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Fuzzy stability of a system of Euler-Lagrange quadratic functional equations via classical methods

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Abstract

This paper aims to establish stability for a system of Euler-Lagrange quadratic functional equations by applying the so-called direct (Hyers) method and a fixed point technique in the setting of fuzzy normed spaces. Comparing the results obtained by the mentioned ways, we find that the fixed point tool not only needs a less conditions for the proofs but also gives us a more exact approximation of approximately Euler-Lagrange-type quadratic mappings in comparison to the direct approach.

Keywords: Fuzzy normed space, fuzzy Hyers-Ulam-Rassias stability, Euler-Lagrange-type quadratic mapping
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1 Introduction

A fuzzy concept after the definition of fuzzy sets which has been introduced by Zadeh [42], is fuzzy norm on linear spaces. A fuzzy norm on a linear space has been defined by Katsaras [24] and almost simultaneously, Wu and Fang [40] presented a concept of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological vector space. Next, Biswas [7] posed a notion of fuzzy inner product spaces in a linear space. Since then, many authors and mathematicians have indicated more discussions of fuzzy norms on a vector space from various points of view [5, 19, 26, 39]. Following Cheng and Mordeson [14] who introduced a definition of fuzzy norm on a vector space and studied this the new concept in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [25], Bag and Samanta [5] modified their definition and then investigated some properties of fuzzy normed spaces [6].

One of the challenging problems in nonlinear analysis and approximation theory, which has a special importance, undoubtedly, is the stability of functional equations, which many authors are working on nowadays. The story of stability theory has been pioneered by Ulam [38] concerning a query stability of homomorphisms on groups. Hyers [23] reacted positively to the mentioned problem for more groups, assuming that Banach spaces are the groups and also homomorphisms are the linear mappings. Let us recall that the stability of a functional equation is concerned

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with approximating an approximate solution by an exact solution. After that, this theory for functional equations and mappings were extended on miscellaneous spaces which are available in many articles and books; see for instance [4], [16], [20], [21], [34], [35] and more references therein. Moreover, the Hyers-Ulam-Rassias stability of the various functional equations in fuzzy normed spaces were studied for example in [27], [28], [29], [30] and [31].

Let us remember that Skof [36] was the first author who introduced and studied the quadratic functional equation

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y). \tag{1.1}$$

Here, we recall that equation (1.1) is the main tool for characterizing of inner product spaces. In fact, this subject comes from here that the parallelogram equality

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+\|y\|^2$$

is valid for a square norm on an inner product space; for more information about the equations above we refer to [1], [3], [18] and references therein. Moreover, some applications of quadratic functional equations for the stability of ternary quadratic derivations on ternary Banach algebras and C^*-ternary rings and also the stability of quadratic double centralizers and quadratic multipliers are presented in [9] and [10]. A generalized form of (1.1), namely, the Euler-Lagrange-type quadratic mapping, presented by J. M. Rassias [33] as follows:

For linear spaces V and W, recall that a mapping $\mathfrak{Q}: V \longrightarrow W$ fulfills the functional equation

$$\mathfrak{Q}(au+bv)+\mathfrak{Q}(bu-av)=(a^2+b^2)[\mathfrak{Q}(u)+\mathfrak{Q}(v)] \tag{1.2}$$

is called *Euler-Lagrange quadratic*, where $u,v\in V$ in which a,b are two fixed real numbers with $a^2+b^2>1$ and $|a|,|b|\neq 1$. Note that if $|a|=|b|=1$, then with a little connivance, equations (1.1) and (1.2) are the same. Moreover, the function $\mathfrak{Q}(v)=cv^2$ is a common solution of (1.1) and (1.2). After that, Xu [41] extended equation (1.2) to multiple variable mappings and proposed the following definition.

Definition 1.1. Let V and W be vector spaces over real numbers, $n\in\mathbb{N}$. A multivariable mapping $f: V^n \simeq \overbrace{V\times\cdots\times V}^{n-\text{times}} \longrightarrow W$ is said to be *n-Euler-Lagrange quadratic* or *multi-Euler-Lagrange quadratic* if f fulfills the general system of quadratic equations

$$\begin{aligned}&f(u_1,\ldots,u_{i-1},au_i+bu_i',u_{i+1},\ldots,u_n)+f(u_1,\ldots,u_{i-1},bu_i-au_i',u_{i+1},\ldots,u_n)\\&=\lambda f(u_1,\ldots,u_{i-1},u_i,u_{i+1},\ldots,u_n)+\lambda f(u_1,\ldots,u_{i-1},u_i',u_{i+1},\ldots,u_n),\end{aligned}$$

for all $i\in\{1,\ldots,n\}$, where $\lambda:=a^2+b^2$ (this notation used here and from now on) in which $\lambda>1$ and $|a|,|b|\neq 1$. Note that each multi-Euler-Lagrange quadratic mapping satisfies equation (1.2) in each of variable.

For a mapping $f: V^n \longrightarrow W$, we consider two properties as follows:

- (H1) f has zero property, that is, $f\left(v^{[n]}\right)=0$ for any $v^{[n]}\in V^n$, with at least one variable equal to zero.
- (H2) f has the *quadratic condition* in all variables, that is, the equality

$$f(u_1,\ldots,u_{j-1},cu_j,u_{j+1},\ldots,u_n)=c^2f(u_1,\ldots,u_{j-1},u_j,u_{j+1},\ldots,u_n),$$

holds for all $u_1,\ldots,u_n\in V$ and $j\in\{1,\ldots,n\}$, where $c\in\{a,b\}$ and $|a|,|b|\neq 1$.

It is clear that if a f has the hypothesis (H2), then it has (H1) but clearly, the converse is not valid in general.

Regarding a multi-Euler-Lagrange quadratic mapping, as a special case of [12, Theorem 5], Bodaghi et al. proved the following theorem.

