

Some extensions of the Eneström-Kakeya theorem for quaternionic polynomials

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Abstract

In this paper, we are interested in finding regions of some quaternionic polynomials with restricted coefficients. We relax the hypothesis and put less restrictive conditions on the coefficients of the quaternionic polynomial, and thereby generalize some known results by virtue of a maximum modulus theorem. The obtained results for the class of quaternionic polynomials generalize some recently proved results about the distribution of zeros of polynomials.

Keywords: Quaternionic Polynomials, Maximum modulus theorem, Eneström-Kakeya theorem

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1 Introduction

The study of the regional locations of the zeros of polynomial in geometric function theory is of relevance to both mathematics and practical fields like physical systems. The distribution of zeros of polynomials with restricted coefficients has received a great deal of attention in the beginning of 20th century, and there have been a number of significant advancements. The first result concerning the location of zeros of a polynomial was probably due to Gauss [6]. In order to obtain better and sharp zero bounds, it is desirable to put some restrictions on the coefficients of the polynomial. Regarding the condition on the coefficients of a polynomial, the below given elegant result was initially put by Eneström and Kakeya independently. The Eneström-Kakeya theorem is one of the most well-known findings regarding the location of zeros with significant implications for geometric function theory, which is stated as follows.

Theorem 1.1. If $P(z) = \sum_{s=0}^n a_s z^s$ is a polynomial of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_0 > 0$, then all the zeros of $P(z)$ lie in $|z| \leq 1$.

The above classical result is particularly important in the study of stability of numerical methods for differential equations and is starting point of rich literature concerning its extensions, generalizations and improvements in several directions (for instance, see papers [2],[4],[14],[5],[9],[10],[11], [12],[13]).

Below result was obtained in 1967, Joyal et al. [11] extended Theorem 1.1 to those complex polynomials whose coefficients are monotonic but not necessarily non-negative.

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Theorem 1.2. If $P(z) = \sum_{s=0}^n a_s z^s$ is a polynomial of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq \dots \geq a_0$, then all the zeros of $P(z)$ lie in $|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$.

Aziz et al. [1] relaxed the hypothesis of Theorem 1.2 and obtained the following.

Theorem 1.3. If $P(z) = \sum_{s=0}^n a_s z^s$ is a polynomial of degree n with real coefficients such that for some $k \geq 1$ satisfying $ka_n \geq a_{n-1} \geq \dots \geq a_0$, then all the zeros of $P(z)$ lie in $|z + k - 1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|}$.

2 Preliminaries

In the past decade, the holomorphic functions of quaternions were invented and developed by Irish mathematician William Rowan Hamilton in 1843 and are essentially a generalization of complex numbers to four dimensions. The set of quaternions is denoted by $\mathbb{H}$ in the honour of Sir Hamilton and they form a non-commutative division ring as multiplication of quaternions is non commutative in general. For $q \in \mathbb{H}$ is of the form $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, and i, j and k are the unit vectors along the three spatial axis and satisfying $i^2 = j^2 = k^2 = ijk = -1$. Every element $q = \alpha + \beta i + \gamma j + \delta k$ is composed of the real part $\Re(\alpha) = q$ and the imaginary part $\Im(q) = i\beta + j\gamma + k\delta$. Since the quaternions contain real numbers (when vector part is zero), so the field of real numbers is isomorphic to a subset of the quaternions. Moreover the set of quaternions forms a four dimensional vector space over $\mathbb{R}$ with $\{1, i, j, k\}$ as a basis. The conjugate of a quaternion $q = \alpha + i\beta + j\gamma + k\delta$ is denoted by q^* and is defined as $q^* = \alpha - i\beta - j\gamma - k\delta$. In addition, the norm (or length) of a quaternion $q = \alpha + i\beta + j\gamma + k\delta$ is given by

$$\|q\| = \sqrt{qq^*} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}.$$

We define the angle θ between two quaternions $q_1 = \alpha_1 + i\beta_1 + j\gamma_1 + k\delta_1$ and $q_2 = \alpha_2 + i\beta_2 + j\gamma_2 + k\delta_2$ as

$$\angle(q_1, q_2) = \cos^{-1} \left(\frac{\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 + \delta_1 \delta_2}{\|q_1\| \|q_2\|} \right).$$

The class of all n^{th} degree quaternionic polynomials is denoted by

$$P_n = \left\{ P(q); P(q) = \sum_{s=0}^n q^s a_s \right\}.$$

In 2020, Carney et al. [4] proved the following extension of Theorem 1.1 to the quaternionic polynomial $p(q)$.

