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On a class of $l(x)$-biharmonic Kirchhoff-type problem

Maryam Mirzapour

Department of Mathematics Education, Farhangian University, P.O. Box 14665-889, Tehran, Iran

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Abstract

In this paper we deal with the multiplicity of solutions for the following Kirchhoff type problem with Navier boundary conditions

$$\mathcal{K}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi|^{l(x)}dx\Big)\Delta(|\Delta\varphi|^{l(x)-2}\Delta\varphi) = \theta|\varphi|^{r(x)-2}\varphi + \eta|\varphi|^{t(x)-2}\varphi \quad \text{in } \Omega,$$
$$\varphi = \Delta\varphi = 0 \quad \text{on } \partial\Omega.$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $\mathcal{K}$ is a continuous Kirchhoff type function, $l(x), r(x)$ and $t(x)$ are continuous functions on $\overline{\Omega}$, and θ and η are parameters. We show the existence of infinitely many solutions for this problem by using the variational methods.

Keywords: Kirchhoff type problem; Fourth-order; Variable exponent; Critical points; Variational methods
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1 Introduction

Recently, the exploration of variational problems and differential equations with l(x)-growth conditions has transformed the subject from nonlinear electrorheological fluids to an interesting area of study. We refer the eager readers about this subject to Ruzicka [25], Zhikov [32] and the reference therein and also see [14, 16, 17, 19].

Fourth order equations are present in various contexts. Applied mathematics and physics have different problems that theses can address, for instance Micro Electro-Mechanical systems, surface diffusion on solids, and flow in Hele-Shaw cells (see [20]). Moreover, these equations can specify the static outcome of a beam's change or the movement of rigid body parts.

The authors in [13] consider the following $p(x)$-biharmonic problem

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{p(x)-2}u + f(x,u) \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

with bounded domains Ω, and $\lambda \leq 0$. Under some restrictions on the Caratheodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, they acquired the presence and variety of solutions. In [1] G.A. Afrouzi et al. have considered problem

$$\begin{cases} M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

Email address: m.mirzapour@cfu.ac.ir (Maryam Mirzapour)

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in two cases when the nonlinearity f has special forms. They have demonstrated the existence of multiple solutions to the problem by utilizing variational methods. Moreover, the interested reader can see [3, 26, 28], in which, by variational approaches some existence results are given.

The works mentioned above have inspired us to explore this topic in the current paper: the existence and multiplicity of weak solutions of the following fourth order elliptic equation

$$\begin{cases} \mathcal{K}\Big(\int_\Omega \frac{1}{l(x)} |\Delta\varphi|^{l(x)} dx \Big) \Delta(|\Delta\varphi|^{l(x)-2}\Delta\varphi) = \theta|\varphi|^{r(x)-2}\varphi + \eta|u|^{t(x)-2}\varphi \quad \text{in } \Omega, \\ \varphi = \Delta\varphi = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$,and $\mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+$, $l(x), r(x)$ and $t(x)$ are continuous functions on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} l(x) > 1, \inf_{x\in\overline{\Omega}} r(x) > 1$, $\inf_{x\in\overline{\Omega}} t(x) > 1$, and θ and η are parameters. Throughout the paper, our assumption is that $\theta^2 + \eta^2 \neq 0$.

The nonlocality of (1.1) is attributed to the existence of $\mathcal{K}$, which means that the equation in (1.1) is not local, indicating that it lacks pointwise identities. The problem is a source of fascination for some mathematic problems.

It's miles well worth bringing up that Kirchhoff in 1883 offered a desk bound version of differential equation, the so-called Kirchhoff equation:

$$\rho \frac{\partial^2\varphi}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial\varphi}{\partial x} \right|^2 dx \right) \frac{\partial^2\varphi}{\partial x^2} = 0, \tag{1.2}$$

which generalizes the classical D'Alembert's wave equation, by taking into account the impact of the string length change during vibration. (1.2) contains the nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial\varphi}{\partial x} \right|^2 dx$ which depends on the average $\frac{1}{2L} \int_0^L \left| \frac{\partial\varphi}{\partial x} \right|^2 dx$, thus, this equation has been discarded as a pointwise identity. After the work of Lions [23], a great deal hobby has been focused on combining this model with many styles of issues due to its nonlocal nature, see e.g. [5]-[12].