Theorem 1.2. Every multi-Euler-Lagrange quadratic mapping $f: V^n \longrightarrow W$ satisfies

$$\sum_{t_1,\ldots t_n\in\{(a,b),(b,a)\}} f\left(S_1^{t_1},\ldots,S_n^{t_n}\right)=\lambda^n\sum_{l_1,\cdots,l_n\in\{1,2\}} f(v_{l_11},\ldots,v_{l_nn}), \tag{1.3}$$

for all $v_i^{[n]}=(v_{i1},v_{i2},\ldots,v_{in})\in V^n$ with $i\in\{1,2\}$, in which

$$S_j^{(a,b)}=av_{1j}+bv_{2j}, \text{ and } S_j^{(b,a)}=bv_{1j}-av_{2j}, \tag{1.4}$$

for all $j\in\{1,\ldots,n\}$. The converse is valid if f has (H2).

It is know from Theorem 1.2 that a mapping $f : V^n \longrightarrow W$ satisfying the system of Euler-Lagrange quadratic functional equations given in Definition 1.1 for each variable i also fulfills equation (1.3) [12, Theorem 5], which will be the primary equation for our stability analysis. We remind that the stability for the multi-Euler-Lagrange quadratic mappings means that functional equation (1.3) is stable. More information about the structures, characterizations and the stability results for various multi-quadratic mappings and equation on miscellaneous spaces such as normed spaces, non-Archimedean spaces and fuzzy normed spaces are available in [2], [8], [11], [13], [15], [32] and [43].

In the current article, we prove Găvruţa and Hyers-Ulam-Rassias stability of the multi-Euler-Lagrange quadratic mappings by using the direct and fixed methods in the setting of fuzzy normed spaces. Furthermore, we find that the fixed point way is a more exact approximation of approximately multi-Euler-Lagrange-type quadratic mappings in comparison to the direct manner.

2 Stability results for Euler-Lagrange-type quadratic mappings

In this section, we prove various stability of multi-Euler-Lagrange quadratic mappings in fuzzy normed spaces by the direct and fixed point methods.

Definition 2.1. Let V be a real linear space. A function $T : V \times \mathbb{R} \longrightarrow [0, 1]$ is called a fuzzy norm on V if for all $u, v \in V$ and all $s, t \in \mathbb{R}$, we have

(T1) $T(v, r) = 0$ for $r \leq 0$;
(T2) $v = 0$ if and only if $T(v, r) = 1$ for all $r > 0$;
(T3) $T(rv, t) = T\left(v, \frac{t}{|r|}\right)$ if $r \neq 0$;
(T4) $T(u + v, s + t) \geq \min\{T(u, s), T(v, t)\}$;
(T5) $T(v, \cdot)$ is a non-decreasing function on $\mathbb{R}$ on and $\lim_{t\to\infty} T(v, t) = 1$;
(T6) For $v \neq 0$, $T(v, \cdot)$ is (upper semi)-continuous on $\mathbb{R}$.

The pair (V, T) is said to be a fuzzy normed linear space. Now, for a fuzzy normed vector space (V, T), we bring some known observations as follows. A sequence $\{v_j\}_j$ in V *converges* to a $v \in V$ if $\lim_{j\to\infty} T(v_j - v, t) = 1$ for all $t \geq 0$. In this case, v is called the limit of the sequence $\{v_j\}_j$ and denoted by $T - \lim_{t\to\infty} v_j = v$. Moreover, a sequence $\{v_j\}$ in V is called *Cauchy* if for each $\delta > 0$ and each $t > 0$ there exists an $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$ and all $k > 0$, we have $T(v_{j+k} - v_j, t) > 1 - \delta$.

Similar all normed spaces, if each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*. For fuzzy normed vector spaces (V, T) and (W, T'), we say that a mapping $f : (V, T) \longrightarrow (W, T')$ is continuous at a point $v_0 \in V$ if for each sequence $\{v_j\}_j$ converging to v_0 in V, the sequence $\{f(v_j)\}_j$ converges to $f(v_0)$. If $f : V \longrightarrow W$ is continuous at each $v \in V$, then it is said to be continuous on V; for more details we refer to [6].

The following example is presented in [31]. In continuation, we apply this example to obtain further results.

Example 2.2. Let (V, T) be a normed linear space. For $v \in V$, the function

$$T(v, t) = \begin{cases} \frac{t}{\|v\|+t} & t > 0, \\ 0 & \text{otherwise}. \end{cases}$$

is a fuzzy norm on V.

Let $n \in \mathbb{N}$ with $n \geq 2$ and $v_i^{[n]} = (v_{i1}, v_{i2}, \ldots, v_{in}) \in V^n$, where $i \in \{1, 2\}$. Here and subsequently, for a mapping $f : V^n \longrightarrow W$, we consider the operator $\mathbf{D}_q f : V^n \times V^n \longrightarrow W$ defined through

$$\mathbf{D}_q f\left(v_1^{[n]}, v_2^{[n]}\right) := \sum_{t_1,\ldots t_n \in \{(a,b),(b,a)\}} f\left(S_1^{t_1}, \ldots, S_n^{t_n}\right) - \lambda^n \sum_{l_1,\cdots,l_n \in \{1,2\}} f(v_{l_1 1}, \ldots, v_{l_n n}),$$

for fixed non-zero real numbers a, b, considered in Definition 1.1, where $S_j^{(a,b)}$ and $S_j^{(b,a)}$ are defined in (1.4) for all $j = 1, \ldots, n$.

Throughout this section, in all theorems, V is a real linear space, (W, T) is a fuzzy Banach space and (Y, T') is a fuzzy normed space. Moreover, it is assumed that all mapping $f : V^n \longrightarrow W$ has the property (H1).

2.1 Stability results: direct method

In this subsection, we prove Găvruţa and Hyers-Ulam-Rassias stability of multi-Euler-Lagrange quadratic mappings in fuzzy normed spaces by means of the direct method. From now on, we set $\mathbf{0}=(\overbrace{0,\ldots,0}^{n-\text{times}})$.