Theorem 2.1. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq \dots \geq a_0 > 0$, then all the zeros of $P(q)$ lie in $|q| \leq 1$.

They also proved the following results in the same paper.

Theorem 2.2. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with real coefficients $0 \leq s \leq n$, such that $a_n \geq a_{n-1} \geq \dots \geq a_0$, then all the zeros of $P(q)$ lie in $|q| \leq \frac{1}{|a_n|}(|a_n| - a_0 + |a_0|)$.

Theorem 2.3. Let $P(q) = \sum_{s=0}^n q^s a_s$ be a quaternionic polynomial of degree n satisfying $\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_0 \geq 0$, $\alpha_n \neq 0$ then all the zeros of $P(q)$ lie in

$$|q| \leq 1 + \frac{2}{\alpha_n} \sum_{s=0}^n (|\beta_s| + |\gamma_s| + |\delta_s|).$$

Theorem 2.4. Let $P(q) = \sum_{s=0}^n q^s a_s$ be a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a nonzero quaternion and suppose $\angle(a_s, b) \leq \theta \leq \frac{\pi}{2}$ for some θ and $s = 0, 1, 2, \dots, n$. Assume $|a_n| \geq |a_{n-1}| \geq \dots \geq |a_0|$. Then all the zeros of $P(q)$ lie in

$$|q| \leq \cos \theta + \sin \theta + \frac{2 \sin \theta}{|a_n|} \sum_{s=0}^{n-1} |a_s|.$$

Theorem 2.5. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with quaternionic coefficients $a_s \in \mathbb{H}$, $0 \leq s \leq n$ such that

$$\begin{aligned} \alpha_n &\geq \alpha_{n-1} \geq \dots \geq \alpha_0, \\ \beta_n &\geq \beta_{n-1} \geq \dots \geq \beta_0, \\ \gamma_n &\geq \gamma_{n-1} \geq \dots \geq \gamma_0, \\ \delta_n &\geq \delta_{n-1} \geq \dots \geq \delta_0, \end{aligned}$$

then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \left[(\alpha_n - \alpha_0 + |\alpha_0|) + (\beta_n - \beta_0 + |\beta_0|) + (\gamma_n - \gamma_0 + |\gamma_0|) + (\delta_n - \delta_0 + |\delta_0|) \right].$$

As remarked in the beginning, the main aim of this paper is to extend various results of Eneström-Kakeya type from complex to quaternionic setting to obtain regions containing zeros of quaternionic polynomial. We shall make use of maximum modulus theorem, factorization theorem to get desired results. The obtained results produce a kind of generalisations of Theorems 2.1, 2.2, 2.3, 2.4, 2.5.

3 Main Results

Theorem 3.1. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with quaternionic coefficients $a_s \in \mathbb{H}$, $0 \leq s \leq n$ and if for some positive integer $\lambda \leq n$ and $k \geq 1$

$$k^{n-\lambda+1} \alpha_n \geq k^{n-\lambda} \alpha_{n-1} \geq \dots \geq k^\lambda \alpha_\lambda \geq \alpha_{\lambda-1} \geq \dots \geq \alpha_1 \geq \alpha_0 \geq 0,$$

then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{\alpha_n} \left[(2k^{n-\lambda+1} - 1) \alpha_n + 2 \left\{ (k^{n-\lambda} - 1) \alpha_{n-1} + (k^{n-\lambda-1} - 1) \alpha_{n-2} + \dots + (k^2 - 1) \alpha_{\lambda+1} \right\} + (2k - 1) \alpha_\lambda + M_1 \right],$$

where, $M_1 = 2 \sum_{s=0}^n [|\beta_s| + |\gamma_s| + |\delta_s|]$.

If we take $\lambda = n$, we obtain the following corollary.