2 Notations and preliminaries

We need to obtain some results for $l(x)$-Laplacian starting from the spaces $L^{l(x)}(\Omega)$ and $W^{k,l(x)}(\Omega)$.
For a bounded domain Ω, let

$$C_+(\overline{\Omega}) = \{s(x);\ s(x) \in C(\overline{\Omega}),\ s(x) > 1,\ \forall x \in \overline{\Omega}\}.$$

For any $s \in C_+(\overline{\Omega})$, set

$$s^+ = \max\{s(x);\ x \in \overline{\Omega}\}, \quad s^- = \min\{s(x);\ x \in \overline{\Omega}\}.$$

For any given $l \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{l(x)}(\Omega) = \Big\{\varphi;\ \varphi \text{ is a real-valued function that can be measured } \int_\Omega |\varphi(x)|^{l(x)} dx < \infty\Big\},$$

endorsed with the so-called *Luxemburg norm*

$$|\varphi|_{l(x)} = \inf\Big\{\zeta > 0;\ \int_\Omega |\frac{\varphi(x)}{\zeta}|^{l(x)} dx \leq 1\Big\},$$

then, $(L^{l(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space. By [18], we see that the space $(L^{l(x)}(\Omega), |\cdot|_{l(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{n(x)}(\Omega)$ where $n(x)$ is the conjugate function of $l(x)$, i.e.,

$$\frac{1}{l(x)} + \frac{1}{n(x)} = 1,$$

for all $x \in \Omega$. Also, the Hölder inequality hold: for $\varphi \in L^{l(x)}(\Omega)$ and $\psi \in L^{n(x)}(\Omega)$, we have

$$\Big|\int_\Omega \varphi\psi dx\Big| \leq \Big(\frac{1}{l^-}+\frac{1}{n^-}\Big)|\varphi|_{l(x)}|\psi|_{n(x)} \leq 2|\varphi|_{l(x)}|\psi|_{n(x)}.$$

Define the variable exponent Sobolev space

$$W^{k,l(x)}(\Omega)=\{\varphi \in L^{l(x)}(\Omega):\ D^\alpha\varphi \in L^{l(x)}(\Omega),\ |\alpha|\leq k\},$$

where $D^\alpha\varphi = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}\varphi$, with $\alpha=(\alpha_1,\ldots,\alpha_N)$ is a multi-index and $|\alpha|=\sum_{i=1}^N \alpha_i$. Then, the space $W^{k,l(x)}(\Omega)$ equipped with the norm

$$\|\varphi\|_{k,l(x)} = \sum_{|\alpha|\leq k}|D^\alpha\varphi|_{l(x)},$$

is a separable and reflexive Banach space. To learn more about this space, the reader can see [15, 18, 24, 29]. Now, set

$$l_k^*(x)=\begin{cases}\frac{Nl(x)}{N-kl(x)} & \text{if} \quad kl(x)<N,\\ +\infty & \text{if} \quad kp(x)\geq N\end{cases}$$

for any $x\in\overline{\Omega}$, $k\geq 1$.

Proposition 2.1 ([18]). If $l,\sigma \in C_+(\overline{\Omega})$ with $\sigma(x)\leq l_k^*(x)$ for any $x\in\overline{\Omega}$, then there is a continuous embedding

$$W^{k,l(x)}(\Omega)\hookrightarrow L^{\sigma(x)}(\Omega).$$

Moreover, by replacing $\leq$ with $<$, it turns to a compact embedding.

Now, let $W_0^{k,l(x)}(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $W^{k,l(x)}(\Omega)$. As we know, the generalized Sobolev

$$X=W^{2,l(x)}(\Omega)\cap W_0^{k,l(x)}(\Omega)$$

encompasses the weak solutions to problem (1.1), which is equipped with the following norm

$$\|\varphi\| = \inf\Big\{\zeta>0: \int_\Omega\Big|\frac{\Delta\varphi(x)}{\zeta}\Big|^{l(x)}dx\leq 1\Big\}.$$

Remark 2.2. According to [30], the norms $\|\cdot\|_{2,l(x)}$ and $|\Delta\cdot|_{l(x)}$ are equivalent. Therefore, we have these equivalent norms $\|\cdot\|_{2,l(x)}$, $\|\cdot\|$ and $|\Delta\cdot|_{l(x)}$.

