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Conceptual metaphors used by educators when teaching graph isomorphism

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Abstract

The isomorphism of graphs is one of the main themes of graph theory. This study aims to investigate the dialectical language used by teachers when teaching the concept of graph isomorphism. In the study, using the qualitative research method, instructional videos of 4 courses on teaching the concept of graph isomorphism were viewed by two educators and interviews were conducted with both educators after teaching the concept. After the content analysis, the conceptual metaphors used by the educators in the interview and class were identified and categorized into four main categories: Sameness, Mapping, Sameness/Mapping and Formal Definition. The research results showed that both educators paid more attention to sameness metaphors in the classroom setting and to the metaphors of mapping and formal definition metaphors in the interview setting.

Keywords: Graph theory, Isomorphism, Conceptual metaphors, Educator
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1 Introduction

Researchers have introduced graph isomorphism as one of the most important topics in graph theory [1]. Although some studies have been done on the applications and theorems of graph isomorphism, less importance has been given to educators' perceptions of the concept of graph isomorphism. Analyzing educators' perceptions of the concept of graph isomorphism can help identify students' learning routes.

Conceptual metaphors connect any phenomenon we want to understand to networks of concepts from another semantic realm, enabling us to reconceptualize it, see from different directions, and illuminate previously overlooked situations [7]. With metaphors, the outward reflection of the inner world of individuals and the effects of how they interpret events and phenomena can be seen [8]. The concept of metaphorical perception is the process of forming thoughts through metaphor [5]. Dickmeyer refers to it as "explaining a phenomenon in familiar terms" [10].

This paper focuses on the language used by two educators to discuss graph isomorphism through the lens of conceptual metaphors. Specifically, two research questions are addressed: (1) How did educators use metaphors to describe and teach isomorphism in interviews and instruction and (2) In what ways were educators' descriptions in interviews and instruction similar and different?

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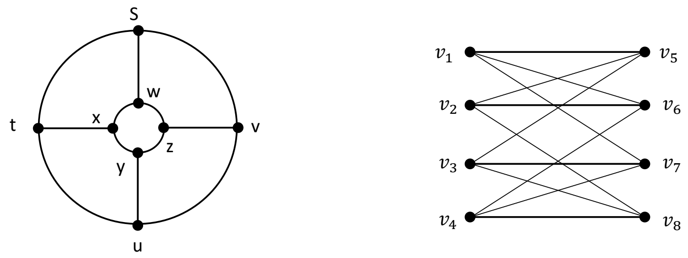

Figure 1: Two drawings of essentially the same graph.

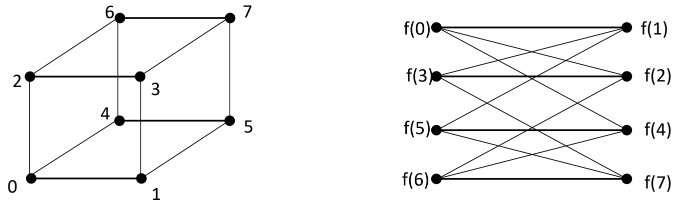

Figure 2: Specifying an isomorphism between two simple graphs.

2 Literature

In this section, we will first define the isomorphism of graphs. We will then discuss the students' perceptions of the concept of isomorphism in graphs, providing a basis for comparison with the teachers' understanding of graph isomorphism.

2.1 Definitions

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection (a one-to-one, onto map) φ from V_1 to V_2 such that for any

$$v, w \in V_1, \quad \{v, w\} \in E_1 \Leftrightarrow \{\varphi(v), \varphi(w)\} \in E_2.$$

In this case, we call φ an isomorphism from G_1 to G_2.

Example 2.1. The names of the vertices and edges of graphs G_1 and G_2 in Figure 1 differ, but these two graphs are strikingly similar.

Example 2.2. Figure 2 specifies an isomorphism between the two simple graphs shown. Checking that the given vertex bijection is structure-preserving is left to the reader.

Researchers have examined conceptions of group isomorphism more than graph isomorphism. For example, the study of Rapnow who categorized conceptual metaphors related to the isomorphism of groups [6].

