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Cohomological properties for certain Banach algebras on locally compact groups

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Abstract

For a locally compact group G, let $L_0^\infty(G)$ be the Banach space of all essentially bounded measurable functions on G vanishing at infinity. Here, we deal with a derivation problem for the Banach algebra $L_0^\infty(G)^*$ equipped with a multiplication of Arens type. We first show that the Singer-Wermer conjecture for $L_0^\infty(G)^*$ is valid only in the case where G is abelian. Also, we characterize various cohomological properties for $L_0^\infty(G)^*$ according to algebraic and topological properties of G; in particular, we obtain diverse properties of G in terms of these notions based on derivations of $L_0^\infty(G)^*$.

Keywords: Amenability, Banach algebra, derivation, essentially bounded measurable functions, locally compact group, vanishing at infinity
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1 Introduction and Preliminaries

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure λ_G. The group algebra $L^1(G)$ is defined as in [11] consisting of all measurable functions on G equipped with the convolution product $*$ and the norm $\|.\|_1$. Also, let $L^\infty(G)$ denote the Lebesgue space as defined in [11] consisting of all locally essentially bounded measurable functions on G equipped with the essential supremum norm $\|.\|_\infty$. Then $L^\infty(G)$ is the dual of $L^1(G)$ for the pairing

$$\langle f, \phi\rangle = \int_G f(s)\ \phi(s)\ d\lambda_G(s).$$

for all $\phi \in L^1(G)$ and $f \in L^\infty(G)$. We denote by $L_0^\infty(G)$ the subspace of $L^\infty(G)$ consisting of all elements $f \in L^\infty(G)$ that vanish at infinity; i.e., for each $\varepsilon > 0$, there is a compact subset K of G for which $\|f\ \chi_{G\setminus K}\|_\infty < \varepsilon$, where $\chi_{G\setminus K}$ denotes characteristic function of $G \setminus K$ on G. For every $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$, we denote by ng the function in $L^\infty(G)$ defined by

$$\langle ng, \phi\rangle = \langle n, \frac{1}{\Delta}\,\widetilde{\phi} * g\rangle$$

for all $\phi \in L^1(G)$, where $\widetilde{\phi}(s) = \phi(s^{-1})$ and Δ denotes the modular function of G. The space $L_0^\infty(G)$ is left introverted in $L^\infty(G)$; i.e., for each $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$, we have $ng \in L_0^\infty(G)$. This lets us to endow $L_0^\infty(G)^*$ with the *first Arens product* $\diamond$ defined by

$$\langle m \diamond n, g\rangle = \langle m, ng\rangle$$

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for all $m, n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$. Then $L_0^\infty(G)^*$ with this product is a Banach algebra which is in relevance to the group algebra $L^1(G)$ of G. For each $\phi \in L^1(G)$, let ϕ also denote the functional in $L_0^\infty(G)^*$ defined by

$$\langle \phi, g\rangle := \int_G \phi(s)\; g(s)\; d\lambda_G(s)$$

for all $g \in L_0^\infty(G)$. Note that this duality defines a linear isometric embedding of $L^1(G)$ into $L_0^\infty(G)^*$, and that $L^1(G) = L_0^\infty(G)^*$ if and only if G is discrete [16]. Moreover, observe that $\phi \diamond \psi = \phi * \psi$ for all $\phi, \psi \in L^1(G)$, and that $L^1(G)$ is a closed ideal in $L_0^\infty(G)^*$; see Lau and Pym [13] for details.

Let us recall that an element $u \in L_0^\infty(G)^*$ is called a mixed identity if $\phi \diamond u = u \diamond \phi = \phi$ for all $\phi \in L^1(G)$. Denote by $\Lambda_0(G)$ the nonempty set of all mixed identities u with norm one in $L_0^\infty(G)^*$, and recall from Ghahramani, Lau and Losert [5] that $u \in \Lambda_0(G)$ if and only if it is a weak*-cluster point of an approximate identity in $L^1(G)$ bounded by one; or equivalently, it is a right identity of $L_0^\infty(G)^*$ with norm one. Moreover, $L_0^\infty(G)^*$ has an identity if and only if G is discrete.

