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On new subclasses of analytic functions associated with generalized Bessel functions

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Abstract

Bessel functions arise in the solution of many physical and mathematical problems. This, with some other special functions, has recently gained increased importance in the study of geometric function theory. The aim of this paper is to establish some geometric properties such as coefficient inequalities, characterization properties and convolution properties for the new subclasses $Q_n(\lambda,\alpha,\beta,\mu,t)$, $P_n(\lambda,\alpha,\beta,\mu,t)$ and $P^*_n(\lambda,\alpha,\beta,\mu,t)$ of univalent functions defined by Opoola Differential Operator in collaboration with generalized Bessel functions.

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1 Introduction and Preliminary

Let A be the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by S the subclass of A consisting of functions which are analytic, univalent in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$.

A function $f(z) \in S$ of the form (1.1) is star-like in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ if it maps a unit disk onto a star-like domain. A necessary and sufficient condition for a function $f(z)$ to be star-like is that

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, (z \in \mathbb{U})$$

The class of all star-like functions is denoted by S^*. An analytic function $f(z)$ is convex if it maps the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ conformally onto a convex domain. Equivalently, a function $f(z)$ is said to be convex if and

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only if it satisfies the following condition;

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ (z\in\mathbb{U}).$$

The class of all convex functions is denoted by K. A function $f(z)\in A$ is said to be in the class $R^{\tau}(A,B),\{\tau\in\mathbb{C}\setminus\{0\}\},-1\leq B<A\leq 1$ if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1,(z\in\mathbb{U}).$$

The Class $\text{Ŗ}^{\tau}(A,B)$ was introduced in [11]. Let $\mathbb{T}$ denote the subclass of S consisting of functions whose non-zero coefficients, from the second on, are negative. That is, an analytic and univalent function $f(z)\in\mathbb{T}$ if it can be expressed as

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k,(a_k\geq 0)$$

see [30].

Definition 1.1. [22] For $t\geq 0$, $0\leq\mu\leq\beta,n\in\mathbb{N}_0,z\in\mathbb{U}$, Opoola Differential Operator $D^n(\mu,\beta,t)f(z):A\to A$ is defined as:

$$\begin{aligned}
&D^0(\mu,\beta,t)f(z)=f(z)\\
&D^1(\mu,\beta,t)f(z)=tzf'(z)-z(\beta-\mu)t+(1+(\beta-\mu-1)t)f(z)\\
&D^n(\mu,\beta,t)f(z)=(D(D^{n-1}(\mu,\beta,t)f(z))).
\end{aligned}$$

From the above definition and for $f(z)$ given by equation (1.1)

$$D^n(\mu,\beta,t)f(z)=z+\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^na_kz^k \tag{1.2}$$

Remark 1.2. The following are some remarks on Opoola Differential Operator.

1. When $t=1$, $\mu=\beta$, then $D^n(\mu,\beta,t)f(z)$ reduces to Sălăgean differential operator. [29]
2. When $\mu=\beta$, then $D^n(\mu,\beta,t)f(z)$ reduces to Al-Oboudi differential operator. [1]

Definition 1.3. A function $f(z)\in A$ is in the class $Q_n(\lambda,\alpha,\beta,\mu,t)$ if

$$\left|\frac{z(D^nf(z))'+\lambda z^2(D^nf(z))''}{D^nf(z)}-1\right|<1-\alpha,(z\in\mathbb{U}). \tag{1.3}$$

where $0\leq\lambda<1,0\leq\alpha<1,n\in\mathbb{N}_0$ and $D^nf(z)$ is the Opoola differential operator for $f(z)$.

A function $f(z)\in A$ belong to the class $P_n(\lambda,\alpha,\beta,\mu,t)$ if

$$\left|\frac{z[z(D^nf(z))'+\lambda z^2(D^nf(z))'']'}{z(D^nf(z))'}-1\right|<1-\alpha,(z\in\mathbb{U}) \tag{1.4}$$

where $0\leq\lambda<1,0\leq\alpha<1,n\in\mathbb{N}_0$ and $D^nf(z)$ is the Opoola differential operator for $f(z)$.