The graph isomorphism problem (GI) is that of determining whether an isomorphism between two given graph? GI has long been a favourite target of algorithm designers, so much so that it was already described as a "disease" in 1976. Although the presented algorithms are perfectly accurate, their complexity is not so ideal and may become exponential in some special scenarios. In fact, most classical graph isomorphism algorithms have a high computational complexity. Therefore, researchers seek to explore an efficient graph isomorphism algorithm with polynomial time complexity. In contrast to previous studies, which focused more on finding specific isomorphism, Weber and Alcock and Weber asked undergraduate and doctoral students to prove theorems related to isomorphism and to prove or disprove specific groups were isomorphic [11]. While both doctoral and undergraduate students were able to prove simple propositions, doctoral students had continued success with more sophisticated propositions. Nardi noted that students' struggles in proving the isomorphism stemmed from three major sources: an inability to recall definitions or a lack of understanding of definitions, poor conceptions of mapping, and not recognizing the purpose of sections of the proof [4].

In general, researchers make occasional nods to isomorphism's importance, difficulty and applications, such as in Somkunwar's study identifying that graph isomorphism has various applications, such as image processing, protein structure, social networks and chemical structure [9]. From this set of research, it can be concluded that researchers have given less importance to students' perceptions of the concept of isomorphism of graphs.

Considering how to introduce the concept of graph isomorphism in class may either support or limit the way students utilize its features, it is necessary to study educators' perceptions of this concept. Furthermore, the conceptual frameworks utilized by educators when implementing the curriculum related to graph isomorphism have not been thoroughly examined to date.

3 Theoretical framework

A theoretical lens for analyzing mappings is the conceptual metaphor construct [6] Lakoff and Johnson posit that people's conceptual systems are metaphorical and that the metaphorical language individuals use can be examined as evidence of the structure of their metaphorical system. Conceptual metaphors reveal the structure of thought, indicating they are a suitable lens for studying the abstract concepts of isomorphism. The function literature also provides insight into several metaphors that students could potentially leverage if attending to the function portion of the graph isomorphism definition. Lakoff and Nunez's categories include function as a machine and function as a collection of objects with directional links [2]. Recently, Zandieh, Ellis, and Rasmussen categorized function metaphors used by linear algebra students including input/output, travelling, mapping, morphing and machine [12]. The commonality across these categories is "an entity 1, an entity2, and a description about how these two are connected". When considering graph isomorphism, this can be likened to transitioning from entity 1 (a vertex in G_1) to entity 2 (a vertex in G_2). We speculate that some of these metaphors could be applicable in the context of graph isomorphism.

Generally, metaphors are defined as a projection from a source domain into a target domain. For example, the familiar context of object collections (source) may serve as a metaphor for the mathematical context of arithmetic (target). Lokoff and Núñez distinguish between grounding metaphors, those metaphors that connect different mathematical domains. For the scope of this paper, educators' grounding metaphors for understanding graph isomorphism are the focus of this research [2].

Because conceptual understanding can be enhanced through metaphors both within and outside the realm of mathematics, one can consider an individual's ways of reasoning as a coherent system depicted through metaphors. When instructing, professors choose which material to emphasize, whether or not they intentionally use specific metaphors. Through instruction, students encounter various metaphors that help them organize their understanding of new concepts. Therefore, this perspective aligns well with the goals of this study.

4 Method

The data for this paper were largely drawn from classroom videos and interviews with two educators from a university in Ardabil. These faculty members were teaching an introductory graph theory class. Classroom data were collected when isomorphism was discussed in class. For Instructor Ali's course, this included portions of three 60-minute class periods, and for Instructor Reza, this included portions of three 45-minute class periods. (Both names are pseudonyms.) We only completely transcribed class segments focused on isomorphism.

In Instructor A's class, students work in groups to complete assignments that are given to them as a team. Regarding graph isomorphism, the students' work began with checking the preservation of adjacencies and non-adjacencies in the specified tasks. Continuing with the lesson, Instructor A explored various representations of the concept of graph isomorphism. The assignments given to the students involved evaluating the connections between these different representations.

Instructor R used a combination of lecture and activity methods. During lecture days, information was mainly presented by the instructor, with occasional questions posed to the entire class. However, without considering various representations of the concept of graph isomorphism, Teacher R began teaching with the definition of isomorphism. He then dedicated his teaching to determining the number of isomorphism graphs under various conditions. Unlike Teacher A's approach, he did not consider the isomorphism of non-simple, directed, or disconnected graphs.

