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Fixed point results of Perov type contractive mappings in generalized F-metric spaces

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Abstract

The purpose of this paper is to present some fixed point results in spaces endowed with a vector-valued $\mathcal{F}$-metric. The results are extensions or generalizations of results proved by Perov [24]. To show the usability of our results, we present two examples.

Keywords: Fixed point, Vector-valued $\mathcal{F}$-metric, Weakly compatible
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1 Introduction

Fixed point theory play a vital role for solving problems in various branches of mathematical, such as nonlinear analysis, integral and differential equations (see [5, 19, 29, 31]). There exist many interesting generalizations of metric spaces (see for example [2, 6, 7, 15, 23, 27]). In particular, Jleli et al. [17] introduced and studied the concept of $\mathcal{F}$-metric space as a generalization of metric space and presented the contraction mapping in $\mathcal{F}$-metric spaces that is generalization of the Banach contraction principle in metric spaces. (see e.g. [4, 9, 10, 11, 22] and references therein). In this work, we introduce the concept of vector-valued $\mathcal{F}$-metrics and establish some of fixed point theorems. The theory is illustrated with some examples.

2 Preliminaries

Let us recall [17] that $\mathcal{F}$ be the family of all functions $f:(0,+\infty)\to\mathbb{R}$ satisfying the following conditions:
$\mathcal{F}_1$) f is non-decreasing, that is, $0<s<t$ implies $f(s)\leq f(t)$.
$\mathcal{F}_2$) For every sequence $\{\alpha_n\}\subset(0,+\infty)$, we have

$$\lim_{n\to+\infty}\alpha_n=0 \text{ if and only if } \lim_{n\to+\infty}f(\alpha_n)=-\infty.$$

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Definition 2.1. [17] Let X be a (nonempty) set. A function $D : X \times X \to [0, +\infty)$ is called a $\mathcal{F}$-metric on X if there exists $(f, L) \in \mathcal{F} \times [0, +\infty)$ such that for all $x, y \in X$ the following conditions hold:

(D_1) $D(x, y) = 0$ if and only if $x = y$.

(D_2) $D(x, y) = D(y, x)$.

(D_3) For every $N \in \mathbb{N}, N \geq 2$ and for every $(u_i)_i^N \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$D(x, y) > 0 \text{ implies } f(D(x, y)) \leq f(\sum_{i=1}^{N-1} D(u_i, u_{i+1})) + L.$$

In this case, D is called an $\mathcal{F}$-metric on X and the pair (X, D) is called an $\mathcal{F}$-metric space.

Example 2.2. [17] Let $X = \mathbb{R}$ and $D : X \times X \to [0, +\infty)$ be defined as follows:

$$D(x, y) = \begin{cases} (x - y)^2 & (x, y) \in [0, 3] \times [0, 3], \\ |x - y| & \text{otherwise}, \end{cases}$$

and let $f(t) = ln(t)$ for all $t > 0$ and $L = ln(3)$. Then, D is a $\mathcal{F}$-metric on X. Since $D(1, 3) = 4 \geq D(1, 2) + D(2, 3) = 2$, Then D is not a metric on X.

Definition 2.3. [17] Let (X, D) be an $\mathcal{F}$-metric space and $\{y_n\}$ be a sequence in X.

1) A sequence $\{x_n\}$ is called $\mathcal{F}$-convergent to $x \in X$, if $\lim_{n \to +\infty} D(x_n, x) = 0$.

2) A sequence $\{x_n\}$ is called $\mathcal{F}$-Cauchy, if $\lim_{n,m \to +\infty} D(x_n, x_m) = 0$.

3) A $\mathcal{F}$-metric space (X, D) is called $\mathcal{F}$-complete, if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to some element in X.

Jleli et al. in [17] proved the following fixed point theorem.

