

Operator inequality related to p -angular distance

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Abstract

Given $p, q \in \mathbb{R}$. The purpose of this paper is to discuss inequalities related to p -angular and q -angular distances for operators. We present some inequalities for absolute value operators which are generalization of inequalities studied by Zou et al. The equality conditions are also investigated.

Keywords: Absolute value operators, Dunkl–Williams inequality, p -angular distance

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1 Introduction

Let $(X, \|\cdot\|)$ be a nontrivial real normed linear space and $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ equipped with the operator norm.

A self-adjoint operator $A \in B(\mathcal{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. We write $A \geq 0$ if A is positive. If $A, B \in B(\mathcal{H})$ are self-adjoint operators such that $B - A \geq 0$ we write $A \leq B$. For $A \in B(\mathcal{H})$, we denote by $|A|$ the absolute value operator of A , that is, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the adjoint operator of A . An operator $A \in B(\mathcal{H})$ is said to be normal if $A^*A = AA^*$.

Clarkson [2] introduced the concept of angular distance for any nonzero vectors x and y in X as $\alpha[x, y] = \|\|x\|^{-1}x - \|y\|^{-1}y\|$. Dunkl and Williams [5] obtained a useful upper bound for the angular distance. They showed that for any nonzero vectors x and y in X ,

$$\alpha[x, y] \leq \frac{4\|x - y\|}{\|x\| + \|y\|}. \quad (1.1)$$

Furthermore, the authors proved that the constant 4 can be replaced by 2 if X is an inner product space. Over the years, many interesting refinements of (1.1) and their reverse inequalities have been obtained. For more information on the well-known Dunkl–Williams inequality and operator versions of it, the reader is referred to [6, 9, 11, 12].

Maligranda [6], considered the p -angular distance ($p \in \mathbb{R}$), as a generalization of the concept of angular distance, between nonzero vectors x and y in X as $\alpha_p[x, y] = \|\|x\|^{p-1}x - \|y\|^{p-1}y\|$. Dehghan [4] introduced the concept of skew angular distance $\beta[x, y] = \|\|y\|^{-1}x - \|x\|^{-1}y\|$ between nonzero vectors x and y in X . In [10], the authors considered the skew p -angular distance ($p \in \mathbb{R}$), as a generalization of the concept of skew angular distance, between nonzero vectors x and y in X as $\beta_p[x, y] := \|\|x\|y\|^{p-1} - y\|x\|^{p-1}\|$.

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Recently Dadipour et al.[3] introduced several operator versions of the Dunkl–Williams inequality with respect to the p -angular distance as a generalization of both the main result of Pečarić and Rajić [9] and that of Saito and Tominaga [12]. Also, Afkhami and Dehghan [1], studied the operator version of $\beta[x, y]$, and also compared the operator versions of $\alpha[x, y]$ and $\beta[x, y]$. For $A, B \in B(\mathcal{H})$, let $A = U|A|$ and $B = V|B|$ be polar decompositions of A and B , respectively. Saito and Tominaga [12, Theorem 2.3] obtained an inequality for absolute value operators as follows:

$$|(U - V)|A||^2 \leq r|A - B|^2 + s(|A| - |B|)^2, \quad (1.2)$$

where $r, s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$.

In this paper, we first introduce an operator version of the skew p -angular distance for Hilbert space operators, and generalize the operator inequality $\alpha < \beta$ presented in [1]. Next, we obtain some inequalities for absolute value operators which are generalization of inequalities due to Zou et al. As a consequence, we obtain some operator inequalities related to p -angular and q -angular distances which are a generalization and also a refinement of the main result of [3], in turn, [9] that have been proven in a different way. Throughout this paper, we assume that $r, s > 1$, with $\frac{1}{r} + \frac{1}{s} = 1$, and

$$\alpha_p[A, B] := |A|A|^{p-1} - B|B|^{p-1}| \quad \text{and} \quad \beta_p[A, B] := |A|B|^{p-1} - B|A|^{p-1}|,$$

where $p, q \in \mathbb{R}$ and $A, B \in B(H)$ are such that $|A|$ and $|B|$ are invertible if necessary.

2 Dunkl–Williams type inequalities related to p -angular distance

In this section, we present operator inequalities associated with the p -angular and skew p -angular distances which are operator versions of the Dunkl–Williams inequality. We start by comparing operator version of $\alpha_p[x, y]$ and $\beta_p[x, y]$ as a generalization of [1, Theorem 2.8]. For this purpose, we need the following lemma.

Lemma 2.1. Let $A, B \in B(\mathcal{H})$ be positive operators such that $AB = BA$.