The Banach algebra $L_0^\infty(G)^*$ has been introduced and studied extensively by Lau and Pym [13]. On the other hand, various concepts of amenability have been described for Banach algebras related to locally compact groups; see for example [6]-[8], [16] and [22].

For a Banach algebra A, a *Banach A-bimodule* is a Banach space X which is algebraically A-bimodule for which there is a constant C such that for each $a \in A$ and $x \in X$, $\|a \cdot x\| \leq C\|a\|\,\|x\|$ and $\|x \cdot a\| \leq C\|a\|\,\|x\|$; in particular, A is a Banach A-bimodule. By a *derivation* $D : A \to X$, we shall mean a linear map satisfying $D(ab) = D(a)\cdot b + a \cdot D(b)$ for all $a, b \in A$. Moreover, a derivation D is called *inner* if there is $x \in X$ such that $D = \mathrm{ad}_x$, where the derivation $\mathrm{ad}_x : A \longrightarrow X$ is defined by $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ for all $a \in A$.

The study of the range of derivations on Banach algebras was initiated by Singer and Wermer [25], since 1955. Having shown that the range of a continuous derivation on a commutative Banach algebra is contained within the radical of algebra, they conjectured that continuity could be ignored. More than thirty years later, Thomas [26] has proved it. So far, there have been several generalizations of Singer-Wermer conjecture presented to non-commutative Banach algebras. Conditions have been investigated under which every derivation on a Banach algebra maps into the radical.

Our aim in this paper is to study derivations on $L_0^\infty(G)^*$ and some related notions. In the second section, we focus on derivations of $L_0^\infty(G)^*$ into Banach A-bimodules and show that the Singer-Wermer conjucture is valid only on the case where G is abelian. In other sections, we deal with cohomological properties of $L_0^\infty(G)^*$. These investigations give us several characterazations of algebraic and topological properties of G in term of amenability properties of $L_0^\infty(G)^*$ such as abelianity, amenability, discreteness and finiteness.

2 Derivations on $L_0^\infty(G)^*$

Mehdipour and Saeedi [19] have proved that the zero map is the only weak*-weak*-continuous derivation on $L_0^\infty(G)^*$ when G is an ablian locally compact group. Our first result is in fact a more general result for all locally compact groups.

First let us recall that the measure algebra $M(G)$ of G is the Banach algebra of all complex Radon measures on G endowed with convolution product $*$ and total variation norm as defined in [11].

Theorem 2.1. Let G be a locally compact group. Then the following statements are equivalent.

(a) Every weak*-weak*-continuous derivation on $L_0^\infty(G)^*$ is zero.

(b) Every derivation from $L_0^\infty(G)^*$ into $L^1(G)$ is zero.

(c) G is abelian.

Proof . (a)⇒(b). This follows from Corollary 2 of [19]. (b)⇒(c). If G is not abelian, then ad_ϕ is a non-zero weak*-weak*-continuous derivation on $L_0^\infty(G)^*$ for some $\phi \in L^1(G) \setminus Z(L^1(G))$, where $Z(L^1(G))$ is the center of $L^1(G)$; indeed if $ad_\phi = 0$ for each $\phi \in L^1(G)$, then $\phi \diamond m = m \diamond \phi$ for all $m \in L_0^\infty(G)^*$ and so, $\phi * \psi = \psi * \phi$ for all $\psi \in L^1(G)$. It follows that $\phi \in Z(L^1(G))$.