Corollary 3.2. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with quaternionic coefficients $a_s \in \mathbb{H}$, $0 \leq s \leq n$ and if for some $k \geq 1$

$$k \alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0 \geq 0$$

then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{\alpha_n} \left[(2k - 1) \alpha_n + M_1 \right],$$

where $M_1 = 2 \sum_{s=0}^n [|\beta_s| + |\gamma_s| + |\delta_s|]$.

Remark 3.3. By taking $k = 1$, Corollary 3.2 reduces to Theorem 2.3.

For $\lambda = n - 1$ and assuming all coefficients are real in Theorem 3.1, we obtain the following result.

Corollary 3.4. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with real coefficients satisfying

$$k^2 a_n \geq k a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0 > 0,$$

for some $k \geq 1$ then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \left[(2k^2 - 1)a_n + 2(k - 1)a_{n-1} \right].$$

Remark 3.5. Taking $k = 1$, Corollary 3.4 reduces to Theorem 2.1.

Theorem 3.6. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with quaternionic coefficients $a_s \in \mathbb{H}$, $0 \leq s \leq n$ and if for some positive integer $\lambda \leq n$ and $k \geq 1$

$$\begin{aligned} k^{n-\lambda+1} \alpha_n &\geq k^{n-\lambda} \alpha_{n-1} \geq \cdots \geq k^\lambda \alpha_\lambda \geq \cdots \alpha_1 \geq \alpha_0, \\ \beta_n &\geq \beta_{n-1} \geq \cdots \geq \beta_0, \\ \gamma_n &\geq \gamma_{n-1} \geq \cdots \geq \gamma_0, \\ \delta_n &\geq \delta_{n-1} \geq \cdots \geq \delta_0. \end{aligned}$$

then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2 \right],$$

where,

$$M_2 = \left[(|\alpha_0| - \alpha_0) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n) \right].$$

Taking $\lambda = n$ in Theorem 3.6, below result is attained.

Corollary 3.7. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with quaternionic coefficients $a_s \in \mathbb{H}$, $0 \leq s \leq n$ satisfying

$$\begin{aligned} k \alpha_n &\geq \alpha_{n-1} \geq \cdots \geq \alpha_0, \\ \beta_n &\geq \beta_{n-1} \geq \cdots \geq \beta_0, \\ \gamma_n &\geq \gamma_{n-1} \geq \cdots \geq \gamma_0, \\ \delta_n &\geq \delta_{n-1} \geq \cdots \geq \delta_0, \end{aligned}$$

for some $k \geq 1$, then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \left\{ (2k - 1)\alpha_n + (|\alpha_0| - \alpha_0) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n) \right\}.$$

Remark 3.8. For $k = 1$ Theorem 3.6 reduces to Theorem 2.5.

Theorem 3.9. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\angle(a_s, b) \leq \theta \leq \frac{\pi}{2}$ for some θ and if for some positive integer $\lambda \leq n$ and $k \geq 1$

$$k^{n-\lambda+1}|a_n| \geq k^{n-\lambda}|a_{n-1}| \geq \cdots \geq k^\lambda|a_\lambda| \geq |a_{\lambda-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \left\{ [k^{n-\lambda+1}|a_n| + k^{n-\lambda}|a_{n-1}| + \cdots + k|a_\lambda|](\cos \theta + \sin \theta) - [|a_{n-1}| + |a_{n-2}| + \cdots + |a_\lambda|](\cos \theta - \sin \theta) + (k^{n-\lambda+1} - 1)|a_n| + (k^{n-\lambda} - 1)|a_{n-1}| + \cdots + (k - 1)|a_{\lambda-1}| + 2 \sin \theta \sum_{s=0}^{\lambda-1} |a_s| + |a_0|(1 - \cos \theta - \sin \theta) \right\}.$$

If we take $\lambda = n$ in Theorem 3.9 following corollary is obtained.