Remark 1.4. For n=0;

1. $Q_n(\lambda,\alpha,\beta,\mu,t)\equiv G(\lambda,\alpha)$.
2. $P_n(\lambda,\alpha,\beta,\mu,t)\equiv K(\lambda,\alpha)$.
   The classes $G(\lambda,\alpha)$ and $K(\lambda,\alpha)$ were introduced and studied by Cho et al in [10].
   For $n=\lambda=0$;
3. $Q_n(\lambda,\alpha,\beta,\mu,t)\equiv S^*(\alpha)$ and $P_n(\lambda,\alpha,\beta,\mu,t)\equiv K(\alpha)$

The classes $S^*(\alpha)$ and $K(\alpha)$ are classes of star-like and convex functions of order α respectively. [30]

Definition 1.5. For $0 \leq \lambda < 1, 0 \leq \alpha < 1, n \in \mathbb{N}_0$. A function $f(z) \in \mathbb{T}$ belongs to the class $P_n^*(\lambda, \alpha, \beta, \mu, t)$ if

$$\left| \frac{z[z(D^n f(z))' + \lambda z^2 (D^n f(z))'']'}{z(D^n f(z))'} - 1 \right| < 1 - \alpha, (z \in \mathbb{U}) \tag{1.5}$$

$D^n f(z)$ is the Opoola differential operator for $f(z) \in \mathbb{T}$.

Bessel function of the first kind of order p: The classical Bessel function of the first kind of order p can be defined by:

$$J_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+1)} \left(\frac{z}{2}\right)^{2k+p} \tag{1.6}$$

$z \in \mathbb{C}, p$ can be real or complex and equation (1.6) satisfies the second order ordinary differential equation

$$z^2 \omega''(z) + z\omega'(z) + (z^2 - p^2)\omega(z) = 0. \tag{1.7}$$

In [5]-[7], Baricz introduced the generalized Bessel function of the first kind of order p which is an extension of the classical Bessel function of the first kind of order p and it is defined by

$$\omega_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k!\Gamma\left(p+k+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2k+p} \tag{1.8}$$

equation (1.8) satisfies the second order linear ordinary differential equation

$$z^2 \omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = 0 \tag{1.9}$$

where $p, b, c \in \mathbb{C}$. On different choices of b and c, classical Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel function of the first kind order p can be obtained respectively from equation (1.9). Next is to consider the function $U_{p,b,c}(z)$ defined by the transformation

$$\begin{aligned} U_{p,b,c}(z) =& 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{-p/2} \omega_{p,b,c}(\sqrt{z}) \\ =& 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{-p/2} \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k!\Gamma\left(p+k+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2k+p} \\ =& \sum_{k=0}^{\infty} \frac{(-1)^k (c/4)^k \Gamma\left(p+\frac{b+1}{2}\right)}{k!\Gamma\left(p+k+\frac{b+1}{2}\right)} z^k \end{aligned}$$

using Pochhammer symbol in terms of Gamma, written as

$$(m)_k = \frac{\Gamma(m+k)}{\Gamma(m)}$$

and for conveniences sake, we write $U_{p,b,c}(z)$ as $U_p(z)$.

$$U_p(z) = \sum_{k=0}^{\infty} \frac{(-c/4)^k}{(m)_k \, k!} z^k \tag{1.10}$$

where $m = \left(p + \frac{b+1}{2}\right) \neq 0, -1, -2, ...U_p(z)$ is called the generalized and normalized Bessel function of the first kind of order p. The function $U_p(z)$ is analytic in $\mathbb{C}$ and satisfies the second order ordinary differential equation

$$4z^2 u''(z) + 4kzu'(z) + czu(z) = 0.$$

Consider the linear operator $I(c,m) : A \longrightarrow A$ defined by

$$I(c,m)f(z) = z(U_p(z)) * f(z) = z + \sum_{k=0}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} a_k z^k$$

for

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

where $z(U_p(z)) * f(z)$ is the Hadarmand product of $z(U_p(z))$ and $f(z)$. If $c < 0$ and $m > 0$, then

$$z(2 - U_p(z)) = z - \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} z^k. \tag{1.11}$$