Theorem 2.4. [17] Let (X, D) be $\mathcal{F}$-complete $\mathcal{F}$-metric space and let $T : X \to X$ be a self-mapping satisfying

$$D(Tx, Ty) \leq \alpha D(x, y),$$

for all $x, y \in X$ where $0 \leq \alpha < 1$. Then T has a unique fixed point.

Definition 2.5. [1] Let T and S be self maps of a set X. Two self-mappings T and S are said to be weakly compatible if they commute at their coincidence points; i.e., if $T(x) = S(x)$ for some $x \in X$, then $T(S(x)) = S(T(x))$.

Let T and S be weakly compatible self maps of a set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S [1].

The classical Banach contraction principle was extended for contraction mappings on spaces endowed with vector-valued metrics by Perov (see e.g. [12, 13, 14, 16, 18, 20, 21, 24, 25, 26, 28, 30] and references therein).

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

1) $d(x, y) \succ \theta$ for all $x, y \in X$; $d(x, y) = \theta$ if and only if $x = y$;

2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A nonempty set X endowed with a vector-valued metric d is called a generalized metric space and it will be denoted by (X, d).

Remark 2.6. If $\alpha, \beta \in \mathbb{R}^m$ with $\alpha = (\alpha_1, ..., \alpha_m), \beta = (\beta_1, ..., \beta_m)$, then by $\alpha \preceq \beta$ (respectively $\alpha \prec \beta$), we mean that $\alpha_i \leq \beta_i$ (respectively $\alpha_i < \beta_i$), for all $i = 1, ..., m$.

Throughout this paper we denote by $M_{m\times m}(R_+)$ the set of all $m \times m$ matrices with positive elements, by 0 the zero $m \times m$ matrix and by I the identity $m \times m$ matrix. A matrix $A \in M_{m\times m}(\mathbb{R}_+)$ is said to be matrix convergent to zero if $A^n \to 0$ as $n \to +\infty$.

Theorem 2.7. [12] Let $A \in M_{m\times m}(R_+)$. Then following assertions are equivalent

1. A is convergent towards zero.

2. $A^n \to 0$ as $n \to +\infty$.

3.The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^n + \cdots .$$

4.The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements.

5. $A^n q \to 0$ and $qA^n \to 0$ as $n \to +\infty$, for each $q \in \mathbb{R}^m$. Where q in the first case is the column of matrices of type $m \times 1$ and in the second case q is the type of matrices of type $1 \times m$.

Example 2.8. Some examples of matrix convergent to zero are

1. Any matrix $A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$.

2. Any matrix $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$.

2. $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}$.

Perov [24] proved the following generalization of Banach contraction principle.

Theorem 2.9. Let (X, d) be a complete generalized metric space and $T : X \to X$ be an self-mapping. Suppose there exists a matrix $A \in M_{m\times m}(\mathbb{R}_+)$ convergent to zero such that

$$d(T(x), T(y)) \preceq Ad(x, y).$$

Then the following statements hold:

1. T has a unique fixed point x^*.

2. The Picard iterative sequence $x_n = T^n(x_0)$, $n \in \mathbb{N}$ converges to x^* for all $x_0 \in X$.

3. $d(x_n, x^*) \preceq A^n(I - A)^{-1}d(x_0, x_1)$, for all $n \in \mathbb{N}$, where $A \in M_{m\times m}(R_+)$ is a matrix convergent to zero.

3 Main results

In this section of the paper, by considering the technique of Jleli et al. [17], we introduce a generalization of $\mathcal{F}$-metric.