(c)⇒(a). If G is abelian, then $L^1(G)$ is commutative and so $D|_{L^1(G)} = 0$ for all derivations $D : L_0^\infty(G)^* \longrightarrow L^1(G)$; this is because that $D|_{L^1(G)} = ad_\mu$, for some $\mu \in M(G)$. Moreover, for each $n \in L_0^\infty(G)^*$, we have

$$D(n) = \lim_\alpha e_\alpha * D(n) = \lim_\alpha [D(e_\alpha \diamond n) - D(e_\alpha) \diamond n] = 0,$$

where (e_α) is a bounded approximate identity of $L^1(G)$. □

In the case where G is arbitrary, we have the following result for derivations on $L_0^\infty(G)^*$.

Proposition 2.2. Let G be a locally compact group. Then every weak*-weak*-continuous derivation on $L_0^\infty(G)^*$ is inner.

Proof . Suppose that $D : L_0^\infty(G)^* \longrightarrow L_0^\infty(G)^*$ is a weak*-weak*-continuous derivation. On the one hand, since $L^1(G) * L^1(G) = L^1(G)$ and

$$D(\phi * \psi) = D(\phi) \diamond \psi + \phi \diamond D(\psi) \in L^1(G)$$

for each $\phi, \psi \in L^1(G)$. Hence, the range of the derivation $D|_{L^1(G)}$ containing in $L^1(G)$. So, there exists a measure $\mu \in M(G)$ such that $D|_{L^1(G)} = \mathrm{ad}_\mu$ by [14, Corollary 1.2].

On the other hand, the restriction map defines a continuous epimorphism $\tau : L_0^\infty(G)^* \longrightarrow M(G)$, and so there exists an element $m \in L_0^\infty(G)^*$ such that $\tau(m) = \mu$. Now, note that the $M(G)$-bimodule $L^1(G)$ is even an $L_0^\infty(G)^*$-bimodule by the actions

$$\phi \diamond n = \phi * \tau(n), \qquad n \diamond \phi = \tau(n) * \phi,$$

for all $n \in L_0^\infty(G)^*$ and, $\phi \in L^1(G)$. Hence,

$$D(\phi) = \phi \diamond m - m \diamond \phi \qquad (\phi \in L^1(G)).$$

So, the result will follow when G is discrete; otherwise,

$$M(G)^* = L^1(G)^* \oplus M_d(G)^* \oplus M_s(G)^*$$

and so, for each $f \in M(G)^*$, we have $f = f_a + f_d + f_s$, where $f_a \in L^1(G)^*$, $f_d \in M_d(G)^*$ and $f_s \in M_s(G)^*$. Now, let $k \in L_0^\infty(G)^*$. Then there exists a net $(\phi_i) \subseteq L^1(G)$ such that $\phi_i \longrightarrow k$ with respect to the weak*-topology in $L^1(G)^{**}$. It follows that

$$\begin{aligned} \lim_i \langle \phi_i,\, f\rangle &= \lim_i \langle \phi_i,\, f_a + f_d + f_s\rangle \\ &= \lim_i \langle \phi_i,\, f_a\rangle \\ &\longrightarrow \langle k,\, f_a\rangle \\ &= \langle k,\, f\rangle. \end{aligned}$$

Therefore, $\phi_i \longrightarrow k$ with respect to the weak*-topology in $M(G)^{**}$.

On the other hand, $Z_t(M(G)^{**}) = M(G)$ and so, $m \diamond \phi_i = \mu * \phi_i \longrightarrow m \diamond k$ with respect to weak*-topology in $M(G)^{**}$ and so, $L_0^\infty(G)^*$. Whence,

$$\begin{aligned} k \diamond m - m \diamond k &= \text{weak}^* - \lim_i\, (\phi_i \diamond m - m \diamond \phi_i) \\ &= \text{weak}^* - \lim_i\, D(\phi_i) \\ &= D(k). \end{aligned}$$

That is, $D = \mathrm{ad}_m$. □

Mehdipour and Saeedi have recently studied derivations on $L_0^\infty(G)^*$ for locally compact groups G [19]. They have shown that the Singer-Wermer conjecture is true for $L_0^\infty(G)^*$ when G is abelian; here, we show that this is an "if and only if" statement. Let us remark that the conjecture is proved for abelian semigroup $\mathbb{R}^+$; see [18].