Theorem 2.3. Let $f:V^n\longrightarrow W$ be a mapping and $\phi:V^n\longrightarrow Y$ be a mapping with the property

(H3) For all $v^{[n]}\in V^n$, $f\left(\lambda v^{[n]}\right)=\alpha f\left(v^{[n]}\right)$ for some real number α with $0<|\alpha|<\lambda^{2n}$,
such that

$$T\left(\mathbf{D}_q f\left(v_1^{[n]},v_2^{[n]}\right),t+s\right)\geq min\left\{T'\left(\phi\left(v_1^{[n]}\right),t\right),T'\left(\phi\left(v_2^{[n]}\right),s\right)\right\},\tag{2.1}$$

for all $v_1^{[n]},v_2^{[n]}\in V^n$ and $t,s>0$. Then, there exists a solution $\mathcal{Q}:V^n\longrightarrow W$ of *(1.3)* such that

$$T\left(f\left(v^{[n]}\right)-\mathcal{Q}\left(v^{[n]}\right),t\right)\geq M\left(v,\frac{\lambda^{2n}-|\alpha|}{2\lambda^{2n}}t\right),\tag{2.2}$$

for all $v^{[n]}\in V^n$ and $t>0$, where

$$\begin{aligned}M(v^{[n]},t)\quad &:= min\Big\{T'\left(\phi\left(v^{[n]}\right),\tfrac{\lambda^n}{4}t\right),T'\left(\phi(\mathbf{0}),\tfrac{\lambda^n}{4}t\right),\\ &T'\left(\phi\left(av^{[n]}\right),\tfrac{\lambda^{2n}}{4}t\right),T'\left(\phi\left(bv^{[n]}\right),\tfrac{\lambda^{2n}}{4}t\right)\Big\}.\end{aligned}\tag{2.3}$$

Moreover, $\mathcal{Q}$ is a unique multi-Euler-Lagrange quadratic mapping satisfying (2.2) provided that the hypothesis (H2) valid for it.

Proof . Putting $v_2^{[n]}=\mathbf{0}$ and $s=t$ in (2.1) and using the assumptions, we have

$$T\left(\widetilde{f}\left(v_1^{[n]}\right)-\lambda^n f\left(v_1^{[n]}\right),2t\right)\geq \min\left\{T'\left(\phi\left(v_1^{[n]}\right),t\right),T'\left(\phi(\mathbf{0}),t\right)\right\},\tag{2.4}$$

for all $v_1^{[n]}\in V^n$ and $t>0$, where

$$\widetilde{f}\left(v_1^{[n]}\right)=\sum_{a_{l_1 1},\cdots,a_{l_n n}\in\{a,b\}} f(a_{l_1 1}v_{11},\ldots,a_{l_n n}v_{1n}).\tag{2.5}$$

For the remainder of the proof, we denote $v_1^{[n]}$ by $v^{[n]}$ unless stated otherwise. It follows from (2.4) that

$$T\left(\lambda^n\widetilde{f}\left(v^{[n]}\right)-\lambda^{2n}f\left(v^{[n]}\right),2t\right)\geq \min\left\{T'\left(\phi\left(v^{[n]}\right),\frac{t}{\lambda^n}\right),T'\left(\phi(\mathbf{0}),\frac{t}{\lambda^n}\right)\right\},\tag{2.6}$$

for all $v^{[n]}\in V^n$ and $t>0$. Once more, switching $\left(v_1^{[n]},v_2^{[n]}\right)$ by $\left(av^{[n]},bv^{[n]}\right)$ and putting $s=t$ in (2.1), we get

$$T\left(f\left(\lambda v^{[n]}\right)-\lambda^n\widetilde{f}\left(v^{[n]}\right),2t\right)\geq \min\left\{T'\left(\phi\left(av^{[n]}\right),t\right),T'\left(\phi\left(bv^{[n]}\right),t\right)\right\},\tag{2.7}$$

for all $v^{[n]}\in V^n$ and $t>0$. It follows from relation (2.6) and (2.7) that

$$\begin{aligned}&T\left(f\left(\lambda v^{[n]}\right)-\lambda^{2n}f\left(v^{[n]}\right),t\right)\\ &\geq \min\left\{T'\left(\phi\left(v^{[n]}\right),\frac{t}{4\lambda^n}\right),T'\left(\phi(\mathbf{0}),\frac{t}{4\lambda^n}\right),T'\left(\phi\left(av^{[n]}\right),\frac{t}{4}\right),T'\left(\phi\left(bv^{[n]}\right),\frac{t}{4}\right)\right\},\end{aligned}$$

for all $v^{[n]}\in V^n$ and $t>0$. From the relation above, applying (T3) with $r=\frac{1}{\lambda^{2n}}$ to the vector inside T yields

$$T\left(\frac{f\left(\lambda v^{[n]}\right)}{\lambda^{2n}}-f\left(v^{[n]}\right),\frac{1}{\lambda^{2n}}t\right)\geq \min\left\{T'\left(\phi\left(v^{[n]}\right),\frac{t}{4\lambda^n}\right),T'\left(\phi(\mathbf{0}),\frac{t}{4\lambda^n}\right),T'\left(\phi\left(av^{[n]}\right),\frac{t}{4}\right),T'\left(\phi\left(bv^{[n]}\right),\frac{t}{4}\right)\right\},$$

for all $v^{[n]} \in V^n$ and $t > 0$. By a change of variable t into $\lambda^{2n}t$ in the last inequality, we obtain

$$T\left(\frac{f\left(\lambda v^{[n]}\right)}{\lambda^{2n}} - f\left(v^{[n]}\right), t\right) \geq M\left(v^{[n]}, t\right), \tag{2.8}$$

for all $v^{[n]} \in V^n$ and $t > 0$, where $M\left(v^{[n]}, t\right)$ is defined in (2.3). Putting $v^{[n]}$ by $\lambda^m v^{[n]}$ in (2.8) and applying the equality

$$M\left(\lambda v^{[n]}, t\right) = M\left(v^{[n]}, \frac{t}{|\alpha|}\right), \tag{2.9}$$

we arrive to

$$\begin{aligned} T\left(\frac{f\left(\lambda^{m+1} v^{[n]}\right)}{\lambda^{2(m+1)n}} - \frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}}, \frac{|\alpha|^m}{\lambda^{2mn}} t\right) &= T\left(\frac{f\left(\lambda^{m+1} v^{[n]}\right)}{\lambda^{2n}} - f\left(\lambda^m v^{[n]}\right), |\alpha|^m t\right) \\ &\geq M\left(\lambda^m v^{[n]}, |\alpha|^m t\right) = M\left(v^{[n]}, t\right), \end{aligned}$$

for all $v^{[n]} \in V^n$, $t > 0$ and $m \geq 0$. Furthermore, for each $m > l > 0$, we get

$$\begin{aligned} T\left(\frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}} - \frac{f\left(\lambda^l v^{[n]}\right)}{\lambda^{2ln}}, \sum_{j=l}^{m-1} \frac{|\alpha|^j}{\lambda^{2jn}} t\right) &= T\left(\sum_{j=l}^{m-1} \frac{f\left(\lambda^{m+1} v^{[n]}\right)}{\lambda^{2(m+1)n}} - \frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}}, \sum_{j=l}^{m-1} \frac{|\alpha|^j}{\lambda^{2jn}} t\right) \\ &\geq \min \bigcup_{j=l}^{m-1} \left\{T\left(\frac{f\left(\lambda^{m+1} v^{[n]}\right)}{\lambda^{2(m+1)n}} - \frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}}, \frac{|\alpha|^j}{\lambda^{2jn}} t\right)\right\} \\ &\geq M\left(v^{[n]}, t\right). \end{aligned} \tag{2.10}$$

