(\mathbb{R}^3)}^p \\
&\geq \frac{1}{4}(\|u\|_E^2 - \|u\|_E^p) \\
&= \frac{2^{p-2}-1}{2^{p+2}} := \delta > 0.
\end{aligned}$$

This completes the proof.

Proof of Theorem 1.1 Let $Y = Y_m$ and $Z = Z_m$. By Lemma 3.1, Lemma 3.5 and Corollary 3.3, all conditions of Theorem 2.8 are satisfied. Thus, the problem (1.1) has infinitely many nontrivial solutions. □

4 Conclusion

In this paper, through the Symmetric Mountain Pass Theorem, we obtained the existence of infinitely many nontrivial solutions in Bessel potential space to a new class of fractional Schrödinger-Maxwell systems driven by distributional Riesz fractional gradient. From our perspective, this paper seems to enrich the related results of systems that involve this fractional operator and corresponding functional space. In addition, since the concept of distributional Riesz fractional gradient is very recent, it will be useful to expand the result of the present paper to the result that involves a singularity. This suggestion will be treated in future studies.

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