Opoola differential operator was introduced by Opoola in [22]. In geometric function theory, some researchers that have studied Opoola differential operator include but not limited [8, 9, 13, 14, 15, 17, 18, 19, 23, 24, 31]. The operator has been used to define various new subclasses of univalent, p-valent and bi-univalent functions, their geometric properties were also established by these researchers. Due to the work of Baricz in [5, 6, 7], Ramachandran et al in [26] and Dixit and Pal in [11], it is observed that the study of generalized Bessel functions of the first kind and it's application to subclasses of analytic functions in geometric function theory started recently. Motivated by [10], Cho et al defined two subclasses of univalent functions and studied the characterization property of function $z(2 - U_p(z))$ (derived from the generalized Bessel functions of the first kind of order p) to be in the new subclasses $G(\lambda, \alpha)$ and $K(\lambda, \alpha)$ of analytic functions. The geometric properties of the new classes of analytic functions defined in inequalities (1.3)-(1.5) will be studied and new results for these classes will also be established by the following theorems.

2 Main Results

Coefficients Inequalities for classes $Q_n(\lambda, \alpha, \beta, \mu, t)$, $P_n(\lambda, \alpha, \beta, \mu, t)$ and $P_n^*(\lambda, \alpha, \beta, \mu, t)$.

Theorem 2.1. A function $f(z) \in A$ belongs to the class $Q_n(\lambda, \alpha, \beta, \mu, t)$ if

$$\sum_{k=2}^{\infty} [k + \lambda k(k-1) - \alpha] M_k |a_n| \leq 1 - \alpha \tag{2.1}$$

where $M_k = [1 + (k + \beta - \mu - 1)t]^n$, $0 \leq \lambda < 1$, $0 \leq \alpha < 1$, $n \in \mathbb{N}_0$, $t \geq 0$, $0 \leq \beta \leq \mu \leq 1$, $z \in \mathbb{U}$.

Proof . Let $f(z) \in A$ belongs to the class $Q_n(\lambda, \alpha, \beta, \mu, t)$, then by the Definition of $Q_n(\lambda, \alpha, \beta, \mu, t)$ in inequality (1.3), it implies that

$$\left| \frac{z(D^n f(z))' + \lambda z^2 (D^n f(z))''}{D^n f(z)} - 1 \right| < 1 - \alpha. \tag{2.2}$$

Inequality (2.2) implies that $\frac{z(D^n f(z))' + \lambda z^2 (D^n f(z))''}{D^n f(z)}$ lies in a circle centered at $v = 1$, radius $(1 - \alpha)$ and

$$\begin{aligned}
\left| \frac{z(D^n f(z))' + \lambda z^2 (D^n f(z))''}{D^n f(z)} - 1 \right| &= \left| \frac{z(D^n f(z))' + \lambda z^2 (D^n f(z))'' - D^n f(z)}{D^n f(z)} \right| \\
&= \left| \frac{\sum_{k=2}^{\infty} [k + \lambda k(k-1) - 1] M_k a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} M_k a_k z^{k-1}} \right| \\
&= \frac{\left| \sum_{k=2}^{\infty} [k + \lambda k(k-1) - 1] M_k a_k z^{k-1} \right|}{\left| 1 + \sum_{k=2}^{\infty} M_k a_k z^{k-1} \right|}
\end{aligned} \tag{2.3}$$

Using the principle of triangle inequality, equation (2.3) can now be written as

$$\frac{\left| \sum_{k=2}^{\infty} [k + \lambda k(k-1) - 1] M_k a_k z^{k-1} \right|}{\left| 1 + \sum_{k=2}^{\infty} M_k a_k z^{k-1} \right|} \leq \frac{\sum_{k=2}^{\infty} [k + \lambda k(k-1) - 1] M_k |a_k| |z|^{k-1}}{1 - \sum_{k=2}^{\infty} M_k |a_k| |z|^{k-1}}.$$

Taking the value of z on the real axis as $|z| < 1$

$$\frac{\sum_{k=2}^{\infty}[k+\lambda k(k-1)-1]M_k|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}M_k|a_k||z|^{k-1}} \leq \frac{\sum_{k=2}^{\infty}[k+\lambda k(k-1)-1]M_k|a_k|}{1-\sum_{k=2}^{\infty}M_k|a_k|}.$$