Take $\epsilon, \delta > 0$. Due to the equality $\lim_{t\to\infty} M\left(v^{[n]}, t\right) = 1$, there is some $t_0 > 0$ such that $M\left(v^{[n]}, t_0\right) > 1 - \epsilon$. Since the series $\sum_{j=0}^{m-1} \frac{|\alpha|^j}{\lambda^{jn}} t_0$ converges, its partial sums are Cauchy and therefore there is some $n_0 \in \mathbb{N}$ such that $\sum_{j=l}^{m-1} \frac{|\alpha|^j}{\lambda^{2jn}} t_0 < \delta$ for all $m > l \geq n_0$. It now deduces that

$$\begin{aligned} T\left(\frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}} - \frac{f\left(\lambda^l v^{[n]}\right)}{\lambda^{2ln}}, \delta\right) &\geq T\left(\frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}} - \frac{f\left(\lambda^l v^{[n]}\right)}{\lambda^{2ln}}, \sum_{j=l}^{m-1} \frac{|\alpha|^j}{\lambda^{2jn}} t_0\right) \\ &\geq M\left(v^{[n]}, t_0\right) \geq 1 - \epsilon. \end{aligned}$$

The relation above shows that the sequence $\left\{\frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}}\right\}_m$ is Cauchy in (W, T). Since (W, T) is a Banach fuzzy space, the sequence $\left\{\frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}}\right\}_m$ converges pointwise to a mapping $\mathcal{Q} : V^n \longrightarrow W$, that is,

$$\mathcal{Q}\left(v^{[n]}\right) = \lim_{m\to\infty} T - \frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}}.$$

Putting $l = 0$ in (2.10), we obtain

$$T\left(\frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}} - f(v^{[n]}), \sum_{j=0}^{m-1} \frac{|\alpha|^j}{\lambda^{jn}} t\right) \geq M(v^{[n]}, t),$$

for all $v^{[n]} \in V^n$ and $t > 0$. The last inequality necessitates that

$$T\left(\frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}} - f(v^{[n]}), t\right) \geq M\left(v^{[n]}, \frac{t}{\sum_{j=0}^{m-1} \frac{|\alpha|^j}{\lambda^{2jn}}}\right), \tag{2.11}$$

for all $v^{[n]} \in V^n$ and $t>0$. It will be shown that the mapping $\mathcal{Q}$ is a solution of (1.3). For this, we have

$$\begin{aligned}
&T\left(\mathbf{D}_q\mathcal{Q}\left(v_1^{[n]}, v_2^{[n]}\right), t\right)\\
&\geq \min\Bigg\{ \bigcup_{t_1,\ldots t_n \in\{(a,b),(b,a)\}} T\left(\mathcal{Q}\left(S_1^{t_1},\ldots,S_n^{t_n}\right) - \frac{f\left(\lambda^m S_1^{t_1},\ldots,\lambda^m S_n^{t_n}\right)}{\lambda^{2mn}}, \frac{t}{(2^n+2)2^n}\right),\\
&\bigcup_{l_1,\ldots,l_n\in\{1,2\}} T\left(\lambda^n\mathcal{Q}\left(v_{l_1 1},\ldots,v_{l_n n}\right) - \lambda^n\frac{f\left(\lambda^m v_{l_1 1},\ldots,\lambda^m v_{l_n n}\right)}{\lambda^{2mn}}, \frac{t}{(2^n+2)2^n}\right),\\
&\bigcup_{t_1,\ldots t_n \in\{(a,b),(b,a)\}}\ \bigcup_{l_1,\ldots,l_n\in\{1,2\}} T\left(\frac{f\left(\lambda^m S_1^{t_1},\ldots,\lambda^m S_n^{t_n}\right)}{\lambda^{2mn}} - \lambda^n\frac{f\left(\lambda^m v_{l_1 1},\ldots,\lambda^m v_{l_n n}\right)}{\lambda^{2mn}}, \frac{t}{(2^n+2)2^n}\right)\Bigg\},
\end{aligned}$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $t>0$. By the definition of $\mathcal{Q}$, each member of the first and second unions on the right hand side of the above inequality goes toward 1 as m tends to infinity and each member of the third unions, by (2.1) is greater than or equal to

$$\begin{aligned}
&\min\left\{T'\left(\phi\left(\lambda^m v_1^{[n]}\right), \frac{\lambda^{2mn}}{(2^n+2)2^{n-1}}t\right), T'\left(\phi\left(\lambda^m v_2^{[n]}\right), \frac{\lambda^{2mn}}{(2^n+2)2^{n-1}}t\right)\right\}\\
&= \min\left\{T'\left(\phi\left(v_1^{[n]}\right), \frac{\lambda^{2mn}}{(2^n+2)2^{n-1}|\alpha|^m}t\right), T'\left(\phi\left(v_2^{[n]}\right), \frac{\lambda^{2mn}}{(2^n+2)2^{n-1}|\alpha|^m}t\right)\right\},
\end{aligned}$$

which goes to 1 when $m\to\infty$. Hence,

$$T\left(\mathbf{D}_q\mathcal{Q}\left(v_1^{[n]}, v_2^{[n]}\right), t\right) = 1,$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $t>0$. This means that $\mathcal{Q}$ is a solution of (1.3). If now $\mathcal{Q}$ has the quadratic condition in each component, then it is a multi-Euler-Lagrange quadratic mapping by Theorem 1.2. Here, we approximate the difference between f and $\mathcal{Q}$ in a fuzzy sense. By inequality (2.11), for each $v^{[n]}\in V^n$ and $t>0$, we have