Definition 3.1. Let $f : \mathbb{R}^m_+ \to \mathbb{R}^m$ be a function which satisfies the following conditions:

$(F1)$ f is strictly increasing; for all $a = (a_i)_{i=1}^m, b = (b_i)_{i=1}^m \in \mathbb{R}^m_+$, where

$$a \prec b \text{ implies } f(a) \preceq f(b),$$

$(F2)$ For each sequence $\{a_n\} = (a_1^{(n)}, a_2^{(n)}, ..., a_m^{(n)})$ of $\mathbb{R}^m_+$, we have

$$\lim_{n\to+\infty} a_i^{(n)} = 0 \text{ if and only if } \lim_{n\to+\infty} b_i^{(n)}, = -\infty,$$

for every $i = 1, 2, ..., m$, where $f(a_1^{(n)}, a_2^{(n)}, ..., a_m^{(n)}) = (b_1^{(n)}, b_2^{(n)}, ..., b_m^{(n)})$. Here, $\mathbb{R}^m_+$ is the set of all $m \times 1$ real matrix with positive elements. The set of all functions f satisfying $(F1 - F2)$ is denoted as $\mathcal{F}^m$.

Remark 3.2. Let $f_i \in \mathcal{F}$, for all $i = 1, 2, ..., m$. Define $f : \mathbb{R}^m_+ \to \mathbb{R}^m$, where $f(a_1, ..., a_m) = (f_1 a_1, f_2 a_2, ..., f_m a_m)$ for each $(a_1, ..., a_m) \in \mathbb{R}^m_+$. Then $f \in \mathcal{F}^m$.

Definition 3.3. Let X be a nonempety set and $D : X \times X \to R^m$ be a mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F}^m \times R^m_+$ such that:

(D1) For all $(x, y) \in X \times X, D(x, y) \succ \theta$ and $D(x, y) = \theta$ if and only if $x = y$ (where $\theta = (0, ..., 0) \in \mathbb{R}^m$).

(D2) For all $(x, y) \in X \times X$, $D(x, y) = D(y, x)$.

(D3) For every $(x, y) \in X \times X$, for every $N \in \mathbb{N}, N \geq 2$, and for every $(u_i)_{i=1}^N \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$D(x, y) \succ 0 \text{ implies } f(D(x, y)) \preceq f(\sum_{i=1}^{N-1} D(u_i, u_{i+1})) + \alpha.$$

Then D is called an vector valued F- metric on X and the pair (X, D) is called an generalized F-metric space. Also, if $m = 1$, we obtain F-metric.

Example 3.4. The following functions $f : R^3_+ \to \mathbb{R}^3$ are the elements of $\mathcal{F}^3$:

(1) $f((\alpha_1, \alpha_2, \alpha_3)) = (ln\, \alpha_1, ln\, \alpha_2, ln\, \alpha_3)$,

(2) $f((\alpha_1, \alpha_2, \alpha_3)) = (ln\, \alpha_1 + \alpha_1, ln\, \alpha_2 + \alpha_2, ln\, \alpha_3 + \alpha_3)$,

(3) $f((\alpha_1, \alpha_2, \alpha_3)) = (ln\, \alpha_1, \frac{-1}{\sqrt{\alpha_2}}, ln\, \alpha_3)$

Theorem 3.5. Let (X, D) be an F-complete generalized F-metric apace and $T : X \to X$ be an self mapping on X such that

$$D(Tx, Ty) \preceq AD(x, y) + BD(y, Tx) + CD(x, Tx), \tag{3.1}$$

for all $x, y \in X$, where $A, B, C \in M_{m\times m}(R_+)$ and $A + C$ converges to zero. Then T has at least one fixed point in X. If additionally, the matrix $A + B$ converges to zero, then T has a unique fixed point in X.

