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HERMITIAN METRIC ON QUANTUM SPHERES

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ABSTRACT. The paper deal with non-commutative geometry. The notion of quantum spheres was introduced by podles. Here we define the quantum hermitian metric on the quantum spaces and find it for the quantum spheres.

1. Introduction

The basic idea of algebraic geometry is the familiar correspondence between geometric spaces and commutative algebras. The extension of this correspondence to the non-commutative case leads to the non-commutative geometry. General tools of functional analysis show that C^*-algebras constitute a natural framework for the non-commutative geometry. Quantum spaces are generalizations of manifolds. They are identified by the C^*-algebra of functions on them. On the other hand, quantum groups (pseudogroups, twisted groups) are quantum spaces endowed with a group structure [4]. Twisted $SU(2)$ groups $(S_\mu U(2), \mu \in (-1,1)\backslash\{0\})$ were introduced in [4]. They act on quantum spaces $S^2_{\mu c}$ $(c \in [0,1])$ in a similar way to $SU(2)$ acting on two-dimensional sphere S^2. Differential calculus on compact matrix pseudogroups is presented in [6] by Woronowicz. Podles in [1] introduced the family of quantum spheres $S^2_{\mu c}$ by their corresponding C^*-algebras. Later, Podles in [2] introduced a differential calculus on quantum 2-spheres $S^2_{\mu c}$ and then, he in [3] classified exterior algebras of differential forms on $S^2_{\mu c}$, for $\mu \in [-1,1]\backslash\{0\}, c \in [0,\infty]$ $(c = 0$ for $\mu = \pm 1)$. A natural question which arises here is the metric tensor on these spaces. We generalize the definition of hermitian metric on complex manifolds to the case of Podles quantum spheres. This metric is defined in such a way that when the deformation parameter tend to 1 it gives us the usual hermitian metric on complex spheres.

2. Quantum spheres

We start this section with definition of compact matrix pseudogroups which are a background for quantum spheres.

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Definition 2.1. *Let A be a C^*-algebra with unity, u be a $N \times N$ matrix with entries belonging to A: $u = (u_{kl})_{kl=1,2,...,N}, u_{kl} \in A$, and $\mathcal{A}$ be the $*$-subalgebra of A generated by the entries of u. We say that (A, u) is a compact matrix pseudogroup if*
1) $\mathcal{A}$ is dense in A.
2) There exists a C^*-homomorphism

$$\Phi : A \longrightarrow A \otimes A, \tag{2.1}$$

such that

$$\Phi(u_{kl}) = \sum_{r=1}^{N} u_{kr} \otimes u_{rl}, \tag{2.2}$$

for any $k =, l = 1, 2, ..., N$.
3) There exists a linear antimultiplicative mapping

$$\kappa : \mathcal{A} \longrightarrow \mathcal{A}, \tag{2.3}$$

such that

$$\kappa(\kappa(a^*)^*) = a \tag{2.4}$$

for any $a \in \mathcal{A}$ and

$$\sum_{r=1}^{N} \kappa(u_{kr})u_{rl} = \delta_{kl} I \tag{2.5}$$

$$\sum_{r=1}^{N} u_{kr}\kappa(u_{rl}) = \delta_{kl} I \tag{2.6}$$

for any $k =, l = 1, 2, ..., N$. δ_{kl} denotes the Kronecker symbol equal to 1 for $k = l$ and 0 otherwise, I is the unity of the algebra A.

Let G be a compact group of $N \times N$ matrices with complex entries: $G \subset M_N(\mathbb{C})$. We denote by $C(G)$ the commutative C^*-algebra of all continuous functions on G. for any $g \in G$ and any $k =, l = 1, 2, ..., N$, we denote by $w_{kl}(g)$ the matrix element of g standing in the k^{th} row and the l^{th} column:

$$g = (w_{kl}(g))_{k=,l=1,2,...,N}. \tag{2.7}$$

Clearly $w_{kl}(g)$ depends continuously on g, i.e., w_{kl} are continuous functions defined on $G : w_{kl} \in C(G)$. Let $w_G = (w_{kl})_{k=,l=1,2,...,N}$. Then the following Theorem is proved in [4].