For context, both educators had taught the course at least once before. educators were recruited at the beginning of the semester from that semester's graph theory teachers. Participants engaged in semi-structured interviews lasting roughly half an hour each. The relevant interview with each instructor occurred as they began teaching isomorphism and focused on definitions and descriptions of isomorphism, as well as their explanations when teaching. Interviews were audio and video recorded and any written work was collected. The interview questions are included in the Appendix. The interviews and videos were transcribed and coded in alignment with the phases of thematic analysis.

The method of content analysis was chosen because the purpose of this study was to examine patterns of language use to determine conceptual metaphors. This method also allows for previous research to be taken into consideration when conducting the analysis.

5 Results and Discussion

After reviewing and analyzing the content, twelve metaphors were extracted for the isomorphism graph. These metaphors are introduced in Table 1.

Table 1: Metaphors with Defining

Metaphor Category	Metaphor Code	Metaphor Definition
Sameness	Generic sameness	Generic references to graphs being the same or similar, whether at the whole graph level or as general statements about relationships between vertices.
	Same properties	Use of properties that are same for all isomorphic graphs. (e.g., cardinality, same degree sequence, Same edge connectivity).
	Disembedding	Structure – focused language to high light (Sub) structures for special inspection by the existence of an isomorphism.
Sameness/mapping	Renaming/Relabeling	Giving new names or labels to vertices to show equivalence between graphs
	Matching	Connecting specific vertices in two graphs or lining up vertices in order to create a specific correspondence that reveals sameness of the paired vertices
Mapping	Generic mapping	Generic reference to an isomorphism as a function or mapping without further details about the mapping or explicit reliance on properties of functions
	Function	Specific use of a function property
	journey	traveling from a starting point to an ending point
	machine	Connections to how a machine works, (e.g. takes inputs and produces outputs).
Formal definition	Literal formal definition	Use of the string of symbols in the formal definition for isomorphism or use of words related to bijective, onto, or one-to-one (or a mapping lacking those properties) to talk about graph isomorphism.
	Structure-preserving	Use of “structure-preserving” or a slight variation without interpretation.
	Operation-preserving	Use of “operation-preserving” or a slight variation without interpretation or use of a specific operation while talking about Preserving (preserving addition).

The results are presented in two main sections. The research findings are presented in two main parts. The first part analyzes the way instructor R uses metaphors to explain the concept of graph isomorphism and to address the research questions. The second part investigates the use of conceptual metaphors by instructor A to answer the raised research questions. The frequency of codes in the interviews is presented because both coaches were asked the same questions in the interviews. In class, instead of focusing on abundance, the presence or absence of metaphor is noted. This is because the time spent on graph isomorphism in the two classes was not the same. Therefore, metaphors used in interviews and classes are not directly comparable. Additionally, since the class videos were selectively recorded and table discussions were not always audible, providing a frequency for the use of metaphors in class can be misleading.

5.1 Metaphors in instructor R’s class

In this section, the language used by instructor R was examined separately in both the interview and the class. During the interview, instructor R frequently used general sameness language. R’s initial description of graph isomorphism was: “When I think of two isomorphic graphs, it means that they are the same graph yet labelled with different names or marked with different operations, but the graphs are essentially the same.” Instructor R further explains the concept of general sameness by discussing the idea of finding a renaming function to represent it. He states, “I confirm the isomorphism between two graphs by finding a renaming function and asserting that the two graphs are identical.” When asked if he thinks about graph isomorphism in the same way he describes it to students, Instructor R expressed his thoughts on the matter as follows: “I want students to understand the formal definition, but I also want them to see it as a renaming function. In addition to the standard explanations I provide for isomorphism, I relate it to a name change.”

When instructor R was asked how to explain the concept of isomorphism to a child, they responded by saying, “I would explain it by using the example of a doll. If I give a doll different names, it is still the same doll, only the name has changed.” By comparing this metaphor to the standard language used by R’s instructor to explain isomorphism, we can see that this choice once again emphasizes the same metaphor, specifically highlighting the idea of changing the name as a way to understand isomorphism.

In the classroom, instructor R used the metaphor of “sameness” as a window to introduce the concept of isomorphism. This metaphor was consistently used when defining isomorphism and verifying the graphs. The idea of mapping was prevalent throughout the discussion, particularly when exploring the nature of isomorphism functions. While the sameness/mapping metaphor was initially emphasized and continued throughout the assignments, it was gradually phased out after the formal definition was established. The formal definition metaphor was predominantly employed in proof fields, while general sameness was applied in more general cases, such as when introducing the official definition.