Corollary 3.10. If $P(q) = \sum_{s=0}^n q^s a_s$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\angle(a_s, b) \leq \theta \leq \frac{\pi}{2}$ for some $\theta, s = 0, 1, \dots, n$. Assume

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

for some positive integer k and $k \geq 1$, then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \left\{ (k - 1)|a_n| + k|a_n|(\cos \theta + \sin \theta) + |a_0|(1 - \cos \theta - \sin \theta) + 2 \sin \theta \sum_{s=0}^{n-1} |a_s| \right\},$$

which can be written in modified form as:

$$|q| \leq \frac{1}{|a_n|} \left\{ (k - 1)|a_n| + k|a_n|(\cos \theta + \sin \theta) + 2 \sin \theta \sum_{s=0}^{n-1} |a_s| \right\},$$

$$\left(\text{using } \cos \theta + \sin \theta \geq 1 \quad \text{when } \theta \in [0, \frac{\pi}{2}] \right).$$

Remark 3.11. Theorem 2.4 is a special case of Corollary 3.10 for $k = 1$.

4 Lemmas

We use the following lemmas in the proof of our results.

Lemma 4.1. [7] Let $f(q) = \sum_{s=0}^{\infty} q^s a_s$ and $g(q) = \sum_{s=0}^{\infty} q^s b_s$, be two given quaternionic power series with radii of convergence greater than R . The regular product of $f(q)$ and $g(q)$ is defined as $(f * g)q_0 = \sum_{s=0}^{\infty} q^s c_s$ where $c_s = \sum_{l=0}^s a_l b_{s-l}$. Let $|q_0| < R$, then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g(f(q_0)^{-1}q_0 f(q_0)) = 0$.

Lemma 4.2. [4] Let $q_1, q_2 \in \mathbb{H}$ where $q_1 = \alpha_1 + i\beta_1 + j\gamma_1 + k\delta_1$ and $q_2 = \alpha_2 + i\beta_2 + j\gamma_2 + k\delta_2$, such that $\angle(q_1, q_2) = 2\theta' \leq 2\theta$, and $|q_1| \leq |q_2|$. Then

$$|q_2 - q_1| \leq (|q_2| - |q_1|) \cos \theta + (|q_2| + |q_1|) \sin \theta.$$

5 Proof of Main Results

Proof of Theorem 3.1. Consider the polynomial $f(q) = \sum_{s=1}^n q^n(a_s - a_{s-1}) + a_0$. Let $P(q) * (1 - q) = f(q) - q^{n+1}\alpha_n$, therefore by Lemma 4.1, $P(q) * (1 - q) = 0$ iff either $P(q) = 0$ or $P(q) \neq 0$ implies $P(q)^{-1}qP(q) - 1 = 0$, that is $P(q)^{-1}qP(q) = 1$. If $P(q) \neq 0$ then $q = 1$. Therefore, the only zeros of $P(q) * (1 - q)$ are $q = 1$ and the zeros of $P(q)$. For $|q| = 1$ we get

$$\begin{aligned} |f(q)| &\leq |a_0| + \sum_{s=1}^n |a_s - a_{s-1}| \\ &\leq |\alpha_0 + i\beta_0 + j\gamma_0 + k\delta_0| + \sum_{s=1}^n |(\alpha_s - \alpha_{s-1}) + i(\beta_s - \beta_{s-1}) + j(\gamma_s - \gamma_{s-1}) + k(\delta_s - \delta_{s-1})| \\ &\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{s=1}^n [|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|] \\ &\leq \alpha_0 + |\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \cdots + |\alpha_\lambda - \alpha_{\lambda-1}| + \cdots + |\alpha_1 - \alpha_0| + 2 \sum_{s=0}^n [|\beta_s| + |\gamma_s| + |\delta_s|] \\ &\leq \alpha_0 + |k^{n-\lambda+1}\alpha_n - \alpha_{n-1} + \alpha_n - k^{n-\lambda+1}\alpha_n| + |k^{n-\lambda}\alpha_{n-1} - \alpha_{n-2} + \alpha_{n-1} - k^{n-\lambda}\alpha_{n-2}| + \cdots + |k\alpha_\lambda - \alpha_{\lambda-1} \\ &\quad + \alpha_\lambda - k\alpha_\lambda| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + \cdots + |\alpha_1 - \alpha_0| + M_1, \end{aligned}$$

where, $M_1 = 2 \sum_{s=0}^n [|\beta_s| + |\gamma_s| + |\delta_s|]$.