Suppose

$$\frac{\sum_{k=2}^{\infty}[k+\lambda k(k-1)-1]M_k|a_k|}{1-\sum_{k=2}^{\infty}M_k|a_k|} \leq 1-\alpha$$

then

$$\sum_{k=2}^{\infty}[k+\lambda k(k-1)-1]M_k|a_k| \leq (1-\alpha)\left\{1-\sum_{k=2}^{\infty}M_k|a_k|\right\}$$

which implies that

$$\sum_{k=2}^{\infty}[k+\lambda k(k-1)-1+1-\alpha]M_k|a_k| \leq 1-\alpha.$$

Hence,

$$\sum_{k=2}^{\infty}[k+\lambda k(k-1)-\alpha]M_k|a_k| \leq 1-\alpha.$$

□

Theorem 2.2. A function $f(z) \in A$ belongs to the class $P_n(\lambda, \alpha, \beta, \mu, t)$ if

$$\sum_{k=2}^{\infty} k[k+\lambda k(k-1)-\alpha]M_k|a_k| \leq 1-\alpha \tag{2.4}$$

where $M_k = [1+(k+\beta-\mu-1)t]^n, 0 \leq \lambda < 1,\ 0 \leq \alpha < 1, n \in \mathbb{N}_0, t \geq 0, 0 \leq \beta \leq \mu \leq 1, z \in \mathbb{U}$.

Proof . Let $f(z) \in A$ be in the class $P_n(\lambda, \alpha, \beta, \mu, t)$ and from the Definition of $P_n(\lambda, \alpha, \beta, \mu, t)$, expansion of inequality (1.4) yields the following equation

$$\left|\frac{z[z(D^n f(z))' + \lambda z^2(D^n f(z))'']'}{z(D^n f(z))'} - 1\right| = \frac{\left|\sum_{k=2}^{\infty}(k^2+\lambda k^2(k-1)-k)M_k a_k z^{k-1}\right|}{|1+\sum_{k=2}^{\infty}kM_k a_k z^{k-1}|}$$

Using the principle of triangle inequality, then

$$\frac{\left|\sum_{k=2}^{\infty}(k^2+\lambda k^2(k-1)-k)M_k a_k z^{k-1}\right|}{|1+\sum_{k=2}^{\infty}kM_k a_k z^{k-1}|} \leq \frac{\sum_{k=2}^{\infty}(k^2+\lambda k^2(k-1)-k)M_k|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}kM_k|a_k||z|^{k-1}}$$

And taking the value of z on the real axis as $|z| < 1$

$$\frac{\sum_{k=2}^{\infty}(k^2+\lambda k^2(k-1)-k)M_k|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}kM_k|a_k||z|^{k-1}} \leq \frac{\sum_{k=2}^{\infty}(k^2+\lambda k^2(k-1)-k)M_k|a_k|}{1-\sum_{k=2}^{\infty}kM_k|a_k|}$$

Suppose

$$\frac{\sum_{k=2}^{\infty}(k^2+\lambda k^2(k-1)-k)M_k|a_k|}{1-\sum_{k=2}^{\infty}kM_k|a_k|} \leq 1-\alpha$$

then

$$\sum_{k=2}^{\infty}(k^2+\lambda k^2(k-1)-k)M_k|a_k| \leq (1-\alpha)\left(1-\sum_{k=2}^{\infty}kM_k|a_k|\right).$$

Hence,

$$\sum_{k=2}^{\infty}k(k+\lambda k(k-1)-\alpha)M_k|a_k| \leq 1-\alpha.$$

□

Theorem 2.3. A function $f(z) \in \mathbb{T}$ belongs to the class $P_n^*(\lambda, \alpha, \beta, \mu, t)$ if

$$\sum_{k=2}^{\infty} k[k + \lambda k(k-1) - \alpha] M_k |a_k| \leq 1 - \alpha \tag{2.5}$$

where

$$M_k = [1 + (k + \beta - \mu - 1)t]^n, 0 \leq \lambda < 1, 0 \leq \alpha < 1, n \in \mathbb{N}_0, t \geq 0, 0 \leq \beta \leq \mu \leq 1, z \in \mathbb{U}.$$

Lemma 2.4. [12] If a function $f(z) \in A$ belong the class $\c{R}^{\tau}(A, B)$, then

$$|a_k| \leq \frac{(A-B)|\tau}{k}. \tag{2.6}$$

Characterization Properties for $z(2 - U_p(z))$ to be in classes $Q_n(\lambda, \alpha, \beta, \mu, t)$ and $P_n(\lambda, \alpha, \beta, \mu, t)$.