Proposition 2.3 ([13]). If we denote $\rho(\varphi)=\int_\Omega|\Delta\varphi|^{l(x)}dx$, then for $\varphi\in X$, we have

(**1**) $\|\varphi\|<1$ (respectively$=1;>1$) $\Longleftrightarrow \rho(u)<1$ (respectively$=1;>1$);
(**2**) if $\|\varphi\|>1$, then $\|\varphi\|^{l^-}\leq\rho(\varphi)\leq\|\varphi\|^{l^+}$;
(**3**) if $\|\varphi\|<1$, then $\|\varphi\|^{l^+}\leq\rho(\varphi)\leq\|\varphi\|^{l^-}$;
(**4**) $\|\varphi\|\to 0$ (respectively $\to\infty$) $\Longleftrightarrow \rho(\varphi)\to 0$(respectively $\to\infty$).

Note that the energy functional related to problem (1.1) is defined by the following

$$I_{\theta,\eta}(\varphi)=\widehat{\mathcal{K}}\Big(\int_\Omega\frac{1}{l(x)}|\Delta\varphi|^{l(x)}dx\Big)-\theta\int_\Omega\frac{1}{r(x)}|\varphi|^{r(x)}dx-\eta\int_\Omega\frac{1}{t(x)}|\varphi|^{t(x)}dx.$$

with $\widehat{\mathcal{K}}(t)=\int_0^t\mathcal{K}(\tau)d\tau$. Now, consider the well defined, even, and C^1 functional $J(\varphi)=\int_\Omega\frac{1}{l(x)}|\Delta\varphi|^{l(x)}dx$. Then, the operator $L=J':X\to X^*$ defined by

$$\langle L(\varphi),\psi\rangle=\int_\Omega|\Delta\varphi|^{l(x)-2}\Delta\varphi\Delta\psi\, dx$$

for any $\varphi,\psi\in X$, is continuous, bounded and strictly monotone, homeomorphism and also a mapping of (S_+) type, namely: $\varphi_n\rightharpoonup u$ and $\limsup_{n\to+\infty}L(\varphi_n)(\varphi_n-\varphi)\leq 0$, implies $\varphi_n\to\varphi$.

Throughout this paper, the letter c_i indicates the positive constant and also, we impose the following conditions on $\mathcal{K}$:

($\mathbf{M_1}$) There exist $m_2\geq m_1>0$ and $\beta\geq\alpha>1$ so that for all $\tau\in\mathbb{R}^+$, $m_1\tau^{\alpha-1}\leq\mathcal{K}(\tau)\leq m_2\tau^{\beta-1}$.
($\mathbf{M_2}$) For any $\tau\in\mathbb{R}^+$, $\widehat{\mathcal{K}}(\tau)\geq\mathcal{K}(\tau)t$.

3 Main results and proofs

First, we state the main result of this paper and then, we prove the multiplicity results.

Theorem 3.1. Suppose that $r(x), t(x) \in C_+(\overline{\Omega})$, with $(l^+)^\alpha < r^- \leq r(x) < l_2^*(x)$, $t^+ < \alpha l^-$ and $\beta l^+ < r^-$ for all $x \in \overline{\Omega}$. Then we have

- **(i)** If $\theta > 0$, $\eta \in \mathbb{R}$, then problem (1.1) has infinitely many solutions $(\pm\varphi_k)$ with $I_{\theta,\eta}(\pm\varphi_k) \to +\infty$ as $k \to +\infty$.
- **(ii)** If $\eta > 0$, $\theta \in \mathbb{R}$, then problem (1.1) has infinitely many solutions $(\pm\psi_k)$ with $I_{\theta,\eta}(\pm\psi_k) < 0$ and $I_{\theta,\eta}(\pm\psi_k) \to 0$ as $k \to +\infty$.

We will use the Fountain and the Dual Fountain theorem (see Willem [27] and El Amrouss et al. [13]) to prove Theorem 3.1. Since X is a reflexive and separable Banach space, then there exist $\{\varrho_j\} \subset X$ and $\{\varrho_j^*\} \subset X^*$ (see Zhao [31]) such that

$$X = \overline{\text{span}\ \{\varrho_j\ : j = 1, 2, \dots\}}, \qquad X^* = \overline{\text{span}\ \{\varrho_j^*\ : j = 1, 2, \dots\}},$$

and

$$\langle \varrho_i, \varrho_j^* \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

Now, we define the following spaces which used in the Fountain theorem and the Dual fountain theorem.

$$X_j = \text{span}\ \{\varrho_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{3.1}$$

Proof of Theorem 3.1

We apply the Fountain theorem to prove conclusion (**i**) in Theorem 3.1.