$$\begin{aligned} |f(q)| &\leq (2k^{n-\lambda+1} - 1)\alpha_n + 2(k^{n-\lambda} - 1)\alpha_{n-1} + 2(k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (2k - 1)\alpha_\lambda + M_1 \\ &\leq (2k^{n-\lambda+1} - 1)\alpha_n + 2 \left[(k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right] + (2k - 1)\alpha_\lambda + M_1. \end{aligned}$$

Since,

$$\max_{|q|=1} |q^n * f(\frac{1}{q})| = \max_{|q|=1} |f(\frac{1}{q})| = \max_{|q|=1} |f(q)|.$$

Therefore, $q^n * f(\frac{1}{q})$ has the same bound on $|q| = 1$ as $f(q)$, so

$$|q^n * f(\frac{1}{q})| \leq (2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_1,$$

for $|q| = 1$. Applying maximum modulus theorem ([8, Theorem 3.4]) for quaternionic polynomials, we get

$$|q^n * f(\frac{1}{q})| \leq (2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_1,$$

for $|q| \leq 1$. This implies

$$|f(\frac{1}{q})| \leq \frac{1}{|q|^n} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_1 \right]$$

for $|q| \leq 1$. Replacing q by $\frac{1}{q}$, we have for $|q| \geq 1$

$$|f(q)| \leq \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_1 \right] |q|^n. \quad (5.1)$$

But

$$\begin{aligned} |P(q) * (1 - q)| &= |f(q) - q^{n+1}\alpha_n| \\ &\geq |\alpha_n||q|^{n+1} - |f(q)|. \end{aligned}$$

Using (5.1), we have for $|q| \geq 1$

$$\begin{aligned} |P(q) * (1 - q)| &\geq |\alpha_n| |q|^{n+1} - \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} \right. \\ &\quad \left. + (2k - 1)\alpha_\lambda + M_1 \right] |q|^n \\ &\geq \left[|\alpha_n| |q| - \left((2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} \right. \right. \\ &\quad \left. \left. + (2k - 1)\alpha_\lambda + M_1 \right) \right] |q|^n. \end{aligned}$$

This implies that $|P(q) * (1 - q)| > 0$, i.e. $P(q) * (1 - q) \neq 0$ if

$$|q| > \frac{1}{|\alpha_n|} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_1 \right].$$

Since the only zeros of $P(q) * (1 - q)$ are $q = 1$ and the zeros of $P(q)$. Therefore, $P(q) \neq 0$ for

$$|q| > \frac{1}{|\alpha_n|} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_1 \right].$$

Hence all the zeros of $P(q)$ lie in

$$|q| \leq \frac{1}{\alpha_n} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_1 \right].$$

This completes the proof of theorem.

Proof of Theorem 3.6. Consider the polynomial $f(q) = \sum_{s=1}^n q^n (a_s - a_{s-1}) + a_0$. Let $P(q) * (1 - q) = f(q) - q^{n+1} a_n$, therefore by Lemma 4.1, $P(q) * (1 - q) = 0$ iff either $P(q) = 0$ or $P(q) \neq 0$ implies $P(q)^{-1} q P(q) - 1 = 0$, that is $P(q)^{-1} q P(q) = 1$. If $P(q) \neq 0$ then $q = 1$. Therefore, the only zeros of $P(q) * (1 - q)$ are $q = 1$ and the zeros of $P(q)$. For $|q| = 1$, we get