Table 2: Metaphors in instructor R’s class

Metaphor Category	Metaphor Code	Frequency in Interview
Sameness	Generic sameness	10
	Same properties	2
Sameness/mapping	Disembedding	
	Renaming/Relabeling	5
	Matching	
Mapping	Generic mapping	1
	Function	1
	Journey	
	Machine	1
Formal definition	Literal formal definition	1
	Structure-preserving	3
	Operation-preserving	

Similar properties were often utilized when the graphs provided in the assignments were not isomorphic. The renaming language was employed when the initial formal definition was presented colloquially to aid in understanding the concept. Before the formal definitions of matching and correspondence were introduced, this language was often used to debate the isomorphism of graphs. The general mapping language was commonly used to represent the concept of isomorphism in a formulaic manner. The instructor spent a lot of time discussing the well-defined properties of functions, such as domain, one-to-one, and surjective functions, as well as Venn diagrams. Before delving into graph isomorphism, it is important to understand the requirements for a function. The language of travel was frequently used to describe the movement of vertices or the entire graph in a particular direction. Machine language refers to the input and output presented in functions. The formal definition language was used to explain why isomorphism should be one-to-one and onto. The operation preservation language was frequently used to summarize the properties of isomorphic graphs. A student in the class introduced the language of structure preservation. At this point, instructor R pointed out that the structural differences indicated that the graphs were not isomorphic. In summary, there was a clear alignment at the conceptual level when discussing graph isomorphism. As demonstrated in Table 2, during the interview and presentation, Instructor R emphasized the general sameness and the metaphor of sameness/mapping and renaming in his understanding of graph isomorphism. In class, the instructor once again utilized the sameness category and sameness/mapping metaphors to explain the concept of isomorphism. However, it is important to note that the matching and correspondence metaphor, rather than renaming, was more effectively conveyed in the classroom through assignments. The greater variety of mapping metaphors in the classroom, as opposed to the interview, can be largely attributed to differences in interview questions and training goals. During the interview, instructor R was asked to define and describe an isomorphism, rather than identify specific isomorphism between graphs or prove theorems. Instead of focusing on specific shapes, more emphasis was placed on developing a general understanding of the concept of isomorphism.

The formal definition of metaphor was discussed in both the interview and the classroom. In the classroom, students were encouraged to consider graph isomorphism in terms of sameness. The interview briefly touched on the official definition but focused more on exploring its implications for sameness.

5.2 Metaphors in instructor A's class

Like teacher Reza, many examples of the same language were used in both the interview and the classroom environment. However, there was more variety in the class. Despite the greater diversity in the class, the conceptual emphasis in both contexts relates to sameness and sameness/mapping.

Table 3: Metaphors in instructor A's class

Metaphor Category	Metaphor Code	Frequency in Interview
Sameness	Generic sameness	7
	Same properties	
	Disembedding	1
Sameness/mapping	Renaming/Relabeling	6
	Matching	
Mapping	Generic mapping	3
	Function	2
	Journey	1
	Machine	
Formal definition	Literal formal definition	1
	Structure-preserving	2
	Operation-preserving	1

In the interview setting, Instructor A utilized metaphorical categories such as formal definition, sameness, and mapping to describe the concept of graph isomorphism. When asked about the initial words or phrases that come to mind when discussing isomorphism, Instructor A responded with "Structure-preserving mapping, structure equivalence, and vertices relabeling." These phrases represent the metaphors of preserving structure, general sameness, and relabeling, respectively. Without further explanation, Instructor A focused on maintaining structure. "I define an isomorphism as a mapping between two graphs that preserves structure."