(**i**) First, we check the (PS) condition for the functional $I_{\theta,\eta}$. Suppose that $(\varphi_n) \subset X$ is (PS) sequence, i.e.,

$$|I_{\theta,\eta}(\varphi_n)| \leq c_9, \quad I'_{\theta,\eta}(\varphi_n) \to 0 \quad \text{as} \ n \to \infty.$$

It is evident that $I'_{\theta,\eta}$ is of type (S_+). Thus, we just need to verify that (φ_n) is bounded. For $\|\varphi_n\| > 1$ with n large enough, we can write

$$\begin{aligned}
c_9 + 1 + \|\varphi_n\| \geq & I_{\theta,\eta}(\varphi_n) - \frac{1}{r^-}\langle I'_{\theta,\eta}(\varphi_n), \varphi_n \rangle \\
= & \Big[\widehat{\mathcal{K}}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi_n|^{l(x)}\,dx\Big) - \theta\int_\Omega \frac{1}{r(x)}|\varphi_n|^{r(x)}\,dx - \eta\int_\Omega \frac{1}{t(x)}|\varphi_n|^{t(x)}\,dx\Big] \\
& - \frac{1}{r^-}\Big[\mathcal{K}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi_n|^{l(x)}dx\Big)\int_\Omega \frac{1}{l(x)}|\Delta\varphi_n|^{l(x)}\,dx - \theta\int_\Omega |\varphi_n|^{r(x)}\,dx - \eta\int_\Omega |\varphi_n|^{t(x)}\,dx\Big] \\
\geq & \Big(\frac{1}{l^+} - \frac{1}{r^-}\Big)\mathcal{K}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi_n|^{l(x)}\,dx\Big)\int_\Omega |\Delta\varphi_n|^{l(x)}\,dx - \theta\int_\Omega |\varphi_n|^{r(x)}\,dx - \eta\int_\Omega |\varphi_n|^{t(x)}\,dx\Big] \\
\geq & \Big(\frac{1}{l^+} - \frac{1}{r^-}\Big)\frac{m_1}{(l^+)^{\alpha-1}}\Big(\int_\Omega |\Delta\varphi_n|^{l(x)}\,dx\Big)^\alpha - \theta\int_\Omega |\varphi_n|^{r(x)}\,dx - \eta\int_\Omega |\varphi_n|^{t(x)}\,dx \\
\geq & \Big(\frac{1}{l^+} - \frac{1}{r^-}\Big)\frac{m_1}{(l^+)^{\alpha-1}}\|\varphi_n\|^{\alpha l^-} - c_{10}\|\varphi_n\|^{t^+}.
\end{aligned} \tag{3.2}$$

Considering $r^- > l^+$ and $\alpha l^- > t^+$, we obtain $\{\varphi_n\}$ is bounded in X. Next, we will establish that when k is sufficiently large, we can opt $\rho_k > r_k > 0$ so that the two conditions in the Fountain theorem ((**A2**) and (**A3**) as in Lemma 3.4 in [2]) are true.

(**A2**) For $\varphi \in Z_k$ with $\|\varphi\| = r_k > 1$ (r_k will be specified below):

$$\begin{aligned}
I_{\theta,\eta}(\varphi) &= \widehat{\mathcal{K}}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi|^{l(x)}\,dx\Big) - \theta\int_\Omega \frac{1}{r(x)}|\varphi|^{r(x)}\,dx - \eta\int_\Omega \frac{1}{t(x)}|\varphi|^{t(x)}\,dx \\
&\geq \frac{m_1}{\alpha(l^+)^\alpha}\Big(\int_\Omega |\Delta\varphi|^{l(x)}\,dx\Big)^\alpha - \frac{\theta}{r^-}\int_\Omega |\varphi|^{r(x)}\,dx - \frac{c_{11}|\eta|}{t^-}\|\varphi\|^{t^+} \\
&\geq \frac{m_1}{\alpha(l^+)^\alpha}\|\varphi\|^{\alpha l^-} - \frac{\theta}{r^-}\int_\Omega |\varphi|^{r(x)}\,dx - \frac{c_{11}|\eta|}{t^-}\|\varphi\|^{t^+}
\end{aligned}$$