Theorem 2.2. *$(C(G), w_G)$ is a compact matrix pseudogroup.*

Let μ be a nonzero real number in the interval [-1,1] and A be the C^*-algebra generated by two elements α and β satisfying the following relations:

$$\alpha^*\alpha + \gamma^*\gamma = I, \quad \alpha\alpha^* + \mu^2\gamma\gamma^* = I. \tag{2.8}$$

$$\gamma^*\gamma = \gamma\gamma^*, \quad \mu\gamma\alpha = \alpha\gamma \quad \mu\gamma^*\alpha = \alpha\gamma^*. \tag{2.9}$$

We consider 2×2 matrix

$$u = \begin{bmatrix} \alpha & -\mu\nu^* \\ \nu & \alpha^* \end{bmatrix}.$$

Then (A, u) is a compact matrix pseudogroup (see [6] for details and proofs). If $\mu = 1$ then A is a commutative and (A, u) is identical with $(C(G), w_G)$, where $G = SU(2)$. In the general case (A, u) is called the twisted $SU(2)$ group and denoted by $S_\mu U(2)$.

Now we consider a family of quantum spaces by corresponding C^*-algebras is introduced by Podles in [1]. $C(X_{\mu\lambda\rho}), \lambda, \rho \in \mathbb{R}$ denote the C^*-algebra generated by three elements e_{-1}, e_0, e_1 satisfying the following relations

$$e_i^* = e_{-i}, \quad i = -1, 0, 1, \tag{2.10}$$

$$(1 + \mu^2)(e_{-1}e_1 + \mu^{-2}e_1e_{-1}) + e_0^2 = \rho I, \tag{2.11}$$

$$e_0e_{-1} - \mu^2 e_{-1}e_0 = \lambda e_{-1}, \tag{2.12}$$

$$(1 + \mu^2)(e_{-1}e_1 + e_1e_{-1}) + (1 - \mu^2)e_0^2 = \lambda e_0, \tag{2.13}$$

$$e_1e_0 - \mu^2 e_0e_1 = \lambda e_1. \tag{2.14}$$

In short the elements e_{-1}, e_0, e_1 satisfying the relation (2.10) and the following relations

$$a_{lm}e_le_m = \rho I, \tag{2.15}$$

$$b_{lmk}e_le_m = (1 - \mu^2)e_k, \tag{2.16}$$

where the real numbers a_{lm}, b_{lmk}, μ, ρ, $(l, m, r = -1, 0, 1)$ are given in (2.11)-(2.15). Also we set

$$\widetilde{e}_{-2} = e_{-1}e_{-1}, \quad \widetilde{e}_{-1} = e_{-1}e_0 + \mu^2 e_0e_{-1},$$
$$\widetilde{e}_0 = e_0e_0 - \mu^{-2}e_{-1}e_1 - \mu^2 e_1e_{-1} \quad \widetilde{e}_1 = e_0e_1 + \mu^2 e_1e_0 \quad \widetilde{e}_2 = e_1e_1$$

i.e.,

$$\widetilde{e}_r = c_{lm,r}e_le_m, \quad r = -2, -1, 0, 1, 2 \tag{2.17}$$

where the real numbers $c_{lm,r}$ are in the above relations.
It is shown in [1] there exists a C^*-homomorphism

$$\sigma_{\mu\lambda\rho} : C(X_{\mu\lambda\rho}) \longrightarrow C(X_{\mu\lambda\rho}) \otimes C(S_\mu U(2))$$

such that $\sigma_{\mu\lambda\rho}e_i = e'_i \cdot \sigma_{\mu\lambda\rho}$ is an action of $S_\mu U(2)$ on $X_{\mu\lambda\rho}$, where

$$e'_i = \sum_{j=-1}^{1} e_j \otimes d_{1,ji}, \quad i = -1, 0, 1,$$

in which d_1 is a representation and has the matrix

$$(d_{1,ij})_{i,j=-1,0,1} = \begin{bmatrix} \alpha^{*2} & -(\mu^2+1)\alpha^*\gamma & -\mu\gamma^2 \\ \gamma^*\alpha^* & I - (\mu^2+1)\gamma^*\gamma & \alpha\gamma \\ -\mu\gamma^{*2} & -(\mu^2+1)\gamma^*\alpha & \alpha^2 \end{bmatrix} \in M_3 \otimes C(S_\mu U(2)).$$

Let

$$S^2_{\mu c} = X_{\mu,1-\mu^2,(1+\mu^2)^2\mu^{-2}c+1}, \quad (c \in \mathbb{R}) \tag{2.18}$$

$$S^2_{\mu\infty} = X_{\mu,0,(1+\mu^2)^2\mu^{-2}}, \tag{2.19}$$

and $\sigma_{\mu c}$ be the corresponding actions. For $0 < \mu < 1$, each $X_{\mu\lambda\rho}$ considered together with the action of $S_\mu U(2)$, is isomorphic to one of the above quantum spaces. $S^2_{\mu c}$ defined by the above algebra of functions on it is called *quantum sphere*.

If we set

$$e_1 = i(x_1 + ix_2), \quad e_{-1} = -i(x_1 - ix_2), \quad e_0 = 2x_3, \tag{2.20}$$

then we can interpret x_i's as the cartesian coordinates on $C(S^2_{\mu c})$ and in this way $S^2_{10} \cong S^2$, the unit sphere.