Theorem 4. If $c < 0, m > 0, t \geq 0, 0 \leq \beta \leq \mu \leq 1, n \in \mathbb{N} \cup \{0\}, 0 \leq \alpha < 1$ and $0 \leq \lambda < 1$, then $z(2 - U_p(z))$ is in $Q_n(\lambda, \alpha, \beta, \mu, t)$ if

$$\lambda H_p''(1) + (1 + 2\lambda) H_p'(1) + \{(1-\alpha) H_p(1)\} \leq (1 - \alpha)(1 + M_1). \tag{2.7}$$

Proof . Let $z(2 - U_p(z))$ be in $Q_n(\lambda, \alpha, \beta, \mu, t)$, then from equation (1.11)

$$z(2 - U_p(z)) = z - \frac{\sum_{k=2}^{\infty} (-c/4)^{k-1}}{(m)_{k-1}(k-1)!} z^n.$$

By virtue of inequality (2.1), it suffices to show that

$$\sum_{k=2}^{\infty} (k + \lambda k(k-1) - \alpha) M_k \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} \leq 1 - \alpha.$$

Let

$$L(v) = \sum_{k=2}^{\infty} (k + \lambda k(k-1) - \alpha) \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} M_k \tag{2.8}$$

writing $k^2 = (k-1)(k-2) + 3(k-1) + 1$ and $k = (k-1) + 1$, equation (2.8) can be re-written as

$$\begin{aligned}
L(v) \leq & \sum_{k=2}^{\infty} \lambda(k-1)(k-2) \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} M_k + (1 + 2\lambda(k-1)) \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} M_k + (1-\alpha) \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} M_k \\
= & \lambda \sum_{k=3}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-3)!} M_k + (1+2\lambda) \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-2)!} M_k + (1-\alpha) \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} M_k \\
= & \lambda \sum_{k=1}^{\infty} \frac{(-c/4)^{k+1}}{(m)_{k+1}(k-1)!} M_{k+2} + (1+2\lambda) \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m)_{k+1} k!} M_{k+2} + (1-\alpha) \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!} M_{k+1} \\
= & \lambda \frac{(-c/4)^2}{m(m+1)} \sum_{k=0}^{\infty} \frac{(-c/4)^k}{(m+2)_k k!} M_{k+3} + (1+2\lambda) \frac{(-c/4)}{m} \sum_{k=0}^{\infty} \frac{(-c/4)^k}{(m+1)_k k!} M_{k+2} + (1-\alpha) \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!} M_{k+2} \\
= & \lambda \frac{(-c/4)^2}{m(m+1)} \left\{ M_3 + M_4 \frac{(-c/4)}{(m+2)} + \sum_{k=2}^{\infty} \frac{(-c/4)^k}{(m+2)_k k!} \right\} + (1+2\lambda) \frac{(-c/4)}{m} \left\{ M_2 + \sum_{k=1}^{\infty} \frac{(-c/4)}{(m+1)_k k!} \right\} \\
& + (1-\alpha) \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!} M_{k+2} \\
= & \lambda \frac{(-c/4)^2}{m(m+1)} \sum_{k=2}^{\infty} \frac{(-c/4)^{k-2}}{(m+2)_{k-2}(k-2)!} M_{k+1} + (1+2\lambda) \frac{(-c/4)}{m} \sum_{k=1}^{\infty} \frac{(-c/4)^{k-1}}{(m+1)_{k-1}(k-1)!} M_{k+1} \\
& + (1-\alpha) \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!} M_{k+2} \\
= & \lambda \sum_{k=2}^{\infty} \frac{(-c/4)^k}{(m)_k (k-2)!} M_{k+1} + (1+2\lambda) \sum_{k=1}^{\infty} \frac{(-c/4)^k}{(m)_k (k-1)!} M_{k+1} + (1-\alpha) \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!} M_{k+2} \\
= & \lambda H_p''(1) + (1+2\lambda) H_p'(1) + (1-\alpha)\{H_p(1) - M_1\}.
\end{aligned}$$