$$\begin{aligned} |f(q)| &\leq |a_0| + \sum_{s=1}^n |a_s - a_{s-1}| \\ &= |\alpha_0 + i\beta_0 + j\gamma_0 + k\delta_0| + \sum_{s=1}^n |(\alpha_s - \alpha_{s-1}) + i(\beta_s - \beta_{s-1}) + j(\gamma_s - \gamma_{s-1}) + k(\delta_s - \delta_{s-1})| \\ &\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{s=1}^n [|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|] \\ &\leq |\alpha_0| + |\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \cdots + |\alpha_\lambda - \alpha_{\lambda-1}| + \cdots + |\alpha_1 - \alpha_0| \\ &\quad + |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \cdots + |\beta_1 - \beta_0| \\ &\quad + |\gamma_n - \gamma_{n-1}| + |\gamma_{n-1} - \gamma_{n-2}| + \cdots + |\gamma_1 - \gamma_0| \\ &\quad + |\delta_n - \delta_{n-1}| + |\delta_{n-1} - \delta_{n-2}| + \cdots + |\delta_1 - \delta_0| \\ &\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + |k^{n-\lambda+1}\alpha_n - \alpha_{n-1} + \alpha_n - k^{n-\lambda+1}\alpha_n| \\ &\quad + |k^{n-\lambda}\alpha_{n-1} - \alpha_{n-2} + \alpha_{n-1} - k^{n-\lambda}\alpha_{n-2}| + \cdots + |k\alpha_\lambda - \alpha_{\lambda-1} + \alpha_\lambda - k\alpha_\lambda| \\ &\quad + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + \cdots + |\alpha_1 - \alpha_0| + (\beta_n - \beta_0) + (\gamma_n - \gamma_0) + (\delta_n - \delta_0) \\ &\leq (2k^{n-\lambda+1} - 1)\alpha_n + 2(k^{n-\lambda} - 1)\alpha_{n-1} + 2(k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (2k - 1)\alpha_\lambda + M_2. \end{aligned}$$

where, $M_2 = (|\alpha_0| - \alpha_0) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)$.

$$|f(q)| \leq (2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2.$$

Since

$$\max_{|q|=1} |q^n * f(\frac{1}{q})| = \max_{|q|=1} |f(\frac{1}{q})| = \max_{|q|=1} |f(q)|,$$

therefore, $q^n * f(\frac{1}{q})$ has the same bound on $|q| = 1$ as $f(q)$, so

$$|q^n * f(\frac{1}{q})| \leq (2k^{n-\lambda+1} - 1)\alpha_n + 2\left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2,$$

for $|q| = 1$. Applying maximum modulus theorem ([8, Theorem 3.4]), for quaternionic polynomials, we get

$$|q^n * f(\frac{1}{q})| \leq (2k^{n-\lambda+1} - 1)\alpha_n + 2\left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2,$$

for $|q| \leq 1$. This implies

$$|f(\frac{1}{q})| \leq \frac{1}{|q|^n} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2\left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2 \right].$$

Replacing q by $\frac{1}{q}$, we have for $|q| \geq 1$

$$|f(q)| \leq \left[2k^{n-\lambda+1} - 1 \right] \alpha_n + 2\left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2 \Big] |q|^n. \quad (5.2)$$

But

$$\begin{aligned} |P(q) * (1 - q)| &= |f(q) - q^{n+1}a_n| \\ &\geq |a_n||q|^{n+1} - |f(q)|. \end{aligned}$$

Using (5.2), we have for $|q| \geq 1$

$$\begin{aligned} |P(q) * (1 - q)| &\geq |a_n||q|^{n+1} - \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2\left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_1 \right] |q|^n \\ &\geq \left[|a_n||q| - \left\{ (2k^{n-\lambda+1} - 1)\alpha_n + 2\left[(k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right] + (2k - 1)\alpha_\lambda + M_2 \right\} \right] |q|^n. \end{aligned}$$

This implies that $|P(q) * (1 - q)| > 0$, that is $P(q) * (1 - q) \neq 0$ if

$$|q| > \frac{1}{|a_n|} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2\left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2 \right].$$

Since the only zeros of $P(q) * (1 - q)$ are $q = 1$ and the zeros of $P(q)$. Therefore, $P(q) \neq 0$ for

$$|q| > \frac{1}{|a_n|} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2\left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2 \right].$$

Hence, all the zeros of $P(q)$ lie in :

$$|q| \leq \frac{1}{|a_n|} \left[(2k^{n-\lambda+1} - 1)\alpha_n + 2 \left\{ (k^{n-\lambda} - 1)\alpha_{n-1} + (k^{n-\lambda-1} - 1)\alpha_{n-2} + \cdots + (k^2 - 1)\alpha_{\lambda+1} \right\} + (2k - 1)\alpha_\lambda + M_2 \right].$$

where,

$$M_2 = (|\alpha_0| - \alpha_0) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_0) + (|\delta_0| - \delta_0 + \delta_n).$$

This completes the proof of theorem.