$$\begin{aligned}
T\left(\mathcal{Q}\left(v^{[n]}\right) - f\left(v^{[n]}\right), t\right) &\geq \min\left\{T\left(\mathcal{Q}\left(v^{[n]}\right) - \frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}}, \frac{t}{2}\right), T\left(\frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}} - f\left(v^{[n]}\right), \frac{t}{2}\right)\right\}\\
&\geq M\left(v^{[n]}, \frac{t}{2\sum_{j=0}^{\infty}\frac{|\alpha|^j}{\lambda^{2jn}}}\right)\\
&= M\left(v^{[n]}, \frac{\lambda^{2n}-|\alpha|}{2\lambda^{2n}}t\right).
\end{aligned}$$

For the uniqueness of $\mathcal{Q}$, assume that $\mathfrak{Q}$ is another multi-Euler-Lagrange quadratic mapping from V^n into W, such that inequality (2.2) is true for them. For each $v^{[n]}\in V^n$ and $t>0$, we reach

$$\begin{aligned}
T\left(\mathcal{Q}\left(v^{[n]}\right) - \mathfrak{Q}\left(v^{[n]}\right), t\right) &\geq \min\left\{T\left(\mathcal{Q}\left(v^{[n]}\right) - f\left(v^{[n]}\right), \frac{t}{2}\right), T\left(\mathfrak{Q}\left(v^{[n]}\right) - f\left(v^{[n]}\right), \frac{t}{2}\right)\right\}\\
&\geq M\left(v^{[n]}, \frac{\lambda^{2n}-|\alpha|}{2\lambda^{2n}}t\right).
\end{aligned}$$

On the other hand, $\mathcal{Q}$ and $\mathfrak{Q}$ are multi-Euler-Lagrange quadratic, and hence

$$\begin{aligned}
T\left(\mathcal{Q}\left(v^{[n]}\right) - \mathfrak{Q}\left(v^{[n]}\right), t\right) &= T\left(\mathcal{Q}\left(\lambda^m v^{[n]}\right) - \mathfrak{Q}\left(\lambda^m v^{[n]}\right), \lambda^{2mn}t\right)\\
&\geq M\left(v^{[n]}, \frac{\lambda^{2n}-|\alpha|}{2\lambda^{2n}}\left(\frac{\lambda^{2n}}{|\alpha|}\right)^m t\right),
\end{aligned}$$

for all $v^{[n]}\in V^n$, $t>0$ and $m\in\mathbb{N}$. Since $\lim_{m\to\infty}\left(\frac{\lambda^{2n}}{|\alpha|}\right)^m = +\infty$, the right hand side of the last inequality tends to 1 as $m\to\infty$. This shows that $\mathcal{Q}\left(v^{[n]}\right) = \mathfrak{Q}\left(v^{[n]}\right)$ for all $v^{[n]}\in V^n$, and therefore the proof is now finished. □

The next theorem is in analogy with the previous theorem, when the assumptions are the same but we obtain different results. Note that the techniques are completely similar to the that of Theorem 2.3, and so we present a sketch of the proof.

Theorem 2.4. Let $f: V^n \longrightarrow W$ be a mapping and $\phi: V^n \longrightarrow Y$ be a function with the hypothesis (H3) such that

$$T\left(\mathbf{D}_q f\left(v_1^{[n]}, v_2^{[n]}\right), t+s\right) \geq min\left\{T'\left(\phi\left(v_1^{[n]}\right), t\right), T'\left(\phi\left(v_2^{[n]}\right), s\right)\right\},$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $t, s > 0$. Then, there exists a solution $\mathcal{Q}: V^n \longrightarrow W$ of (1.3) such that

$$T\left(f\left(v^{[n]}\right) - \mathcal{Q}\left(v^{[n]}\right), t\right) \geq N\left(v^{[n]}, \frac{|\alpha| - \lambda^{2n}}{2|\alpha|} t\right), \tag{2.12}$$

for all $v^{[n]} \in V^n$ and $t > 0$, where

$$\begin{aligned} N\left(v^{[n]}, t\right) \quad &:= min\Big\{T'\left(\phi\left(v^{[n]}\right), \tfrac{|\alpha|}{4\lambda^n} t\right), T'\left(\phi(\mathbf{0}), \tfrac{|\alpha|}{4\lambda^n} t\right), \\ &T'\left(\phi\left(av^{[n]}\right), \tfrac{|\alpha|}{4} t\right), T'\left(\phi\left(bv^{[n]}\right), \tfrac{|\alpha|}{4} t\right)\Big\}. \end{aligned} \tag{2.13}$$

In addition, $\mathcal{Q}$ is a unique multi-Euler-Lagrange quadratic mapping satisfying (2.12) if it satisfies the hypothesis (H2).

Proof . Note that the assumption (H3) for ϕ implies that $\phi\left(\frac{v^{[n]}}{\lambda}\right) = \frac{1}{|\alpha|}\phi\left(v^{[n]}\right)$ in which $|\alpha| > \lambda^{2n}$. Similar way to that proof of Theorem 2.3, we have

$$\begin{aligned} &T\left(f\left(\lambda v^{[n]}\right) - \lambda^{2n} f\left(v^{[n]}\right), t\right) \\ &\geq \min\left\{T'\left(\phi\left(v^{[n]}\right), \frac{t}{4\lambda^n}\right), T'\left(\phi(\mathbf{0}), \frac{t}{4\lambda^n}\right), T'\left(\phi\left(av^{[n]}\right), \frac{t}{4}\right), T'\left(\phi\left(bv^{[n]}\right), \frac{t}{4}\right)\right\}, \end{aligned}$$

for all $v^{[n]} \in V^n$ and $t > 0$. A direct result from the inequality above is as follows:

$$T\left(f\left(v^{[n]}\right) - \lambda^{2n} f\left(\frac{v^{[n]}}{\lambda}\right), t\right) \geq N\left(v^{[n]}, t\right), \tag{2.14}$$