Hence, the last expression is bounded above by $1-\alpha$ if

$$\lambda H_p''(1)+(1+2\lambda)H_p'(1)+(1-\alpha)H_p(1)\leq(1-\alpha)(1+M_1)$$

where

$$H_p(z)=\sum_{k=0}^{\infty}\frac{(-c/4)^k}{(m)_k k!}M_{k+1}z^k.$$

□

Corollary 2.5. By taking $\lambda=0$, inequality (2.7) reduces to

$$H_p'(1)+(1-\alpha)[H_p(1)]\leq(1-\alpha)(1+M_1). \tag{2.9}$$

Remark 2.6. When $n=0$;

- $H_p(z)$ reduces to $U_p(z)$
- Inequality (2.7) reduces to inequality (2.1) in [11].

Theorem 2.7. If $c<0, m>0, t\geq 0, 0\leq\beta\leq\mu\leq 1, n\in\mathbb{N}_0, 0\leq\alpha<1$ and $0\leq\lambda<1$, then $z(2-U_p(z))$ is in $P_n(\lambda,\alpha,\beta,\mu,t)$ if

$$\lambda H_p'''(1)+(1+5\lambda)H_p''(1)+(3+4\lambda-\alpha)H_p'(1)+(1-\alpha)H_p(1)\leq(1-\alpha)(1+M_1). \tag{2.10}$$

Proof . Let $z(2-U_p(z))\in P_n(\lambda,\alpha,\beta,\mu,t)$. Since

$$z(2-U_p(z))=z-\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}z^k$$

and by the condition of Theorem 2, it is sufficient to show that

$$\sum_{k=2}^{\infty}(k^3+\lambda k^2(k-1)-k\alpha)M_k\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}\leq 1-\alpha.$$

Let

$$H(v)=\sum_{k=2}^{\infty}(k^3+\lambda k^2(k-1)-k\alpha)M_k\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}. \tag{2.11}$$

Writing $k^3=(k-1)(k-2)(k-3)+6(k-1)(k-2)+7(k-1)+1$ and $k^2=(k-1)(k-2)+3(k-1)+1$, equation (2.11) can then be written as

$$\begin{aligned}
H(v)\leq&\lambda\sum_{k=2}^{\infty}(k-1)(k-2)(k-3)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}+(1+5\lambda)\sum_{k=2}^{\infty}(k-1)(k-2)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}\\
&+(3+4\lambda-\alpha)\sum_{k=2}^{\infty}(k-1)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}+(1-\alpha)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}\\
=&\frac{\lambda(-c/4)^3}{m(m+1)(m+2)}\sum_{k=0}^{\infty}\frac{(-c/4)^kM_{k+4}}{(m+3)_k k!}+(1+5\lambda)\frac{(-c/4)^2}{m(m+1)}\sum_{k=0}^{\infty}\frac{(-c/4)^kM_{k+3}}{(m+2)_k k!}\\
&+(3+4\lambda-\alpha)\sum_{k=1}^{\infty}\frac{(-c/4)^kM_{k+1}}{(m)_k(k-1)!}+(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}M_{k+2}}{(m)_{k+1}(k+1)!}\\
=&\lambda\sum_{k=3}^{\infty}\frac{(-c/4)^kM_{k+1}}{(m)_k(k-3)!}+(1+5\lambda)\sum_{k=2}^{\infty}\frac{(-c/4)^kM_{k+1}}{(m)_k(k-2)!}+(3+4\lambda-\alpha)\sum_{k=1}^{\infty}\frac{(-c/4)^kM_{k+1}}{(m)_k(k-1)!}\\
&+(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}M_{k+2}}{(m)_{k+1}(k+1)!}\\
=&\lambda H_p'''(1)+(1+5\lambda)H_p''(1)+(3+4\lambda-\alpha)H_p'(1)+(1-\alpha)[H_p(1)-M_1].
\end{aligned}$$

The last equation is bounded above by $1-\alpha$ if

$$\lambda H_p'''(1)+(1+5\lambda)H_p''(1)+(3+4\lambda-\alpha)H_p'(1)+(1-\alpha)H_p(1)\leq(1-\alpha)(1+M_1),$$

where

$$H_p(z)=\sum_{k=0}^{\infty}\frac{(-c/4)^k}{(m)_k k!}M_{k+1}z^k.$$

□

Corollary 2.8. when $\lambda=0$, inequality (2.10) reduces to

$$H_p''(1)+(3-\alpha)H_p'(1)+(1-\alpha)[H_p(1)]\leq(1-\alpha)(1+M_1).$$

Remark 2.9. When $n=0$

- $H_p(z)$ reduces to $U_p(z)$.
- Inequality (2.11) reduces to inequality (2.4) in [11].