Theorem 2.3. Let G be a locally compact group. Then the image of every derivation on $L_0^\infty(G)^*$ is contained in the radical of $L_0^\infty(G)^*$ if and only if G is abelian.

Proof . Suppose that the image of every derivation on $L_0^\infty(G)^*$ is contained in the radical of $L_0^\infty(G)^*$. Then for each $\phi \in L^1(G)$ and $n \in L_0^\infty(G)^*$, we have

$$\mathrm{ad}_\phi(n) \in \mathrm{rad}(L_0^\infty(G)^*) \cap L^1(G).$$

But the right annihilator of $L_0^\infty(G)^*$ and its radical coincide with $C_0(G)^\perp$; refer to Theorem 2.1 and Corollary 2.2 of [15]. So,

$$\mathrm{ad}_\phi(n) \in \mathrm{rad}(L_0^\infty(G)^*) \cap L^1(G) = \{0\}.$$

It follows that $L^1(G)$ is commutative, whence G is abelian. As we have mentioned above the converse is [19, Corollary 1]. □

Corollary 2.4. Let G be a locally compact group. Then every derivation on $L_0^\infty(G)^*$ is zero if and only if G is discrete and abelian.

Proof . The result follows immediately from Theorem 2.3 together with the fact that $\mathrm{ran}(L_0^\infty(G)^*) = \ker\tau$, where τ is the restriction epimorphism of $L_0^\infty(G)^*$ into $M(G)$; see, also [19, Theorem 6]. □

3 Amenability and contractibility of $L_0^\infty(G)^*$

Let A be a Banach algebra and X be a Banach A-bimodule. Then X^* is also a Banach A-bimodule with the actions induced via $\langle a\cdot\xi, x\rangle = \langle \xi, x\cdot a\rangle$ and $\langle \xi\cdot a, x\rangle = \langle \xi, a\cdot x\rangle$ for all $a\in A$, $x\in X$ and $\xi\in X^*$.

According to Johnson's definition [12], A is called *amenable* if every continuous derivation from A into the dual Banach A-bimodule X^* is inner for all Banach A-bimodules X. Also, A is called *contractible* if for every Banach A-bimodule X, every continuous derivation $D : A \to X$ is inner.

Ghahramani and et.al. in [6] and [7] have introduced and studied the notion of approximate amenability. They called A *approximately amenable* when for each Banach A-bimodule X, every continuous derivation D from A into X^* is approximately inner; that is, there exists a net $(x_\alpha) \subseteq X$ such that $D(a) = \lim_\alpha \mathrm{ad}_{x_\alpha}(a)$ in X for all $a\in A$.

Moreover, they called A *boundedly approximately amenable* if, in addition, the net (x_α) can be chosen such that the net (ad_{x_α}) is uniformly bounded.

They showed that the following notions are equivalent.

(a) approximate contractibility,

(b) approximate amenability,

(c) weak*-approximate amenability.

Furthermore, they proved that A is amenable if and only if A is uniformly approximately amenable; i.e., every continuous derivation from A into any dual Banach A-bimodule may be approximated uniformly on the unit ball of A by inner derivations. It is known from the same paper that

(1) $L^1(G)$ is approximately amenable if and only if G is amenable,

(2) $M(G)$ is approximately amenable if and only if G is discrete and amenable,

(3) $L^1(G)^{**}$ is approximately amenable if and only if G is finite;

let us recall that G is called *amenable* if there exists a left invariant mean on $L^\infty(G)$; that is, a positive norm one functional m in $L^\infty(G)^*$ for which $\langle m, L_x g\rangle = \langle m, g\rangle$ for all $g\in L^\infty(G)$ and $x\in G$, where $(L_x g)(y) = g(x^{-1}y)$ for all $y\in G$.

We commence with the following characterization of approximate amenability for $L_0^\infty(G)^*$.

Theorem 3.1. Let G be a locally compact group. Then the following assertions are equivalent.

(a) $L_0^\infty(G)^*$ is amenable.