Proof of Theorem 3.9. Consider the polynomial

$$f(q) = \sum_{s=1}^n q^s (a_s - a_{s-1}) + a_0.$$

Let $P(q) \star (1 - q) = f(q) - q^{n+1}a_n$, therefore by Lemma 4.1, $P(q) \star (1 - q) = 0$ iff either $P(q) = 0$ or $P(q) \neq 0$ implies $P(q)^{-1}qP(q) - 1 = 0$, that is $P(q)^{-1}qP(q) = 1$. If $P(q) \neq 0$ then $q = 1$. Therefore, the only zeros of $P(q) \star (1 - q)$ are $q = 1$ and the zeros of $P(q)$. For $|q| = 1$ we get

$$\begin{aligned} |f(q)| &\leq |a_0| + \sum_{s=1}^n |a_s - a_{s-1}| \\ &\leq |a_0| + |a_n - a_{n-1}| + \cdots + |a_{\lambda+1} - a_\lambda| + |a_\lambda - a_{\lambda-1}| + \cdots + |a_1 - a_0| \\ &\leq |a_0| + |k^{n-\lambda+1}a_n - a_{n-1} + a_n - k^{n-\lambda+1}a_n| + \cdots + |k^\lambda a_\lambda - a_{\lambda-1} + a_\lambda - k^\lambda a_\lambda| + |a_{\lambda-1} - a_{\lambda-2}| \\ &\quad + \cdots + |a_2 - a_1| + |a_1 - a_0| \\ &\leq |a_0| + |k^{n-\lambda+1}a_n - a_{n-1}| + (k^{n-\lambda+1} - 1)|a_n| + |k^{n-\lambda}a_{n-1} - a_{n-2}| + (k^{n-\lambda} - 1)|a_{n-1}| \\ &\quad + \cdots + |k^\lambda a_\lambda - a_{\lambda-1}| + (k^\lambda - 1)|a_\lambda| + |a_{\lambda-1} - a_{\lambda-2}| + \cdots + |a_2 - a_1| + |a_1 - a_0|. \end{aligned}$$

Using Lemma 4.2, it follows that

$$\begin{aligned} |f(q)| &\leq |a_0| + (k^{n-\lambda+1}|a_n| - |a_{n-1}|) \cos \theta + (k^{n-\lambda+1}|a_n| + |a_{n-1}|) \sin \theta + (k^{n-\lambda+1} - 1)|a_n| \\ &\quad + \cdots + (k|a_\lambda| - |a_{\lambda-1}|) \cos \theta + (k|a_\lambda| + |a_{\lambda-1}|) \sin \theta + (k - 1)|a_\lambda| + \\ &\quad (|a_{\lambda-1}| - |a_{\lambda-2}|) \cos \theta + (|a_{\lambda-1}| + |a_{\lambda-2}|) \sin \theta + \cdots + (|a_2| - |a_1|) \cos \theta + \\ &\quad (|a_2| + |a_1|) \sin \theta + (|a_1| - |a_0|) \cos \theta + (|a_1| + |a_0|) \sin \theta \\ &\leq k^{n-\lambda+1}|a_n|(\cos \theta + \sin \theta) - |a_{n-1}|(\cos \theta - \sin \theta) + k^{n-\lambda}|a_{n-1}|(\cos \theta + \sin \theta) \\ &\quad - |a_{n-2}|(\cos \theta - \sin \theta) + (k^{n-\lambda} - 1)|a_{n-1}| + \cdots + k^2|a_{\lambda+1}|(\cos \theta + \sin \theta) \\ &\quad - |a_\lambda|(\cos \theta - \sin \theta) + (k^2 - 1)|a_{\lambda+1}| + k|a_\lambda|(\cos \theta + \sin \theta) + (k - 1)|a_{\lambda-1}| \\ &\quad + 2 \sin \theta \sum_{s=0}^{\lambda-1} |a_s| + |a_0|(1 - \sin \theta - \cos \theta). \end{aligned}$$