Convolution Properties for $I(c,m)$ to be in the classes $P_n(\lambda,\alpha,\beta,\mu,t)$ and $P_n^*(\lambda,\alpha,\beta,\mu,t)$

Theorem 2.10. Let $c<0, m>0, t\geq 0, 0\leq\mu\leq\beta, n\in\mathbb{N}\cup\{0\}, 0\leq\alpha<1$ and $0\leq\lambda<1$. If

$$f\in\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1,\qquad(z\in\mathbb{U})$$

and

$$(A-B)|\tau|[\lambda H_p''(1)+(1+2\lambda)H_p'(1)+(1-\alpha)[H_p(1)-M_1]\leq(1-\alpha)(1+M_1)\tag{2.12}$$

is satisfied, then $I(c,m)f(z)\in P_n(\lambda,\alpha,\beta,\mu,t)$.

Proof . Let $f(z)\in\c{R}^{\tau}(A,B)$, it can be recalled from equation (1.11) that

$$z(2-U_p(z))=z-\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}z^k.$$

Also,

$$I(c,m)f(z)=z(U_p(z))*f(z)=z+\sum_{k=0}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}a_kz^k$$

for

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k$$

To show that $I(c,m)f(z)\in P_n(\lambda,\alpha,\beta,\mu,t)$ is to show that by Theorem 2 and Inequality (2.4)

$$\sum_{k=2}^{\infty}k(k^2\lambda+k(1-\lambda)-\alpha)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}|a_k|\leq 1-\alpha.\tag{2.13}$$

Since by the condition of the class $\c{R}^{\tau}(A,B)$ which is defined by

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1,\qquad(z\in\mathbb{U})$$

and applying Lemma 1, inequality (2.13) gives

$$\sum_{k=2}^{\infty}k(k^2\lambda+k(1-\lambda)-\alpha)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}|a_k|\leq\sum_{k=2}^{\infty}k(k^2\lambda+k(1-\lambda)-\alpha)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}\frac{(A-B)}{k}|\tau|$$

Let

$$F(k)=\sum_{k=2}^{\infty}(k^2\lambda+k(1-\lambda)-\alpha)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}(A-B)|\tau|.$$

To show that $F(k)\leq 1-\alpha$, we write $k^2=(k-1)(k-2)+3(k-1)+1$, then $F(k)$ can be re-written as;

$$\begin{aligned}
F(k)\leq&(A-B)|\tau|\sum_{k=2}^{\infty}\lambda(k-1)(k-2)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}+(A-B)|\tau|(1+2\lambda)\sum_{k=2}^{\infty}(k-1)\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}\\
&+(A-B)|\tau|(1-\alpha)\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}M_k}{(m)_{k-1}(k-1)!}\\
=&(A-B)|\tau|\lambda\frac{(-c/4)^2}{m(m+1)}\sum_{k=2}^{\infty}\frac{(-c/4)^{k-2}M_{k+1}}{(m+2)_{k-2}(k-2)!}+(A-B)|\tau|(1+2\lambda)\frac{(-c/4)}{m}\sum_{k=1}^{\infty}\frac{(-c/4)^{k-1}M_{k+1}}{(m+1)_k(k-1)!}\\
&+(A-B)|\tau|(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}M_{k+2}}{(m)_{k+1}(k+1)!}\\
=&(A-B)|\tau|\lambda\sum_{k=2}^{\infty}\frac{(-c/4)^{k}M_{k+1}}{(m)_{k}(k-2)!}+(A-B)|\tau|(1+2\lambda)\sum_{k=1}^{\infty}\frac{(-c/4)^{k}M_{k+1}}{(m)_{k}(k-1)!}\\
&+(A-B)|\tau|(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}M_{k+2}}{(m)_{k+1}(k+1)!}\\
=&(A-B)|\tau|[\lambda H_p''(1)+(1+2\lambda)H_p'(1)+(1-\alpha)[H_p(1)-M_1]]
\end{aligned}\tag{2.14}$$