Proof . Let x_0 be an arbitrary point in X. We can define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}\cup\{0\}$. From (3.1), we have

$$\begin{aligned} D(x_n, x_{n+1}) &= D(Tx_{n-1}, Tx_n) \\ &\preceq AD(x_{n-1}, x_n) + BD(x_n, Tx_{n-1}) + CD(x_{n-1}, Tx_{n-1}) \\ &= (A + C)D(x_{n-1}, x_n), \end{aligned}$$

for all $n \in \mathbb{N}$. which yields

$$D(x_n, x_{n+1}) \preceq (A + C)D(x_{n-1}, x_n) \preceq (A + C)^2 D(x_{n-2}, x_{n-1}) \preceq ... \preceq (A + C)^n D(x_0, x_1), n \in \mathbb{N}. \tag{3.2}$$

Using (3.2), we can write

$$\begin{aligned} \sum_{k=n}^{m-1} D(x_k, x_{k+1}) &\preceq (A + C)^n (I + (A + C) + (A + C)^2 + ... + (A + C)^{m-n-1}) D(x_0, x_1) \\ &\preceq (A + C)^n (I - (A + C))^{-1} D(x_0, x_1), m > n. \end{aligned}$$

Since $\lim_{n\to+\infty}(A + C)^n (I - (A + C))^{-1} D(x_0, x_1) = \theta$, for any $\delta > 0$ there exists some $n' \in \mathbb{N}$ such that

$$\theta \prec (A + C)^n (I - (A + C))^{-1} D(x_0, x_1) \prec \delta, \;\; n \geq n'. \tag{3.3}$$

Furthermore, let $\varepsilon \succ \theta$ be fixed. Since $(f, \alpha) \in \mathcal{F}^m \times \mathbb{R}^m_+$ satisfies (D3), by $(F2)$ it follows that there is some $\delta \succ \theta$ such that

$$\theta \prec t \prec \delta \text{ implies } f(t) \prec f(\varepsilon) - \alpha. \tag{3.4}$$

By (3.3) and (3.4), we get

$$f(\sum_{k=n}^{m-1} D(x_k, x_{k+1})) \leq f((A+C)^n(I-(A+C))^{-1}D(x_0, x_1)) < f(\varepsilon) - \alpha, m > n \geq n'.$$

By (D3) and the above inequality, we obtain

$$D(x_n, x_m) \succ \theta,\ m > n > n' \text{ implies } f(D(x_n, x_m)) \prec f(\varepsilon).$$

This shows

$$D(x_n, x_m) \prec \varepsilon, m > n \geq n'.$$

Hence, we showed that (x_n) is an $\mathcal{F}$-Cauchy sequence in X. Since X is $\mathcal{F}$-complete, there exists $x^* \in X$ such that (x_n) is $\mathcal{F}$-convergent to x^*, i.e, $\lim_{n\to+\infty} D(x_n, x^*) = \theta$. We shall prove that x^* is a fixed point of T. Suppose $D(Tx^*, x^*) \succ \theta$. From (D_3) for all $n \in \mathbb{N}$, we have,

$$f(D(x^*, Tx^*)) \preceq f(D(x^*, Tx_n) + D(Tx_n, Tx^*)) + \alpha.$$

Using (3.1) and $\mathcal{F}_1$, we obtain

$$f(D(x^*, Tx^*)) \preceq f(D(x^*, x_{n+1}) + AD(x_n, x^*) + BD(x^*, Tx_n) + CD(x_n, Tx_n)) + \alpha.$$

Since $\lim_{n\to+\infty}(D(x^*, x_{n+1}) + AD(x_n, x^*) + BD(x^*, Tx_n) + CD(x_n, Tx_n)) = 0$, from $\mathcal{F}_2$, we have

$$\lim_{n\to+\infty} f(D(x^*, x_{n+1}) + AD(x_n, x^*) + BD(x^*, Tx_n) + CD(x_n, Tx_n)) + \alpha = -\infty,$$

which is a contradiction. Therefore, we have $D(Tx^*, x^*) = \theta$, i.e. $Tx^* = x^*$. Finally, we shall show that the fixed point is unique. To this end, we assume that there exists another fixed point z^* and $D(x^*, z^*) \succ \theta$. From (3.1), we have

$$\begin{aligned} D(x^*, z^*) &= D(Tx^*, Tz^*) \\ &\preceq AD(x^*, z^*) + BD(z^*, Tx^*) + CD(x^*, Tx^*) \\ &= (A+B)D(x^*, z^*). \end{aligned}$$

Thus $(I - A - B)D(x^*, z^*) \preceq \theta$. Since $A + B$ converges to zero, we get that $I - A - B$ is non-singular and $(I - A - B)^{-1} \in M_{mm}(\mathbb{R}_+)$. Hence $D(x^*, z^*) \preceq \theta$ which is a contradiction and hence $x^* = z^*$. □

Corollary 3.6. Putting $B = C = 0$, Theorem 3.5 reduces to Perov Theorem 2.9.