(b) $L_0^\infty(G)^*$ is boundedly approximately amenable.

(c) $L_0^\infty(G)^*$ is approximately amenable.

(d) G is discrete and amenable.

Proof . That (d) implies (a) follows from the classical result of Johnson that $L^1(G)$ is amenable whenever G is amenable by Theorem 2.5 of [12]. Since, (a) implies (b) and (b) implies (c) trivially, to complete the proof, we need only to show that (c) implies (d); to that end, suppose that $L_0^\infty(G)^*$ is approximately amenable. It follows from Lemma 2.2 in [6] that $L_0^\infty(G)^*$ has a left approximate identity (u_γ). For every $u \in \Lambda_0(G)$ we have

$$\|u_\gamma - u\| = \|u_\gamma \diamond u - u\| \to 0.$$

This shows that for any $m \in L_0^\infty(G)^*$,

$$\begin{aligned} \|u \diamond m - m\| &\le \|(u - u_\gamma) \diamond m\| + \|u_\gamma \diamond m - m\| \\ &\le \|u - u_\gamma\| \, \|m\| + \|u_\gamma \diamond m - m\| \to 0. \end{aligned}$$

That is, u is a left identity for $L_0^\infty(G)^*$. Since $L_0^\infty(G)^*$ has always a right identity, $L_0^\infty(G)^*$ has an identity. Now, invoke Proposition 3.1 from [16] to conclude that G is discrete. Furthermore, $L^1(G) = L_0^\infty(G)^*$, and thus $L^1(G)$ is approximately amenable. Hence, G is also amenable. □

In our next result, we characterize essential amenability of $L_0^\infty(G)^*$. First let us recall that a Banach algebra A is called *essentially amenable* (resp. *approximately essentially amenable*) if every continuous derivation $D : A \longrightarrow X^*$ is inner (resp. approximately inner) for all A-bimodules X which is neo-unital; that is,

$$X = \{a \cdot y \cdot b : a, b \in A, y \in X\}.$$

Theorem 3.2. Let G be a locally compact group. Then the following assertions are equivalent.

(a) $L_0^\infty(G)^*$ is essentially amenable.

(b) $L_0^\infty(G)^*$ is approximately essentially amenable.

(c) G is discrete and amenable.

Proof . That (c) implies (a) follows from Theorem 3.1. Also, (a) implies (b) trivially. Now, suppose that $L_0^\infty(G)^*$ is essentially approximately amenable. Choose $u \in \Lambda_0(G)$, and note that the map $m \mapsto u \diamond m$ is a continuous epimorphism from $L_0^\infty(G)^*$ onto $u \diamond L_0^\infty(G)^*$. Next recall from Theorem 2.11 of [13] that $u \diamond L_0^\infty(G)^*$ is isometrically isomorphic to the measure algebra of $M(G)$. Thus, there exists a continuous epimorphism $\tau : L_0^\infty(G)^* \to M(G)$. For each $M(G)$-bimodule X, let Y be the Banach space X considering as a neo-unital $L_0^\infty(G)^*$-bimodule with actions

$$m \bullet \xi = \tau(m) \cdot \xi \quad \text{and} \quad \xi \bullet m = \xi \cdot \tau(m)$$

for all $\xi \in Y$ and $m \in L_0^\infty(G)^*$. So, if $D : M(G) \to X^*$ is a continuous derivation, then $D \circ \tau : L_0^\infty(G)^* \to Y^*$ is a continuous derivation, and there exists a net $(\xi_\alpha) \subseteq X^*$ such that

$$\begin{aligned} D(\tau(m)) &= (D \circ \tau)(m) \\ &= \lim_\alpha \, (m \bullet \xi_\alpha - \xi_\alpha \bullet m) \\ &= \lim_\alpha \, (\tau(m) \cdot \xi_\alpha - \xi_\alpha \cdot \tau(m)) \end{aligned}$$

for all $m \in L_0^\infty(G)^*$. Therefore, D is approximately inner. This shows that $M(G)$ is approximately amenable, and (a) holds by Theorem 3.1 of [6]. □

We end the section by descriptions of contractibility and essential contractibility of $L_0^\infty(G)^*$.