Since $\alpha l^- > t^+$, there exists $r_0 > 0$ large enough such that $\frac{c_{11}|\eta|}{t^-}\|\varphi\|^{t^+} \leq \frac{m_1}{2(l^+)^\alpha}\|\varphi\|^{\alpha l^-}$ as $r = \|\varphi\| \geq r_0$. If $|\varphi|_{r(x)} \leq 1$ then $\int_\Omega |\varphi|^{r(x)}dx \leq |\varphi|_{r(x)}^{r^-} \leq 1$. However, if $|\varphi|_{r(x)} > 1$ then $\int_\Omega |\varphi|^{r(x)}\,dx \leq |\varphi|_{r(x)}^{r^+} \leq (\beta_k\|\varphi\|)^{r^+}$. So, we conclude that

$$\begin{aligned}
I_{\theta,\eta}(\varphi) &\geq \begin{cases} \frac{m_1}{2(l^+)^\alpha}\|\varphi\|^{\alpha l^-} - \frac{\theta c_{12}}{r^-} & \text{if } |\varphi|_{r(x)} \leq 1, \\ \frac{m_1}{2(l^+)^\alpha}\|\varphi\|^{\alpha l^-} - \frac{\theta}{r^-}(\beta_k\|\varphi\|)^{r^+} & \text{if } |\varphi|_{r(x)} > 1. \end{cases} \\
&\geq \frac{m_1}{2(l^+)^\alpha}\|\varphi\|^{\alpha l^-} - \frac{\theta}{r^-}(\beta_k\|\varphi\|)^{r^+} - c_{13},
\end{aligned}$$

choose $r_k = \Big(\frac{2\theta}{m_1 r^-} r^+ \beta_k^{r^+}\Big)^{\frac{1}{\alpha p^- - r^+}}$, we have

$$I_{\theta,\eta}(\varphi) = \frac{m_1}{2}\Big(\frac{1}{(l^+)^\alpha} - \frac{1}{r^+}\Big)r_k^{\alpha p^-} - c_{13} \to \infty \quad \text{as} \quad k \to \infty,$$

since $(l^+)^\alpha < r^- \leq r^+$ and $\beta_k \to 0$ (Lemma 3.3 in [2]).

(**A3**) If $\varphi \in Y_k$ with $\|\varphi\| = \rho_k > r_k > 1$, we have

$$\begin{aligned}
I_{\theta,\eta}(\varphi) &= \widehat{\mathcal{K}}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi|^{l(x)}\,dx\Big) - \theta\int_\Omega \frac{1}{r(x)}|\varphi|^{r(x)}\,dx - \eta\int_\Omega \frac{1}{t(x)}|\varphi|^{t(x)}\,dx \\
&\leq \frac{m_2}{\beta}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi|^{l(x)}\,dx\Big)^\beta - \frac{\theta}{r^+}\int_\Omega |\varphi|^{r(x)}\,dx + \frac{|\eta|}{t^-}\int_\Omega |\varphi|^{t(x)}\,dx \\
&\leq \frac{m_2}{\beta(l^-)^\beta}\|\varphi\|^{\beta l^+} - \frac{\theta}{r^+}\int_\Omega |\varphi|^{r(x)}\,dx + \frac{|\eta|}{t^-}\int_\Omega |\varphi|^{t(x)}\,dx.
\end{aligned}$$

Using the fact that all norms are equivalent in in Y_k, due to dim$Y_k < \infty$, it follows

$$I_{\theta,\eta}(\varphi) \leq \frac{m_2}{\beta(l^-)^\beta}\|\varphi\|^{\beta l^+} - \frac{\theta}{r^+}\|\varphi\|^{r^-} + \frac{|\eta|}{t^-}\|\varphi\|^{t^+}.$$

We get that $I_{\theta,\eta}(\varphi) \to -\infty$ as $\|\varphi\| \to +\infty$, since $r^- > \beta l^+$ and $t^+ < \alpha l^-$. Considering (**A2**) and (**A3**), the option to choose $\rho_k > r_k > 0$ is available for us. Also, $I_{\theta,\eta}$ is even and the first assertion in Theorem 3.1 is completed.