where

$$H_p(z)=\sum_{k=0}^{\infty}\frac{(-c/4)^k}{(m)_k k!}M_{k+1}z^k.$$

□

Corollary 2.11. When $\lambda=0$, Inequality (2.12) reduces to

$$(A-B)|\tau|[H_p'(1)+(1-\alpha)[H_p(1)-M_1]]\leq 1-\alpha.$$

Remark 2.12. When $n=0$;

- $H_p(z)$ reduces to $U_p(z)$.
- Inequality (2.14) reduces to inequality (2.3) in [11].

Theorem 2.13. For $t\geq 0, 0\leq\mu\leq\beta, n\in\mathbb{N}\cup\{0\}, 0\leq\alpha<1$ and $0\leq\lambda<1$. Let $c<0$, and $m>0$. Then

$$\int_0^z(2-U_p(t))dt$$

is in the class $P_n^*(\lambda,\alpha,\beta,\mu,t)$ if and only if the inequality

$$\lambda H_p''(1)+(1+2\lambda)H_p'(1)+(1-\alpha)\{H_p(1)-M_1\}\leq 1-\alpha\tag{2.15}$$

is satisfied.

Proof . If the generalized Bessel function is given as

$$U_p(z)=\sum_{k=2}^{\infty}\frac{(-c/4)^k}{(m)_k k!}z^k$$

then, let

$$2-U_p(z)=1-\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}z^{k-1}$$

which implies that

$$\int_0^z (2-u_p(t))dt = \int_0^z \left(1-\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}t^{k-1}\right)dt$$
$$= z-\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}k!}z^k.$$

To show that

$$\int_0^z (2-u_p(t))dt \in P_n^*(\lambda,\alpha,\beta,\mu,t)$$

is to show that by the condition of the class $P_n^*(\lambda,\alpha,\beta,\mu,t)$ and Theorem 3,

$$\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}}k[k^2\lambda+k(1-\lambda)-\alpha]M_k \le 1-\alpha.$$

Let

$$L(c,m,z)=\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}}k[k^2\lambda+k(1-\lambda)-\alpha]M_k$$

then it is sufficient to prove that $L(c,m,z)\le 1-\alpha$

$$L(c,m,z)=\sum_{k=2}^{\infty}k(k^2\lambda+k(1-\lambda)-\alpha)M_k\frac{(-c/4)^{k-1}}{(m)_{k-1}k!}$$

following the same steps for the proofs of Theorems 5 and 6, then

$$L(c,m,z)=\lambda H_p''(1)+(1+2\lambda)H_p'(1)+(1-\alpha)\left[H_p(1)-M_1\right] \tag{2.16}$$

Equation (2.16) is bounded above by $1-\alpha$ if

$$\lambda H_p''(1)+(1+2\lambda)H_p'(1)+(1-\alpha)\{H_p(1)-M_1\}\le 1-\alpha \tag{2.17}$$

where

$$H_p(z)=\sum_{k=0}^{\infty}\frac{(-c/4)^k}{(m)_k k!}M_{k+1}z^k.$$

□

Corollary 2.14. When $\lambda=0$, inequality (2.15) reduces to $H_p'(1)+(1-\alpha)\{H_p(1)-M_1\}\le 1-\alpha$

Remark 2.15. When $n=0$;

- $H_p(z)$ reduces to $U_p(z)$.
- Inequality (2.15) reduces to inequality (3.5) in [11].

Concluding Remark

The geometric properties of the newly defined classes $Q_n(\lambda,\alpha,\beta,\mu,t)$, $P_n(\lambda,\alpha,\beta,\mu,t)$ and $P_n^*(\lambda,\alpha,\beta,\mu,t)$ of analytic univalent functions were established. The result obtained are generalization of the existing results in literature specifically Cho et al in [10] and it also represents a step forward in understanding geometric function theory of complex analysis.

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