Theorem 3.3. Let G be a locally compact group. Then the following assertions are equivalent.

(a) $L_0^\infty(G)^*$ is contractible.

(b) $L_0^\infty(G)^*$ is uniformly approximately contractible.

(c) $L_0^\infty(G)^*$ is uniformly approximately essentially contractible.

(d) $L_0^\infty(G)^*$ is essentially contractible.

(e) G is finite.

Proof . Suppose that $L_0^\infty(G)^*$ is essentially contractible. Then $L_0^\infty(G)^*$ is essentially amenable, and therefore G is discrete by Theorem 3.2. This shows that $L_0^\infty(G)^*$ is equal to $L^1(G)$, and $L^1(G)$ is contractible. It follows that G is finite; see for example [24]. We have shown that (d) implies (e). The other implications are trivial. Note that the implication $(a) \Longleftrightarrow (e)$ has been proved in [17]. □

Theorem 3.4. Let G be a locally compact group. Then $L_0^\infty(G)^*$ is approximately essentially contractible if and only if G is discrete and amenable.

Proof . Suppose that $L_0^\infty(G)^*$ is approximately essentially contractible. Then $L_0^\infty(G)^*$ is approximately essentially amenable, and hence G is discrete and amenable by Theorem 3.2.

Now, suppose that G is discrete and amenable. Then $L_0^\infty(G)^*$ is approximately amenable which is equivalent to approximate contractibility; therefore, $L_0^\infty(G)^*$ is approximately essentially contractible. □

4 Weak amenability and ideal amenability of $L_0^\infty(G)^*$

A Banach algebra A is called *weakly amenable* (resp. *approximately weakly amenable*) if every continuous derivation $D : A \to A^*$ is inner (resp. approximately inner).

Theorem 4.1. Let G be a locally compact group. Then the following assertions are equivalent.

(a) $L_0^\infty(G)^*$ is weakly amenable.

(b) $L_0^\infty(G)^*$ is approximately weakly amenable.

(c) G is discrete.

Proof . Clearly, (a) implies (b). That (c) implies (a) follows from the known fact that $L^1(G)$ is always weakly amenable; see [2].

Now, suppose that (b) holds and G is not discrete. Then $M(G)$ admits a non-zero continuous point derivation d at a character ρ of $M(G)$; that is, a linear functional satisfying $d(\mu * \nu) = d(\mu)\rho(\nu) + \rho(\mu)d(\nu)$ for all $\mu, \nu \in M(G)$; see [1]. So, $\rho \circ \tau$ is a character of $L_0^\infty(G)^*$ and $d \circ \tau$ is a non-zero continuous point derivation at $\rho \circ \tau$, where τ is the epimorphism of $L_0^\infty(G)^*$ into $M(G)$. Therefore, $L_0^\infty(G)^*$ is not approximately weakly amenable by Proposition 2.1 of [6]. □

Let I be a closed two-sided ideal of a Banach algebra A. Then A is called *approximately I-weakly amenable* (resp. *I-weakly amenable*) if every continuous derivation $D : A \longrightarrow I^*$ is approximately inner (resp. inner). Moreover, A is called *approximately ideally amenable* (resp. *ideally amenable*) if A is approximately I-weakly amenable (resp. I-weakly amenable) for every closed two-sided ideal I of A; see [3] for details.

Proposition 4.2. Let G be a locally compact group. Then the following statments hold.

(a) $L_0^\infty(G)^*$ is always $L^1(G)$-weakly amenable.

(b) If every two-sided closed ideal of $L_0^\infty(G)^*$ is weakly amenable, then $L_0^\infty(G)^*$ is ideally amenable.