- (i) If $p \geq 0$, then $(A - B)(A^p - B^p) \geq 0$.
- (ii) If $p < 0$ and A, B are invertible, then $(A - B)(A^p - B^p) \leq 0$.

Proof . By exchanging A and B with $A + \epsilon I$ and $B + \epsilon I$ respectively, we can assume that $A, B > 0$. First, suppose that $p = \frac{m}{n}$ is a rational number in which $m, n \geq 1$ are integer numbers. We have

$$\begin{aligned} (A - B)(A^p - B^p) &= (A - B)(A^{\frac{m}{n}} - B^{\frac{m}{n}}) \\ &= (A - B)(A^{\frac{1}{n}} - B^{\frac{1}{n}})(A^{\frac{m-1}{n}} + A^{\frac{m-2}{n}}B^{\frac{1}{n}} + \dots + B^{\frac{m-1}{n}}) \\ &= \frac{(A - B)^2(A^{\frac{m-1}{n}} + A^{\frac{m-2}{n}}B^{\frac{1}{n}} + \dots + B^{\frac{m-1}{n}})}{A^{\frac{n-1}{n}} + A^{\frac{n-2}{n}}B^{\frac{1}{n}} + \dots + B^{\frac{n-1}{n}}} \geq 0. \end{aligned}$$

- (i) Now let $p \geq 0$, be arbitrary. There exists a sequence $p_n \geq 0$ of rational numbers such that $p_n \rightarrow p$, when $n \rightarrow \infty$. Thus

$$(A - B)(A^p - B^p) = \lim_{n \rightarrow \infty} (A - B)(A^{p_n} - B^{p_n}) \geq 0.$$

- (ii) For $p < 0$, we have

$$(A - B)(A^p - B^p) = A^p B^p (A - B)(B^{-p} - A^{-p}) = -A^p B^p (A - B)(A^{-p} - B^{-p}) \leq 0.$$

□

Theorem 2.2. Let A and B be normal operators such that $AB = BA$, and $|A|$ and $|B|$ are invertible.

- (i) If $p \geq 1$, then

$$\alpha_p^2[A, B] \geq \beta_p^2[A, B]. \quad (2.1)$$

- (ii) If $p = 1$, then

$$\alpha[A, B] = \beta[A, B].$$

- (iii) If $p < 1$, then

$$\alpha_p^2[A, B] < \beta_p^2[A, B]. \quad (2.2)$$

Equality holds if and only if $|A| = |B|$.

Proof . We have

$$\begin{aligned} \beta_p^2[A, B] - \alpha_p^2[A, B] &= |A|B|^{p-1} - B|A|^{p-1}|^2 - |A|A|^{p-1} - B|B|^{p-1}|^2 \\ &= (A|B|^{p-1} - B|A|^{p-1})^*(A|B|^{p-1} - B|A|^{p-1}) - (A|A|^{p-1} - B|B|^{p-1})^*(A|A|^{p-1} - B|B|^{p-1}) \\ &= |B|^{p-1}|A|^2|B|^{p-1} - |B|^{p-1}A^*B|A|^{p-1} - |A|^{p-1}B^*A|B|^{p-1} + |A|^{p-1}|B|^2|A|^{p-1} - |A|^{2p} \\ &\quad + |A|^{p-1}A^*B|B|^{p-1} + |B|^{p-1}B^*A|A|^{p-1} - |B|^{2p} \\ &= |A|^2|B|^{2p-2} + |B|^{p-1}(B^*A - A^*B)|A|^{p-1} + |A|^{p-1}(A^*B - B^*A)|B|^{p-1} + |B|^2|A|^{2p-2} - |A|^{2p} - |B|^{2p} \\ &= -(|A|^2 - |B|^2) (|A|^{2p-2} - |B|^{2p-2}). \end{aligned}$$

Now, according to Lemma 2.1, if $p \geq 1$, we get (2.1). If $p < 1$, we get (2.2). The assertion in the case of $p = 1$ and also equality condition is clear. $\square$

In the following, we obtain some new operator inequalities by using the polar decomposition of operators. Equality conditions are also presented. The next lemma will be used to prove equality conditions.

Lemma 2.3. Let $V \in B(\mathcal{H})$ be such that $|V| \leq I$. Then for any operator $A \in B(\mathcal{H})$ which $|A|^2 = A^*|V|^2A$, we have $A = |V|^2A$.

Proof . $A^*A = A^*V^*VA$ implies $A^*(I - V^*V)A = 0$, and so $A^*(I - V^*V)^{\frac{1}{2}}(I - V^*V)^{\frac{1}{2}}A = 0$. Then, we have $|(I - V^*V)^{\frac{1}{2}}A|^2 = 0$. This implies that $(I - V^*V)^{\frac{1}{2}}A = 0$, and hence, $(I - V^*V)A = 0$. This means that $A = V^*VA$. $\square$

The following theorem is a generalization of [13, Theorem 2.1] (when $p = 0, q = 1$), and thus it is a generalization and refinement of (1.2).