In order to support Theorem 3.5, we present the following example:

Example 3.7. Let $X = [0, 1] \times [0, 1]$ and $D : X \times X \to \mathbb{R}^2$ be defined as follows:

$$D(x, y) = D((x_1, x_2), (y_1, y_2)) = \begin{cases} (e^{|x_1-y_1|}, e^{|x_2-y_2|}) & \text{if } (x_1, x_2) \neq (y_1, y_2) \\ 0 & \text{if } (x_1, x_2) = (y_1, y_2), \end{cases}$$

for each $x, y \in X$. Take $f(t_1, t_2) = (-\frac{1}{t_1}, -\frac{1}{t_2}), t_1, t_2 > 0$ and $\alpha = (1, 1)$. Then (X, D) is an generalized F-metric space. Define $T : X \to X$ by $T((x_1, x_2)) = (\frac{1}{2}(x_1 + 1), \frac{x_2}{3})$. Suppose that $A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$, $B = 0$ and $C = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$. Thus, Theorem 3.5 implies that T has a unique fixed point in X. Note that $(1, 0)$ is fixed point of T.

Theorem 3.8. Let (X, D) be an F-complete generalized F-metric apace and let the mappings $T, S : X \to X$ be self-mappings on X which satisfy,

$$D(Sx, Sy) \preceq AD(Tx, Ty), \tag{3.5}$$

for all $x, y \in X$, where $A \in M_{mm}(R_+)$ be a nonzero matrix convergent to zero. Let T and S be weakly compatible, $S(X) \subseteq T(X)$ and $T(X)$ is an $\mathcal{F}$-complete subset of X. Then T and S have a unique common fixed point in X.

Proof . We choose elements $x_0, x_1 \in X$ such that $T(x_1) = S(x_0)$. Since $S(X) \subseteq T(X)$, we can define a sequence $\{x_n\}$ such that $T(x_n) = S(x_{n-1})$ for each $n \in \mathbb{N} \cup \{0\}$. From (3.5), we have

$$\begin{aligned} D(T(x_{n+1}), T(x_n)) &= D(S(x_n), S(x_{n-1})) \\ &\preceq AD(T(x_n), T(x_{n-1})), \end{aligned}$$

for all $n \in \mathbb{N}$. Inductively, we get

$$D(T(x_{n+1}), T(x_n)) \preceq A^n D(T(x_1), T(x_0)), \tag{3.6}$$

for all $n \in \mathbb{N}$. Using (3.6), we can write

$$\begin{aligned} \sum_{k=n}^{m-1} D(T(x_{k+1}, x_{k)}) &\preceq A^n(I + A + A^2 + ... + A^{m-n-1})D(T(x_1), T(x_0)) \\ &\preceq A^n(I-A)^{-1}D(T(x_1), T(x_0)),\ m > n. \end{aligned}$$

Since $\lim_{n\to+\infty} A^n(I-A)^{-1}D(T(x_1), T(x_0)) = \theta$, for any $\delta > 0$ there exists some $n' \in \mathbb{N}$ such that

$$\theta \prec A^n(I-A)^{-1}D(T(x_1), T(x_0)) \prec \delta,\ n \geq n' \tag{3.7}$$