Proof . These are a special case of a more general result that a Banach algebra A is I -weakly amenable if the closed two-sided ideal I is weakly amenable; see [3], Lemma 2.1. □

Remark 4.3. Let G be a locally compact group and let I be a two-sided closed ideal I of $L^1(G)$. Since $L^1(G)$ is a two-sided ideal of $L_0^\infty(G)^*$ with a bounded approximate identity, I is a two-sided closed ideal of $L_0^\infty(G)^*$. Let $D : L^1(G) \longrightarrow I^*$ be a continuous derivation. Then $\tilde{D} : L_0^\infty(G)^* \longrightarrow I^*$ with

$$\tilde{D}(m) = \text{weak}^* - \lim_{\alpha} D(m \diamond e_\alpha) - m \cdot D(e_\alpha)$$

is a continuous derivation, where (e_α) is a bounded approximate identity of $L^1(G)$; this follows from [20]. Hence, the following statements are equivalent.

(a) $L_0^\infty(G)^*$ is approximately I-weakly amenable (resp. I-weakly amenable).

(b) $L^1(G)$ is approximately I-weakly amenable (resp. I-weakly amenable).

In particular, if $L_0^\infty(G)^*$ is approximately ideally amenable (resp. ideally amenable), then so is $L^1(G)$.

Corollary 4.4. Let G be an amenable locally compact group. Then the following assertions are equivalent.

(a) $L_0^\infty(G)^*$ is ideally amenable,

(b) $L_0^\infty(G)^*$ is approximately ideally amenable,

(c) G is discrete.

Proof . That (b) implies (c) follows from Theorem 4.1 together with the fact that any approximately ideally amenable Banach algebra is approximately weakly amenable. That (c) implies (a) follows from Johnson' result; in fact $L^1(G)$ is amenable and coincides with $L_0^\infty(G)^*$ when G is discrete ane amenable. □

5 Connes-amenability of $L_0^\infty(G)^*$

Recall that a Banach algebra A is called *dual* if it is a dual space such that multiplication in A is separately weak*-weak*-continuous. The class of dual Banach algebras was introduced and studied by Runde in [20].

Lemma 5.1. Let G be a locally compact group. Then $L_0^\infty(G)^*$ is a dual Banach algebra if and only if G is discrete.

Proof . Note that, for $m \in L_0^\infty(G)^*$, the map $n \mapsto m\diamond n$ is not weak*-weak*-continuous on $L_0^\infty(G)^*$ unless $m \in L^1(G)$ by [13], Theorem 2.11. So, $L_0^\infty(G)^*$ is not a dual Banach algebra except for the case where $L_0^\infty(G)^*$ coincides $L^1(G)$; but the later coincideness is equivalent to that G is discrete by [16]. □

Let A be a dual Banach algebra. A dual Banach A-bimodule X is called *normal* if for each $x \in X$, the maps $a \longmapsto a\cdot x$ and $a \longmapsto x\cdot a$ from A into X are weak*-weak*-continuous. A dual Banach algebra A is called *Connes-amenable* (resp. *approximately Connes-amenable*) if for every normal, dual Banach A-bimodule X, every weak*-weak*-continuous derivation $D : A \longrightarrow X$ is inner (resp. *approximately inner*).

Theorem 5.2. Let G be a locally compact group. Then the following assertions are equivalent.

(a) $L_0^\infty(G)^*$ is Connes-amenable dual Banach algebra.

(b) $L_0^\infty(G)^*$ is approximately Connes-amenable dual Banach algebra.

(c) G is discrete and amenable.

Proof . That (a) implies (b) is trivial. Now, suppose that $L_0^\infty(G)^*$ is an approximately Connes-amenable dual Banach algebra. It follows that G is discrete by Lemma 5.1, and so $L_0^\infty(G)^* = M(G)$ is approximately Connes-amenable. Thus, G is amenable by Proposition 5.1 of [4].