This implies

$$\begin{aligned} |f(q)| &\leq [k^{n-\lambda+1}|a_n| + k^{n-\lambda}|a_{n-1}| + \cdots + k|a_\lambda|](\cos \theta + \sin \theta) - [|a_{n-1}| + |a_{n-2}| + \cdots + |a_\lambda|](\cos \theta - \sin \theta) \\ &\quad + (k^{n-\lambda+1} - 1)|a_n| + (k^{n-\lambda} - 1)|a_{n-1}| + \cdots + (k - 1)|a_{\lambda-1}| + 2 \sin \theta \sum_{s=0}^{\lambda-1} |a_s| + |a_0|(1 - \cos \theta - \sin \theta). \end{aligned}$$

Proceeding likewise as in the proof of Theorem 3.1, we finally arrive at

$$\begin{aligned} |f(q)| &\leq \left\{ [k^{n-\lambda+1}|a_n| + k^{n-\lambda}|a_{n-1}| + \cdots + k|a_\lambda|](\cos \theta + \sin \theta) - [|a_{n-1}| + |a_{n-2}| + \cdots + |a_\lambda|](\cos \theta - \sin \theta) \right. \\ &\quad \left. + (k^{n-\lambda+1} - 1)|a_n| + (k^{n-\lambda} - 1)|a_{n-1}| + \cdots + (k - 1)|a_{\lambda-1}| + 2 \sin \theta \sum_{s=0}^{\lambda-1} |a_s| + |a_0|(1 - \cos \theta - \sin \theta) \right\} \end{aligned} \quad (5.3)$$

for $|q| \geq 1$. Using 5.3, we have for $|q| \geq 1$

$$\begin{aligned} |P(q) \star (1-q)| &\geq |a_n|q^{n+1} - |f(q)| \\ &\geq |a_n|q^{n+1} - \left[[k^{n-\lambda+1}|a_n| + k^{n-\lambda}|a_{n-1}| + \cdots + k|a_\lambda|](\cos \theta + \sin \theta) \right. \\ &\quad \left. - [|a_{n-1}| + |a_{n-2}| + \cdots + |a_\lambda|](\cos \theta - \sin \theta) + (k^{n-\lambda+1} - 1)|a_n| + (k^{n-\lambda} - 1)|a_{n-1}| + \cdots \right. \\ &\quad \left. + (k-1)|a_{\lambda-1}| + 2 \sin \theta \sum_{s=0}^{\lambda-1} |a_s| + |a_0|(1 - \cos \theta - \sin \theta) \right] |q|^n \quad \text{for } |q| \geq 1. \end{aligned}$$

This implies that $|P(q) \star (1-q)| > 0$, i.e., $P(q) \star (1-q) \neq 0$ if

$$\begin{aligned} |q| &> \frac{1}{|a_n|} \left[[k^{n-\lambda+1}|a_n| + k^{n-\lambda}|a_{n-1}| + \cdots + k|a_\lambda|](\cos \theta + \sin \theta) - [|a_{n-1}| + |a_{n-2}| + \cdots + |a_\lambda|](\cos \theta - \sin \theta) \right. \\ &\quad \left. + (k^{n-\lambda+1} - 1)|a_n| + (k^{n-\lambda} - 1)|a_{n-1}| + \cdots + (k-1)|a_{\lambda-1}| + 2 \sin \theta \sum_{s=0}^{\lambda-1} |a_s| + |a_0|(1 - \cos \theta - \sin \theta) \right]. \end{aligned}$$

But by Lemma 4.1, $P(q) \star (1-q) = 0$ if and only if either $q = 1$ or $P(q) = 0$. Hence all zeros of $P(q)$ lie in

$$\begin{aligned} |q| &\leq \frac{1}{|a_n|} \left\{ [k^{n-\lambda+1}|a_n| + k^{n-\lambda}|a_{n-1}| + \cdots + k|a_\lambda|](\cos \theta + \sin \theta) - [|a_{n-1}| + |a_{n-2}| + \cdots + |a_\lambda|](\cos \theta - \sin \theta) \right. \\ &\quad \left. + (k^{n-\lambda+1} - 1)|a_n| + (k^{n-\lambda} - 1)|a_{n-1}| + \cdots + (k-1)|a_{\lambda-1}| + 2 \sin \theta \sum_{s=0}^{\lambda-1} |a_s| + |a_0|(1 - \cos \theta - \sin \theta) \right\}. \end{aligned}$$

as claimed. This completes the proof of theorem.

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