Theorem 2.4. Let $A, B \in B(\mathcal{H})$ be operators with polar decomposition $A = U|A|$ and $B = V|B|$ and let $q \geq p \geq 0$. Then

$$\begin{aligned} |(U|A|^p - V|B|^p)|A|^{q-p}|^2 &\leq |U|A|^q - V|B|^q|^2 + ||B|^p(|A|^{q-p} - |B|^{q-p})|^2 - (T + T^*) \\ &\leq r|U|A|^q - V|B|^q|^2 + s||B|^p(|A|^{q-p} - |B|^{q-p})|^2, \end{aligned} \quad (2.3)$$

where $T = (|A|^{q-p} - |B|^{q-p})|B|^pV^*(U|A|^q - V|B|^q)$. Equality in the first part of (2.3) holds if and only if

$$|B|^p(|A|^{q-p} - |B|^{q-p}) = V^*V|B|^p(|A|^{q-p} - |B|^{q-p}), \quad (2.4)$$

and equality in the second part of (2.3) holds if and only if

$$|B|^p(|A|^{q-p} - |B|^{q-p}) = V^*V|B|^p(|A|^{q-p} - |B|^{q-p}),$$

and

$$r(U|A|^q - V|B|^q) = sV|B|^p(|B|^{q-p} - |A|^{q-p}). \quad (2.5)$$

Proof . Since $V^*V \leq I$, we have

$$\begin{aligned} |(U|A|^p - V|B|^p)|A|^{q-p}|^2 &= |U|A|^q - V|B|^q - V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ &= |U|A|^q - V|B|^q|^2 + |V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 - (T + T^*) \\ &\leq |U|A|^q - V|B|^q|^2 + ||B|^p(|A|^{q-p} - |B|^{q-p})|^2 - (T + T^*). \end{aligned}$$

This proves the first part of (2.3). Next, we prove the second part of (2.3). To prove the second one, since $(r-1)(s-1) = 1$ and $V^*V \leq I$, we have

$$\begin{aligned} & r|U|A|^q - V|B|^q|^2 + s||B|^p(|A|^{q-p} - |B|^{q-p})|^2 - (|U|A|^q - V|B|^q|^2 + ||B|^p(|A|^{q-p} - |B|^{q-p})|^2 - (T + T^*)) \\ &= (r-1)|U|A|^q - V|B|^q|^2 + (s-1)||B|^p(|A|^{q-p} - |B|^{q-p})|^2 + (T + T^*) \\ &\geq (r-1)|U|A|^q - V|B|^q|^2 + (s-1)|V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 + (T + T^*) \\ &= \left| \sqrt{r-1}(U|A|^q - V|B|^q) + \sqrt{s-1}V|B|^p(|A|^{q-p} - |B|^{q-p}) \right|^2 \geq 0. \end{aligned}$$

This implies the second part inequality. By the above proof, equality holds in the first part on (2.3) if and only if

$$||B|^p(|A|^{q-p} - |B|^{q-p})|^2 = |V|B|^p(|A|^{q-p} - |B|^{q-p})|^2,$$

which according to the Lemma 2.3 it is equivalent to (2.4). Equality in the second part of (2.3) holds if and only if we have (2.4), and also

$$\sqrt{r-1}(U|A|^q - V|B|^q) = \sqrt{s-1}V|B|^p(|A|^{q-p} - |B|^{q-p}),$$

which is equivalent to (2.5). $\square$

Remark 2.5. By interchanging the roles of operators A and B in (2.3), for all $q \geq p \geq 0$, we have

$$\begin{aligned} |(U|A|^p - V|B|^p)|B|^{q-p}|^2 &\leq |U|A|^q - V|B|^q|^2 + ||A|^p(|A|^{q-p} - |B|^{q-p})|^2 - (T + T^*) \\ &\leq r|U|A|^q - V|B|^q|^2 + s||A|^p(|A|^{q-p} - |B|^{q-p})|^2, \end{aligned} \quad (2.6)$$

where $T = (|A|^{q-p} - |B|^{q-p})|A|^p U^*(U|A|^q - V|B|^q)$. Equality in the first part of (2.6) holds if and only if