Now, suppose that (c) holds. Then $L_0^\infty(G)^* = M(G)$ and G is amenable; therefore $L_0^\infty(G)^*$ is Connes-amenable by Theorem 1 of [22]. □

6 Pseudo-amenability of $L_0^\infty(G)^*$

Let A be a Banach algebra. An *approximate diagonal* for A is a net (M_i) in the Banach A-bimodule $A\widehat{\otimes}A$ such that $a\cdot M_i - M_i\cdot a \longrightarrow 0$ and $a\pi(M_i) \longrightarrow a$ for all $a \in A$, where $A\widehat{\otimes}A$ denotes the projective tensor product of A with itself and π denotes the bounded linear map from $A\widehat{\otimes}A$ into A specified by $\pi(a\widehat{\otimes}b) = ab$ for all $a, b \in A$. The Banach algebra A is called *pseudo-amenable* if it has an approximate diagonal.

Theorem 6.1. Let G be a locally compact group. Then $L_0^\infty(G)^*$ is pseudo-amenable if and only if G is discrete and amenable.

Proof . Suppose that $L_0^\infty(G)^*$ is pseudo-amenable. It follows that $L_0^\infty(G)^*$ has an approximate identity (u_γ). As in the proof of Theorem 3.1, it follows that G is discrete, and of course $L^1(G) = L_0^\infty(G)^*$. Therefore, $L^1(G)$ is pseudo-amenable whence G is also amenable by Proposition 4.1 of [8]. □

The Banach algebra A is called *pseudo-contractible* if it has an approximate diagonal (M_i) with $a\cdot M_i = M_i\cdot a$ for all $a \in A$; see [8] for more details.

Theorem 6.2. Let G be a locally compact group. Then $L_0^\infty(G)^*$ is pseudo-contractible if and only if G is finite.

Proof . Pseudo-contractibility of $L_0^\infty(G)^*$ implies pseudo-amenability of $L_0^\infty(G)^*$, and so G is discrete and amenable by Theorem 6.1. Thus, $L_0^\infty(G)^*$ has a unit, and so it is contractible by Theorem 2.4 of [8]. Hence, G is finite by Theorem 3.3. The converse is trivial. □

7 Biflatness and biprojectivity of $L_0^\infty(G)^*$

A Banach algebra A is called *biflat* if there is a bounded A-bimodule homomorphism $\rho : A \longrightarrow (A\widehat{\otimes}A)^{**}$ such that $\pi^{**} \circ \rho$ is the canonical embedding of A into A^{**}, where $\pi : A\widehat{\otimes}A \longrightarrow A$ is defined by $\pi(a \otimes b) = ab$ for all $a, b \in A$.

Theorem 7.1. Let G be a locally compact group. Then $L_0^\infty(G)^*$ is biflat if and only if G is discrete and amenable.

Proof . It is known that a Banach algebra A is amenable if and only if A is biflat and has a bounded approximate identity; see Exercise 4.3.15 of [21]. Suppose that $L_0^\infty(G)^*$ is biflat. Then $L_0^\infty(G)^*$ is weakly amenable; this is because that any biflat Banach algebra is weakly amenable by [9]. Therefore, G is discrete by Theorem 4.1. Hence, $L_0^\infty(G)^* = L^1(G)$ and so $L^1(G)$ is biflat which implies that $L^1(G)$ is amenable; this follows from the fact that $L^1(G)$ has a bounded approximate identity and is biflat. Thus, G is amenable by Theorem 2.5 of [12].

Conversely, if G is discrete and amenable, then $L_0^\infty(G)^* = L^1(G)$ and G is amenable; that is $L_0^\infty(G)^*$is amenable by Theorem 3.1. Hence, $L^1(G)$ is biflat. □

A Banach algebra A is called *biprojective* if π has a bounded right inverse which is an A-bimodule homomorphism; see [23] for detailes. Automatically, biprojective Banach algebras are biflat. This fact together withTheorem 7.1 and Theorem 5.1 of [10] verifies the following result. Also, it has been proved in Theorem 4.5 of [4].

Theorem 7.2. Let G be a locally compact group. Then $L_0^\infty(G)^*$ is biprojective if and only if G is finite.

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