3. Hermitian Structure on Podles Quantum spheres

A pre-Hilbert module over a C^*-algebra A is a complex linear space E which is a left A-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in \mathbb{C}$, $a \in A$, and $x \in E$) equipped with an A-valued inner product $\langle .,. \rangle : E \times E \to A$ satisfying:

1. $\langle x, x\rangle \geq 0$,
2. $\langle x, x\rangle = 0 \Leftrightarrow x = 0$,
3. $\langle x + \lambda y, z\rangle = \langle x, z\rangle = +\lambda\langle y, z\rangle$,
4. $\langle x, y\rangle^* = \langle y, x\rangle$,
5. $\langle ax, y\rangle = a\langle x, y\rangle$,

For all $x, y, z \in E, a \in A$, and $\lambda \in \mathbb{C}$.

A pre-Hilbert A-module is called a Hilbert A-module or Hilbert C^*-module over A, if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. If the closed linear span of the set $\{\langle x, y\rangle : x, y \in E\}$ is dense in A then E is called full. For example every C^*-algebra is a full Hilbert A-module whenever we define $\langle x, y\rangle = xy^*$.
Let A be a C^*-algebra and E be pre-Hilbert module over A. Recall that a *Hermitian structure* on A-bimodule E ia a sesquilinear functional $\langle .,. \rangle : E \times E \to A$ in which $\langle x, ya\rangle = \langle x, y\rangle a$, for all $x, y \in E, a \in A$. It follows the last equality for each $x, y \in E, a \in A$, we have

$$\langle xa, y\rangle = \langle y, xa\rangle^* = (\langle y, x\rangle a)^* = a^*\langle y, x\rangle^* = a^*\langle x, y\rangle.$$

Let $S^\wedge = \bigoplus_{n=0}^\infty S^{\wedge n}$, where

$$S^{\wedge n} = span\{a_0 da_1 \wedge ...da_n : a_0, a_1, ..., a_n \in \beta\}$$

is the bimodule of exterior differential forms on S^2 of n^{th} degree, which are generated by the base β of the set of smooth functions on S^2. We denote the exterior derivative by $d : S^\wedge \longrightarrow S^\wedge$. Let $* : S^\wedge \longrightarrow S^\wedge$ be the complex conjugation:

$$(a_0 da_1 \wedge ...da_n)^* = a_0^* d(a_1^*) \wedge ...d(a_n^*), \quad a_0, a_1, ..., a_n \in \beta.$$

Now let $\mathcal{A} \subset C(S_\mu U(2))$ is the $*$-algebra of polynomials on $S_\mu U(2)$ and $\sigma^\wedge : S^\wedge \longrightarrow S^\wedge \otimes \mathcal{A}$ is a graded homomorphism such that

$$(id \otimes e)\sigma^\wedge = id, \quad (\sigma^\wedge \otimes id)\sigma^\wedge = (id \otimes \Phi)\sigma^\wedge, \quad \sigma^{\wedge 0} = \sigma_{|\beta},$$

where Φ is the map in Theorem 2.2 and $\wedge$ denotes multiplication in $S^{\wedge}$. In the following we assume that $\mu \in [-1,1]\backslash\{0\}, c \in [0,\infty]$ ($c = 0$ for $\mu = \pm 1$) and $\mathcal{A}_\mu$ is the $*$-algebra of polynomials on $S^2_{\mu c}$ generated by e_{-1}, e_0, e_1, and the $*$-homomorphism $\sigma_{\mu c} : \mathcal{A}_\mu \longrightarrow \mathcal{A}_\mu \otimes \mathcal{A}$ describes the action of $S_\mu U(2)$ on $S^2_{\mu c}$, $\beta = \mathcal{A}_\mu, \sigma = \sigma_{\mu c}$.

Let $S^{\wedge}$ be as in the above and $P = a_{kl}e_k de_l$(a_{kl} were used in (2.15)). Using (5) of [2] we get $\sigma^{\wedge}P = P \otimes I$, i.e., P is $\sigma^{\wedge 1}$-invariant. Moreover, we can check that P is unique (up to a scalar) $\sigma^{\wedge 1}$-invariant element of $\mathcal{A}_\mu$·span$\{d_{-1}, de_0, de_1\}$ (see [3]). If $\mu = 1$, then $P \equiv 4x_k dx_k$. In this case $dx_k, k = 1, 2, 3$, generate the left module $S^{\wedge 1}$ with only one constraint(see (2.3)), namely $P = 0$. Now put $A = C(S^2_{\mu c}$. Then $\Omega^1(A)$, the module of one forms on $S^2_{\mu c}$ is generated as a left A-module by $\beta = \{de_k | k = -1, 0, 1\}$ which the following formulas hold [3]:

$$a_{lm}(de_l)e_m = 0, \tag{3.1}$$

$$b_{lm,k}(de_l)e_m = (1-\mu^2)de_k - b_{lm,k}e_l de_m, \quad k = -1, 0, 1, \tag{3.2}$$

$$c_{lm,r}(de_l)e_m = c_{lm,r}e_l[de_m + \mu^{-2}(1-\mu^2)b_{kn,m}e_k de_n], \quad r = -2, -1, ..., 2., \tag{3.3}$$

$$(de_k)^* = de_{-1}, \quad k = -1, 0, 1, \tag{3.4}$$

where $a_{lm}, b_{lm,k}, c_{lm,r}, l, m = -1, 0, 1$, were in (2.15),(2.16), and (2.17). We can show that with the above relations $\Omega^1(A)$ becomes a bimodule over A ([3, Theorem]), and makes $\Omega^1(A)$ into a pre-Hilbert A-module.

Theorem 3.1. *There is a Hermitian structure on $\Omega^1(C(S^2_{\mu c}))$ as a finitely generated $C(S^2_{\mu c})$-module. In particular, for $\mu = 1, c = 0$, this structure is the usual hermitian metric on complex unit sphere S^2.*

Proof. We consider the sesquilinear map

$$T = \langle ., . \rangle : \Omega^1(C(S^2_{\mu c})) \times \Omega^1(C(S^2_{\mu c})) \longrightarrow C(S^2_{\mu c}),$$

satisfying the following equations:

$$T(\phi, \phi^*) \geq 0, \quad T(\phi, \eta) = (T(\eta, \phi))^*, \quad T(\phi a, \eta b) = a^* T(\phi, \eta) b \tag{3.5}$$

for all $\phi, \eta \in \Omega^1(C(S^2_{\mu c}))$, and $a, b \in C(S^2_{\mu c})$. Put $T_{i,j} = \langle de_i, de_j \rangle$. Then, from the relation (2.1) and (3.1), we have

$$(1+\mu^2)T_{i,-1}e_1 + (1+\mu^{-2})T_{i,1}e_{-1} + T_{i,0}e_0 = 0. \tag{3.6}$$

for all $i = -1, 0, 1$. Using from (2.12), (3.2) and (3.5), we conclude

$$T_{i,0}e_1 - T_{i,1}e_0 - (1-\mu^2)T_{i,1} = 0, \tag{3.7}$$

and

$$\mu^2 T_{i,-1}e_0 - \mu^2 T_{i,0}e_{-1} + (1-\mu^2)T_{i,-1} = 0. \tag{3.8}$$

for all $i = -1, 0, 1$. Also from (2.13), (3.2) and (3.5), we get

$$(1+\mu^2)T_{i,-s}e_s - (1+\mu^2)T_{i,s}e_s - (1-\mu^2)T_{i,0} = 0, \tag{3.9}$$

and

$$2\mu^2 T_{i,0}e_0 + (1-\mu^2)T_{0,0} = 0. \tag{3.10}$$

for all $i=-1,0,1$ and $s=-1,1$. Now, from (2.14) and (3.2), we deduce the following relations

$$T_{i,1}e_0 - T_{i,0}e_{-1} - (1-\mu^2)T_{i,1} = 0, \tag{3.11}$$

and

$$\mu^2 T_{i,0}e_1 - \mu^2 T_{i,1}e_0 + (1-\mu^2)T_{i,1} = 0. \tag{3.12}$$

for all $i=-1,0,1$. Using from the above relations which is lengthy and tedious, we obtain the following matrix

$$T = (T_{ij})_{i,j=-1,0,1} = \begin{bmatrix} \frac{-\mu^2 e_0 - (1-\mu^2)I}{1+\mu^2} & e_1 & 0 \\ e_{-1} & (1-\mu^2)(e_0 - I) & -\mu^2 e_1 \\ 0 & -\mu^2 e_{-1} & \frac{e_0-(1-\mu^2)I}{1+\mu^{-2}} \end{bmatrix}, \tag{3.13}$$

where obviously we have $T^*_{i,j} = T_{j,i}$. Setting $\mu = 1$ in (3.13) and using from (2.20), we have

$$T = (T_{ij})_{i,j=-1,0,1} = \begin{bmatrix} x_3 & -x_2 + ix_1 & 0 \\ -x_2 - ix_1 & 0 & x_2 - ix_1 \\ 0 & x_2 + ix_1 & x_3 \end{bmatrix},$$

which is a C^*-algebra of functions on unit sphere S^2. $\square$

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