for all $v^{[n]} \in V^n$ and $t > 0$, where $N(v, t)$ is defined in (2.13). Interchanging $v^{[n]}$ into $\lambda^m v^{[n]}$ in (2.14) and applying the property $N\left(\frac{v^{[n]}}{\lambda}, t\right) = N\left(v^{[n]}, |\alpha| t\right)$, we get

$$T\left(\lambda^{2mn} f\left(\frac{v^{[n]}}{\lambda^m}\right) - \lambda^{(m+1)n} f\left(\frac{v^{[n]}}{\lambda^{m+1}}\right), \frac{\lambda^{2mn}}{|\alpha|^m} t\right) \geq N\left(v^{[n]}, t\right),$$

for all $v^{[n]} \in V^n$, $t > 0$ and $m \geq 0$. Moreover, for each $m > l > 0$, we obtain

$$T\left(\lambda^{2ln} f\left(\frac{v^{[n]}}{\lambda^l}\right) - \lambda^{2mn} f\left(\frac{v^{[n]}}{\lambda^m}\right), \sum_{j=l}^{m-1} \frac{\lambda^{2jn}}{|\alpha|^j} t\right) \geq N\left(v^{[n]}, t\right), \quad \left(v^{[n]} \in V^n\right). \tag{2.15}$$

Let $\epsilon, \delta > 0$. Since $\lim_{t\to\infty} N\left(v^{[n]}, t\right) = 1$, and so there is some $t_0 > 0$ such that $N\left(v^{[n]}, t_0\right) > 1 - \epsilon$. On the other hand $\sum_{j=0}^{m-1} \frac{\lambda^{2jn}}{|\alpha|^j} t_0 < \infty$, and so for some $n_0 \in \mathbb{N}$, we get $\sum_{j=l}^{m-1} \frac{\lambda^{2jn}}{|\alpha|^j} t_0 < \delta$ for all $m > l \geq n_0$. These arguments imply that

$$T\left(\lambda^{2ln} f\left(\frac{v^{[n]}}{\lambda^l}\right) - \lambda^{2mn} f\left(\frac{v^{[n]}}{\lambda^m}\right), \delta\right) \geq N\left(v^{[n]}, t_0\right) \geq 1 - \epsilon.$$

Thus, the sequence $\left\{\lambda^{2mn} f\left(\frac{v^{[n]}}{\lambda^m}\right)\right\}$ is Cauchy in (W, T). Due to the completeness of (W, T'), this sequence converges pointwise to a mapping $\mathcal{Q}: V^n \longrightarrow W$ such that

$$\mathcal{Q}\left(v^{[n]}\right) = \lim_{m\to\infty} T - \lambda^{2mn} f\left(\frac{v^{[n]}}{\lambda^m}\right), \quad \left(v^{[n]} \in V^n\right).$$

Putting $l=0$ in (2.15), we find

$$T\left(f\left(v^{[n]}\right)-\lambda^{2mn}f\left(\frac{v^{[n]}}{\lambda^m}\right),\sum_{j=0}^{m-1}\frac{\lambda^{2jn}}{|\alpha|^j}t\right)\geq N\left(v^{[n]},t\right),$$

for all $v^{[n]}\in V^n$ and $t>0$. It now follows from the last inequality that

$$T\left(f\left(v^{[n]}\right)-\lambda^{2mn}f\left(\frac{v^{[n]}}{\lambda^m}\right),t\right)\geq N\left(v^{[n]},\frac{t}{\sum_{j=0}^{m-1}\frac{\lambda^{2jn}}{|\alpha|^j}}\right),$$

or all $v^{[n]}\in V^n$ and $t>0$. The proof of being a solution of (1.3) and the uniqueness of $\mathcal{Q}$ can be a standard fashion taken from the proof of Theorem 2.3. □

A direct consequence of Theorems 2.3 and 2.4 is Rassias stability for the multi-Euler-Lagrange quadratic mappings as follows.

Example 2.5. Let V be a real normed space, (W,T) be a fuzzy Banach space and (Y,T') be a fuzzy normed space, where T and T' are fuzzy norm as considered in Example 2.2. Suppose that $f:V^n\longrightarrow W$ is a mapping satisfies

$$T\left(\mathbf{D}_qf\left(v_1^{[n]},v_2^{[n]}\right),t+s\right)\geq\min\left\{T'\left(\textstyle\sum_{j=1}^n\|v_{1j}\|^ry_0,t\right),T'\left(\textstyle\sum_{j=1}^n\|v_{2j}\|^ry_0,s\right)\right\},$$

for all $v_1^{[n]},v_2^{[n]}\in V^n$ and $t,s>0$, where y_0 is a fixed vector in Y. If $0<r\neq 2n$ and $|a|\geq|b|$, then by Theorem 2.3 and Theorem 2.4 there exists a solution $\mathcal{Q}:V^n\longrightarrow W$ of (1.3) such that

$$T\left(f\left(v^{[n]}\right)-\mathcal{Q}\left(v^{[n]}\right),t\right)\geq\begin{cases}\frac{(\lambda^{2n}n-\lambda^r)t}{(\lambda^{2n}-\lambda^r)t+8\lambda^n\|y_0\|\sum_{j=1}^n\|v_{1j}\|^r} & 0<r<2n,\\ \frac{(\lambda^rn-\lambda^{2n})t}{(\lambda^r-\lambda^{2n})t+8\lambda^n\|y_0\|\sum_{j=1}^n\|v_{1j}\|^r} & r>2n,\lambda^{2n}\geq|a|^r,\\ \frac{(\lambda^rn-\lambda^{2n})t}{(\lambda^r-\lambda^{2n})t+8|a|^r\|y_0\|\sum_{j=1}^n\|v_{1j}\|^r} & r>2n,\lambda^{2n}\leq|a|^r,\end{cases}\tag{2.16}$$

for all $v^{[n]}:=v_1^{[n]}\in V^n$ and $t>0$. Moreover, if $\mathcal{Q}$ has the property (H2), then it is a unique multi-Euler-Lagrange quadratic mapping satisfying (2.16).

It is easily seen that if in the definition of multi-Euler-Lagrange quadratic mapping, we consider a large number for n, then the right side of inequality (2.16) goes to 1 and fuzzy difference closes to zero. In the upcoming example, we prove the Hyers' stability of multi-Euler-Lagrange quadratic mappings.