Matching was brought when considering how to describe the concept of isomorphism to a child. When asked about their thoughts on graph isomorphism, they used metaphors of sameness and relabeling, along with metaphors of formal definition and conservation of operations. In the classroom setting, Instructor A frequently used the general sameness metaphor to describe graphs that are "essentially the same," particularly when introducing this concept. The use of the same properties was evident during the tasks. Labelling was employed to identify isomorphism when students were searching for common features that two graphs must possess to be considered isomorphic. Mapping was used to offer students quick strategies while working on assignments. Matching was utilized multiple times, as students were tasked with identifying which vertices corresponded to vertices from another graph when creating a map. General mapping was employed when students wanted to demonstrate a rule for isomorphism. However, the main focus of the class was not to present the concept of isomorphism on the function. Instead, the journey metaphor was frequently used, illustrating the process of moving from one graph vertex to its corresponding vertex in another graph. The literal formal definition metaphor was also employed to demonstrate one-to-one and onto mappings. Structure-preserving metaphor was introduced before providing a formal definition. In summary, there was a relatively clear conceptual alignment between the interview and classroom settings for Instructor Ali. The greater diversity of the mapping metaphor in the classroom compared to the interviews is probably due to the difference in the types of questions asked and educational goals. Similar to Instructor Reza, the difference between metaphors in the interview and the class can be explained by the overarching conceptual questions asked in the interview versus the types of activities in the class. A formal definition of isomorphism was provided in both classroom and interview settings. However, the main emphasis in both settings was not on the formal definition itself. The approach to understanding isomorphism in the classroom still revolved around the metaphor of sameness. During the interview, the formal definition was briefly mentioned but more time was dedicated to delving into its meaning, particularly concerning isomorphism. The language of structure preservation used in the interviews appeared to be connected to the same perspectives discussed in both the interviews and the classroom. In this research, conceptual metaphors used by educators in interviews and classrooms were identified and classified into four main categories: formal definition, mapping, sameness, and a combination of mapping and sameness. The findings of the research showed that both educators paid more attention to the same metaphors in the classroom environment and metaphors of mapping and formal definition in the interview environment.

6 Conclusion

In this study, conducted to investigate the conceptual metaphors used by educators when teaching graph isomorphism, the metaphors of sameness, mapping, sameness/mapping, and formal definition were identified. First, information obtained from interviews and classroom observations was analyzed. Then, to address the second research question, the differences and similarities between the metaphors used in interviews and classroom environments were explored. Upon revisiting the research questions, it was found that both educators primarily used the same ideas to discuss isomorphism in both the interviews and teaching environments. These included calling isomorphic graphs "essentially the same" and using renaming and labelling to discuss how an isomorphism function represents sameness. In the classroom environment, Instructor R mostly used matching and correspondence, which was a less abstract version of name change. This does not appear to be a deliberate change but could have been influenced by the emphasis on correspondence in the interview setting. Both educators used the formal definition metaphor in a secondary capacity. Both mainly used the mapping metaphor when asked to define an isomorphism. However, they did not seem to consider this language as the main conceptual point of isomorphism. This may be because they believed that the structures themselves were more important than the mappings that connected them. Upon examining the research questions, the differences between the interview environment and the classroom for teaching isomorphism became more apparent. Both educators began teaching with a literal formal definition and eventually developed a more complex understanding of the concept. In the interview, instructor R presented a detailed view of isomorphism through equivalence classes, but this concept was only discussed in the last session on isomorphism. The similarities and differences between the findings of the interview and training indicate a variety of ways to understand the concept of graph isomorphism. It also highlights the importance of researching the content knowledge of educators in multiple fields. The sameness metaphor was originally highlighted by [9]. The general similarity metaphor highlighted by Weber and Alcock occurred repeatedly here as well [3]. However, in this study, the main focus of the trainers was to establish a correspondence between the vertices of one graph and its corresponding vertices in another graph. Additionally, students' utilization of common features in isomorphic graphs when completing homework has influenced the classroom explanations provided by educators. Educators utilized various concepts of sameness to explore isomorphism. Their perspectives on graph isomorphism, which included notions of general similarity and renaming or relabeling, are closely aligned with those outlined by Weber and Alcock. Furthermore, this study prompts inquiries into how students interpret their educators' teachings. Subsequent research could delve into the perspectives of both educators and students on isomorphism across various contexts, such as rings or modules, and examine the parallels and distinctions in their viewpoints within different settings.

Acknowledgment

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Appendix: Questions asked during the interview with the educators:

1. When you hear the word "isomorphism," what words or expressions come to your mind?
2. How would you define isomorphism?
3. How would you describe "isomorphism" to a 10-year-old child?
4. In what ways are your personal thoughts on isomorphism similar or different from the ways you explain them to students?

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