(**ii**) The Dual Fountain theorem is aaplied to establish the claim (**ii**). Now we show, when k is large, we can choose $\rho_k > r_k > 0$ so that if k is large enough, the three conditions in the Dual Fountain theorem ((**B1**), (**B2**) and (**B3**) as in Lemma 3.5 in [2]) are satisfied.

(**B1**) Let $\varphi \in Z_k$ we can write

$$\begin{aligned}
I_{\theta,\eta}(\varphi) &= \widehat{\mathcal{K}}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi|^{l(x)}\,dx\Big) - \theta\int_\Omega \frac{1}{r(x)}|\varphi|^{r(x)}\,dx - \eta\int_\Omega \frac{1}{t(x)}|\varphi|^{t(x)}\,dx \\
&\geq \frac{m_1}{\alpha(l^+)^\alpha}\Big(\int_\Omega |\Delta\varphi|^{l(x)}\,dx\Big)^\alpha - \frac{|\theta|}{r^-}\int_\Omega |\varphi|^{r(x)}\,dx - \frac{\eta}{t^-}\int_\Omega |\varphi|^{t(x)}\,dx \\
&\geq \frac{m_1}{\alpha(l^+)^\alpha}\|\varphi\|^{\alpha l^+} - \frac{c_{14}|\theta|}{r^-}\|\varphi\|^{r^-} - \frac{\eta}{t^-}\int_\Omega |\varphi|^{t(x)}\,dx
\end{aligned}$$

Considering $r^- > \alpha l^+$, there exists $\rho_0 > 0$ small enough such that $\frac{c_{14}|\theta|}{r^-}\|\varphi\|^{r^-} \le \frac{m_1}{2\alpha(l^+)^\alpha}\|\varphi\|^{\alpha l^+}$ as $0 < \rho = \|\varphi\| \le \rho_0$. As discussed above, we get

$$I_{\theta,\eta}(\varphi) \ge \begin{cases} \frac{m_1}{\alpha(l^+)^\alpha}\|\varphi\|^{\alpha l^+} - \frac{\eta c_{15}}{t^-} & \text{if } |\varphi|_{t(x)} \le 1, \\ \frac{m_1}{\alpha(l^+)^\alpha}\|\varphi\|^{l^+} - \frac{\eta}{t^-}(\theta_k\|\varphi\|)^{t^+} & \text{if } |\varphi|_{t(x)} > 1. \end{cases} \tag{3.3}$$

Choose $\rho_k = \left(\frac{2\eta}{m_1 t^-}(l^+)^\alpha \theta_k^{t^+}\right)^{\frac{1}{\alpha l^+ - t^+}}$, then

$$I_{\theta,\eta}(\varphi) = \frac{m_1}{2(l^+)^\alpha}(\rho_k)^{\alpha l^+} - \frac{m_1}{2(l^+)^\alpha}(\rho_k)^{\alpha l^+} = 0.$$

Due to $\alpha l^- > t^+$, $\theta_k \to 0$ (Lemma 3.3 in [2]), we have $\rho_k \to 0$ as $k \to \infty$.

(**B2**) If $\varphi \in Y_k$ with $\|\varphi\| \le 1$, we deduce

$$\begin{aligned} I_{\theta,\eta}(\varphi) &= \widehat{\mathcal{K}}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi|^{l(x)}\,dx\Big) - \theta\int_\Omega \frac{1}{r(x)}|\varphi|^{r(x)}\,dx - \eta\int_\Omega \frac{1}{t(x)}|\varphi|^{t(x)}\,dx \\ &\le \frac{m_2}{\beta(l^-)^\beta}\Big(\frac{1}{l(x)}|\Delta\varphi|^{l(x)}\,dx\Big)^\beta + \frac{|\theta|}{r^-}\int_\Omega |\varphi|^{r(x)}dx - \frac{\eta}{t^+}\int_\Omega |\varphi|^{t(x)}\,dx \\ &\le \frac{m_2}{\beta(l^-)^\beta}\|\varphi\|^{\beta l^-} + \frac{|\theta|}{r^-}\int_\Omega |\varphi|^{r(x)}\,dx - \frac{\eta}{t^+}\int_\Omega |\varphi|^{t(x)}\,dx. \end{aligned}$$

By $\dim Y_k = k$, relations $t^+ < \alpha l^- < \beta l^- < \beta(l^-)^\beta$ and $\beta l^+ < r^-$ then, there exists a $r_k \in (0,\rho_k)$ such that $I_{\theta,\eta}(\varphi) < 0$ when $\|\varphi\| = r_k$. So we conclude

$$\max_{\varphi\in Y_k,\ \|\varphi\|=r_k} I_{\theta,\eta}(\varphi) < 0,$$

i.e., (**B2**) is true.