Example 2.6. Given $\delta>0$. Let V be a real normed space, (W,T) be a fuzzy Banach space and (Y,T') be a fuzzy normed space. Suppose that $f:V^n\longrightarrow W$ is a mapping satisfies

$$T\left(\mathbf{D}_qf\left(v_1^{[n]},v_2^{[n]}\right),t+s\right)\geq\min\{T'\left(\delta y_0,t\right),T'\left(\delta y_0,s\right)\},$$

for all $v_1^{[n]},v_2^{[n]}\in V^n$ and $t,s>0$, where y_0 is a fixed vector in Y. By Theorem 2.3, there exists a solution $\mathcal{Q}:V^n\longrightarrow W$ of (1.3) such that

$$T\left(f\left(v^{[n]}\right)-\mathcal{Q}\left(v^{[n]}\right),t\right)\geq T'\left(y_0,\frac{\lambda^{2n}-1}{8\delta\lambda^n}t\right),\tag{2.17}$$

for all $v^{[n]}\in V^n$ and $t>0$. In particular, if $\mathcal{Q}$ has the property (H2), then it is a unique multi-Euler-Lagrange quadratic mapping fulfilling (2.17).

2.2 Stability results: the fixed point method

In this subsection, we present the Hyers-Ulam-Rassias stability of multi-Euler-Lagrange quadratic mappings by a fixed point method. Before presenting the main results in this subsection, we recall that the only essential difference of the generalized metric d on a set X from the metric d' on X is that the range of generalized metric includes the infinity. In the next theorem, we bring a fundamental result in fixed point theory from [17], is useful to our goals in this subsection; we remind that an extension of the result was stated in [37].

Theorem 2.7. Suppose that (Δ, d) is a complete generalized metric space and $\mathcal{T} : \Delta \longrightarrow \Delta$ is a mapping with Lipschitz constant (its definition is available in the literature) $0 < L < 1$. Then, for each element $\eta \in \Delta$, one of the following happen:

- $d(\mathcal{T}^n\eta, \mathcal{T}^{n+1}\eta) = \infty$ for all $n \geq 0$;

or

- there exists a $n_0 \in \mathbb{N}$ such that

(1) $d\left(\mathcal{T}^n\eta, \mathcal{T}^{n+1}\eta\right) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{\mathcal{T}^n\eta\}$ converges to an element η^* of $\mathcal{T}$;
(3) η^* is the unique fixed point of $\mathcal{J} \in \Delta_0 = \{\eta \in \Delta : d(\mathcal{T}^{n_0}y, y) < \infty\}$;
(4) $d(\eta, \eta^*) \leq \frac{1}{1-L} d(\eta, \mathcal{T}\eta)$ for all $\eta \in \Delta_0$.

Theorem 2.8. Let V be a real linear space, (W, T) be a fuzzy Banach space and (Y, T') be a fuzzy normed space. Suppose that $f : V^n \longrightarrow W$ is a mapping and $\phi : V^n \longrightarrow Y$ is a function satisfying (2.1). If the property (H3) is true for ϕ, then there exists a solution $\mathcal{Q} : V^n \longrightarrow W$ of (1.3) such that

$$T\left(f\left(v^{[n]}\right) - \mathcal{Q}\left(v^{[n]}\right), t\right) \geq M\left(v^{[n]}, \frac{\lambda^{2n} - |\alpha|}{\lambda^{2n}} t\right), \tag{2.18}$$

for all $v^{[n]} \in V^n$ and $t > 0$, where $M\left(v^{[n]}, t\right)$ is defined in (2.3). In addition, if $\mathcal{Q}$ has the property (H2), then it is a unique multi-Euler-Lagrange quadratic mapping fulfilling (2.18).

Proof . Consider the set

$$\Delta := \{f : V^n \longrightarrow W : f \text{ has the property (H1)}\}.$$

Let us define a mapping $d_F : V^n \times V^n \longrightarrow [0, \infty]$ via

$$d_F(g, h) := \inf\left\{\mu \in [0, \infty) : T\left(g(v^{[n]}) - h\left(v^{[n]}\right), \mu t\right) \geq M\left(v^{[n]}, t\right), \quad \forall v^{[n]} \in V^n, t > 0\right\},$$

where, as usual, $\inf\emptyset = +\infty$, for which $g, h \in \Delta$, where $M\left(v^{[n]}, t\right)$ is defined in (2.3). It can be easily verified that d_F is a generalized metric on Δ. Moreover, (Δ, d_F) is a complete generalized metric space. Suppose that a sequence $\{g_m\}_m$ is d_F-Cauchy. Fix $v^{[n]} \in V^n$. A similar way to that the proof of [27, Lemma 2.1], one can show that $\left\{g_m\left(v^{[n]}\right)\right\}_m$ is Cauchy in W. Since (W, T) be a fuzzy Banach space, there exists a mapping $g : V^n \longrightarrow W$ satisfying the property (H1) such that $\left\{g_m\left(v^{[n]}\right)\right\}_m$ converges to $g\left(v^{[n]}\right)$. Now, by repeating the second part of the proof of [27, Lemma 2.1] we can show that $\{g_m\}_m$ is d_F-convergent (see also the proof of [22, Lemma 2.6]). Define the mapping $\mathcal{T} : \Delta \longrightarrow \Delta$ through

$$\mathcal{T}f\left(v^{[n]}\right) := \frac{1}{\lambda^{2n}} f\left(\lambda v^{[n]}\right), \qquad \left(v^{[n]} \in V^n\right).$$