Furthermore, let $\varepsilon \succ \theta$ be fixed. Since $(f, \alpha) \in \mathcal{F}^m \times \mathbb{R}_+^m$ satisfies (D3), by F2) it follows that there is some $\delta \succ \theta$ such that

$$\theta \prec t \prec \delta \text{ implies } f(t) \prec f(\varepsilon) - \alpha. \tag{3.8}$$

By (3.7) and (3.8), we write

$$f(\sum_{k=n}^{m-1} D(T(x_{k+1}, x_{k)})) \preceq f(A^n(I-A)^{-1}D(T(x_1), T(x_0)) \prec f(\varepsilon) - \alpha, m > n \geq n'.$$

By (D3) and the above inequality, we obtain

$$D(T(x_n), T(x_m)) \succ \theta,\ m > n > n' \text{ implies } f(D(T(x_n), T(x_m))) \prec f(\varepsilon).$$

This shows

$$D(T(x_n), T(x_m)) \prec \varepsilon,\ m > n \geq n'.$$

Hence, $\{T(x_n)\}$ is an $\mathcal{F}$-Cauchy sequence in $T(X)$. Since $T(X)$ is $\mathcal{F}$-complete, there exists $x^* \in X$ such that $\lim_{n\to+\infty} D(T(x_n), T(x^*)) = \theta$ and $z = T(x^*)$. We show that $S(x^*) = z$. Using (D3) and (3.5), we obtain

$$\begin{aligned} f(D(T(x^*), S(x^*))) &= f((D(T(x^*), T(x_n)) + D(T(x_n), S(x^*)) + \alpha \\ &\preceq f((D(T(x^*), T(x_n)) + D(S(x_{n-1}), S(x^*)) + \alpha \\ &\preceq f((D(T(x^*), T(x_n)) + AD(T(x_{n-1}), T(x^*))) + \alpha \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to+\infty}(D(T(x^*), T(x_n)) + AD(T(x_{n-1}), T(x^*))) = \theta$, from $\mathcal{F}_2$, we have

$$\lim_{n\to+\infty} f(D(T(x^*), T(x_n)) + AD(T(x_{n-1}), T(x^*))) + \alpha = -\infty.$$

This is a contraction, unless $D(T(x^*), S(x^*)) = \theta$, i.e $T(x^*) = S(x^*) = z$ and x^* is a coincidence point and z is a point of coincidence of T and S. Now we show that T and S have a unique point of coincidence. For this, assume that there exists another point q in X such that $z_1 = Tz^* = Sz^*$. Suppose, to the contrary, $D(z, z_1) \succ \theta$. Using (3.5), we have

$$\begin{aligned} D(z, z_1) &= D(S(x^*), S(z^*)) \\ &\preceq AD(T(x^*), T(z^*)) \\ &= AD(z, z_1). \end{aligned}$$

Thus $(I-A)D(z,z_1) \preceq \theta$. Since A converges to zero, we get that $I-A$ is non-singular and $(I-A)^{-1} \in M_{mm}(\mathbb{R}_+)$. Hence $D(z,z_1) \preceq \theta$, which is a contradiction and hence $D(z,z_1) = \theta$ and we get that $z = z_1$. Therefore, z is the unique point of coincidence of T and S. Now, if T and S are weakly compatible then by Proposition 1, T and S have a unique common fixed point. $\square$

Corollary 3.9. In the case that $T = I_X$ the identity mapping on X, we obtain Perov Theorem 2.9.

Example 3.10. Let $X = [0,1] \times [2,3]$ and vector valued F- metric $D : X \times X \to \mathbb{R}^2$ be defined as Example 3.7. Define $T : X \to X$ by $T((x_1,x_2)) = (\frac{3x_1-1}{2}, 2x_2-2)$ and $S((x_1,x_2)) = (\frac{x_1+1}{2}, x_2)$. Suppose that $A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$. Thus, Theorem 3.8 implies that T and S have a unique common fixed point in X. Note that $(1,2)$ is common fixed point of T and S.

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