(**B3**) Due to $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we get

$$d_k = \inf_{\varphi\in Z_k,\|\varphi\|\le\rho_k} I_{\theta,\eta}(\varphi) \le b_k = \max_{\varphi\in Y_k,\|\varphi\|=r_k} I_{\theta,\eta}(\varphi) < 0.$$

Using (3.3) , for $\varphi \in Z_k$, $\|\varphi\| \le \rho_k$ small enough we arrive at

$$\begin{aligned} I_{\theta,\eta}(\varphi) &\ge \frac{m_1}{2(l^+)^\alpha}\|\varphi\|^{\alpha l^+} - \frac{\eta}{t^-}\theta_k^{t^+}\|\varphi\|^{t^+} \\ &\ge -\frac{\eta}{t^-}\theta_k^{t^+}\|\varphi\|^{t^+}, \end{aligned}$$

Due to $\theta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$, (**B3**) is true. Finally, we establish the $(PS)_c^*$ condition (Definition 3.6 in [2]). Let $\{\varphi_{n_j}\} \subset X$ with

$$n_j \to +\infty, \quad \varphi_{n_j} \in Y_{n_j}, \quad I_{\theta,\eta}(\varphi_{n_j}) \to c_{16} \quad \text{and} \quad (I_{\theta,\eta}|_{Y_{n_j}})'(\varphi_{n_j}) \to 0.$$

If $\theta \ge 0$, similar to (3.2), we have the boundedness of $\|\varphi_{n_j}\|$. Suppose $\|\varphi_{n_j}\| \ge 1$. When $\theta < 0$, if n is large enough, we can conclude that

$$\begin{aligned} c_{16} + 1 + \|\varphi_{n_j}\| \ge & I_{\theta,\eta}(\varphi_{n_j}) - \frac{1}{r^+}\langle I'_{\theta,\eta}(\varphi_{n_j}), \varphi_{n_j}\rangle \\ =& \Big[\widehat{\mathcal{K}}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi_{n_j}|^{l(x)}\,dx\Big) - \theta\int_\Omega \frac{1}{r(x)}|\varphi_{n_j}|^{r(x)}\,dx - \eta\int_\Omega \frac{1}{t(x)}|\varphi_{n_j}|^{t(x)}\,dx\Big] \\ & - \frac{1}{r^+}\Big[\mathcal{K}\Big(\int_\Omega \frac{1}{l(x)}|\Delta\varphi_{n_j}|^{l(x)}dx\Big)\int_\Omega \frac{1}{l(x)}|\Delta\varphi_{n_j}|^{l(x)}\,dx - \theta\int_\Omega |\varphi_{n_j}|^{r(x)}\,dx - \eta\int_\Omega |\varphi_{n_j}|^{t(x)}\,dx\Big] \\ \ge& \Big(\frac{1}{l^+} - \frac{1}{r^+}\Big)\frac{m_1}{(l^+)^{\alpha-1}}\Big(\int_\Omega |\Delta\varphi_{n_j}|^{l(x)}\,dx\Big)^\alpha - \theta\int_\Omega |\varphi_{n_j}|^{r(x)}\,dx - \eta\int_\Omega |\varphi_{n_j}|^{t(x)}\,dx\Big] \\ \ge& \Big(\frac{1}{l^+} - \frac{1}{r^+}\Big)\frac{m_1}{(l^+)^{\alpha-1}}\|\varphi_{n_j}\|^{\alpha l^-} - c_{17}\|\varphi_{n_j}\|^{t^+}. \end{aligned}$$

Due to $\alpha l^- > t^+$ and $l^+ < (l^+)^\alpha < r^-$, we deduce $\{\varphi_{n_j}\}$ is bounded in X.

Similar to the proof in [2], we can obtain that $I_{\theta,\eta}$ satisfies the $(PS)_c^*$ condition.

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