Take $g, h \in \Delta$, $v^{[n]} \in V^n$ and $\mu \in [0, \infty]$ with $d_F(g, h) \leq \mu$. Then, $T\left(g\left(v^{[n]}\right) - h\left(v^{[n]}\right), \mu t\right) \geq M\left(v^{[n]}, t\right)$, and so by the property (2.9), we have

$$\begin{aligned} T\left(\mathcal{T}g(v^{[n]}) - \mathcal{T}h(v^{[n]}), \frac{|\alpha|}{\lambda^{2n}}\mu t\right) &= T\left(\frac{1}{\lambda^{2n}} g\left(\lambda v^{[n]}\right) - \frac{1}{\lambda^{2n}} h\left(\lambda v^{[n]}\right), \frac{|\alpha|}{2^n}\mu t\right) \\ &= T\left(g\left(\lambda v^{[n]}\right) - h\left(\lambda v^{[n]}\right), |\alpha|\mu t\right) \\ &\geq M\left(\lambda v^{[n]}, |\alpha| t\right) = M\left(v^{[n]}, t\right), \end{aligned}$$

for all $v^{[n]} \in V^n$ and $t > 0$. Therefore, $d_F(\mathcal{T}g, \mathcal{T}h) \leq \frac{|\alpha|}{\lambda^{2n}}\mu$. This shows that $d_F(\mathcal{T}g, \mathcal{T}h) \leq \frac{\alpha}{\lambda^{2n}} d_F(g,h)$, and indeed, we showed that $\mathcal{T}$ is a strictly contractive operator with the Lipschitz constant $\frac{|\alpha|}{\lambda^{2n}}$. Similar to the proof of Theorem 2.3, we get

$$T\left(\frac{f\left(\lambda v^{[n]}\right)}{\lambda^{2n}} - f\left(v^{[n]}\right), t\right) \geq M\left(v^{[n]}, t\right),$$

for all $v^{[n]} \in V^n$ and $t > 0$, where $M\left(v^{[n]}, t\right)$ is defined in (2.3). Therefore, $d_F(\mathcal{T}f, f) \leq 1$. We can now apply Theorem 2.7 for the space (Δ, d_F), the operator $\mathcal{T}$, to deduce that the sequence $(\mathcal{T}^m f)_{m\in\mathbb{N}}$ is convergent in (Δ, d_F) and its pointwise limit, namely, $\mathcal{Q}$ is a fixed point of $\mathcal{T}$, and additionally, $\mathcal{Q}(v^{[n]}) = \frac{1}{\lambda^{2n}}\mathcal{Q}\left(\lambda v^{[n]}\right)$. Moreover, we have $d_F\left(\mathcal{T}^m f, \mathcal{Q}\right) \to 0$, which implies that

$$\mathcal{Q}(v^{[n]}) = T - \lim_{l\to\infty} \frac{f\left(\lambda^m v^{[n]}\right)}{\lambda^{2mn}},$$

for all $v^{[n]} \in V^n$. Furthermore, $d_F(f, \mathcal{Q}) \leq \frac{1}{1-L} d_F(f, \mathcal{T}f)$, which necessitates that

$$d_F(f, \mathcal{Q}) \leq \frac{1}{1 - \frac{|\alpha|}{\lambda^{2n}}} = \frac{\lambda^{2n}}{\lambda^{2n} - |\alpha|}.$$

If $\{\delta_l\}$ is a decreasing sequence converging to $\frac{\lambda^{2n}}{\lambda^{2n}-|\alpha|}$, then $T\left(f\left(v^{[n]}\right) - \mathcal{Q}\left(v^{[n]}\right), \delta_l t\right) \geq M\left(v^{[n]}, t\right)$, for all $v^{[n]} \in V^n$, $t > 0$ and $l \in \mathbb{N}$. It concludes that

$$T\left(f\left(v^{[n]}\right) - \mathcal{Q}\left(v^{[n]}\right), t\right) \geq M\left(v^{[n]}, \frac{t}{\delta_l}\right),$$

for all $v^{[n]} \in V^n$, $t > 0$ and $l \in \mathbb{N}$. Since M is left continuous by (T6), we deduce that

$$T\left(f\left(v^{[n]}\right) - \mathcal{Q}\left(v^{[n]}\right), t\right) \geq M\left(v^{[n]}, \frac{\lambda^{2n}}{\lambda^{2n} - |\alpha|}\right),$$

for all $v^{[n]} \in V^n$, $t > 0$ and $l \in \mathbb{N}$. Here, it can be proven like in Theorem 2.3 that $\mathcal{Q}$ is a solution of (1.3). Besides, the uniqueness of $\mathcal{Q}$ follows from the fact that $\mathcal{Q}$ is the unique fixed point of $\mathcal{T}$ with the property that there exists $\zeta \in (0, \infty)$ such that

$$T\left(f\left(v^{[n]}\right) - \mathcal{Q}\left(v^{[n]}\right), \zeta t\right) \geq M\left(v^{[n]}, t\right),$$

for all $v^{[n]} \in V^n$ and $t > 0$. This completes the proof. □

It follows from the proof of Theorem 2.8 that for proving the uniqueness of the solution, we do not need the property (H2) for a mapping $f : V^n \longrightarrow W$ and this property is used only for being multi-Euler-Lagrange quadratic mapping, and so the property (H2) is a redundant condition in Theorem 2.8.

The upcoming result is analogous to Theorem 2.8 in which we find a different approximation for a approximately multi-Euler-Lagrange quadratic equations in fuzzy Banach spaces. Since the proof is similar, it is omitted.

Theorem 2.9. Let V be a real linear space, (W, T) be a fuzzy Banach space and (Y, T') be a fuzzy normed space. Suppose that $f : V^n \longrightarrow W$ is a mapping and $\phi : V^n \longrightarrow Y$ is a function satisfying (2.1). If the hypothesis (H3) holds for ϕ, then there exists a solution $\mathcal{Q} : V^n \longrightarrow W$ of (1.3) such that

$$T\left(f\left(v^{[n]}\right) - \mathcal{Q}\left(v^{[n]}\right), t\right) \geq N\left(v^{[n]}, \frac{|\alpha| - \lambda^{2n}}{|\alpha|} t\right), \tag{2.19}$$

for all $v^{[n]} \in V^n$ and $t > 0$, where $N(v^{[n]}, t)$ is defined in (2.13). Furthermore, if $\mathcal{Q}$ has the property (H2), then it is a unique multi-Euler-Lagrange quadratic mapping satisfying (2.19).

In view of results in Theorem 2.3 and Theorem 2.8, we observe that $M\left(v, \frac{\lambda^{2n}-|\alpha|}{\lambda^{2n}} t\right) > M\left(v, \frac{\lambda^{2n}-|\alpha|}{2\lambda^{2n}} t\right)$ and so we find that the fixed point method give us more exact approximation in comparison to direct method. Such a comparison can also be seen for Theorem 2.4 and Theorem 2.9.

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