$$|A|^p(|A|^{q-p} - |B|^{q-p}) = U^*U|A|^p(|A|^{q-p} - |B|^{q-p}),$$

and equality in the second part of (2.6) holds if and only if

$$|A|^p(|A|^{q-p} - |B|^{q-p}) = U^*U|A|^p(|A|^{q-p} - |B|^{q-p})$$

and

$$r(U|A|^q - V|B|^q) = sU|A|^p(|B|^{q-p} - |A|^{q-p}).$$

Corollary 2.6. Let $A, B \in B(H)$ be such that $|A|$ and $|B|$ are invertible, and $p, q \in \mathbb{R}$. Then

$$\alpha_p^2[A, B] \leq |A|^{p-q} \left(r\alpha_q^2[A, B] + s||B|^p(|A|^{q-p} - |B|^{q-p})|^2 \right) |A|^{p-q}, \quad (2.7)$$

$$\alpha_p^2[A, B] \leq |B|^{p-q} \left(r\alpha_q^2[A, B] + s||A|^p(|A|^{q-p} - |B|^{q-p})|^2 \right) |B|^{p-q}. \quad (2.8)$$

Moreover equality holds in (2.7) if and only if

$$(r-1)(A|A|^{q-1} - B|B|^{q-1})|A|^{p-q} = B|B|^{q-1}(|A|^{p-q} - |B|^{p-q}),$$

and equality holds in (2.8) if and only if

$$(r-1)(A|A|^{q-1} - B|B|^{q-1})|B|^{p-q} = A|A|^{q-1}(|A|^{p-q} - |B|^{p-q}).$$

Proof . Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B respectively. According to (2.3), we have

$$|(U|A|^p - V|B|^p)|A|^{q-p}|^2 \leq r|U|A|^q - V|B|^q|^2 + s||B|^p(|A|^{q-p} - |B|^{q-p})|^2. \quad (2.9)$$

Since $|A|$ and $|B|$ are invertible, the condition $q \geq p \geq 0$ is not necessary and (2.9) is equivalent to

$$|A|^{q-p} |(A|A|^{p-1} - B|B|^{p-1})|^2 |A|^{q-p} \leq r|A|A|^{q-1} - B|B|^{q-1}|^2 + s||B|^p(|A|^{q-p} - |B|^{q-p})|^2.$$

From which we get (2.7). By the same method and using (2.6), we can easily get (2.8). By the above proof, equality conditions are obvious. $\square$

Remark 2.7. Obviously, the inequalities (2.7) and (2.8) are generalization of the inequalities which are due to Dadipour [3], in turn, Pečarić and Rajić [9], and have proved by different method.

Theorem 2.8. Let $A, B \in B(\mathcal{H})$ be operators with polar decomposition $A = U|A|$ and $B = V|B|$ and let $q \geq p \geq 1$. Then

$$\begin{aligned} |(A|B|^{p-1} - B|A|^{p-1})|B|^{q-p}|^2 &\leq \beta_q^2[A, B] + ||B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 + T + T^* \\ &\leq r\beta_q^2[A, B] + s||B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2, \end{aligned} \quad (2.10)$$

where $T = (|A|^{q-p} - |B|^{q-p})|A|^{p-1}|B|V^*(A|B|^{q-1} - B|A|^{q-1})$. Equality in the first part of (2.10) holds if and only if

$$|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p}) = V^*V|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p}), \quad (2.11)$$

and equality in the second part of (2.10) holds if and only if

$$|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p}) = V^*V|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p}),$$

and

$$r(A|B|^{q-1} - B|A|^{q-1}) = sV|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p}). \quad (2.12)$$

Proof . Since $V^*V \leq I$, we get

$$\begin{aligned} |(A|B|^{p-1} - B|A|^{p-1})|B|^{q-p}|^2 &= |(A|B|^{q-1} - B|A|^{q-1}) + V|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 \\ &= \beta_q^2[A, B] + |V|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 + T + T^* \\ &\leq \beta_q^2[A, B] + ||B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 + T + T^*, \end{aligned}$$

which is the first inequality. To prove the second one, since $(r-1)(s-1) = 1$ and $V^*V \leq I$, we have

$$\begin{aligned} r\beta_q^2[A, B] + s||B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 - (\beta_q^2[A, B] + ||B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 + T + T^*) \\ = (r-1)\beta_q^2[A, B] + (s-1)||B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 - (T + T^*) \\ \geq (r-1)\beta_q^2[A, B] + (s-1)|V|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 - (T + T^*) \\ = |\sqrt{r-1}(A|B|^{q-1} - B|A|^{q-1}) - \sqrt{s-1}V|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 \geq 0. \end{aligned}$$

According to the above proof, equality in the first part of (2.10) holds if and only if

$$|V|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 = ||B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2,$$

which according to the Lemma 2.3, it is equivalent to (2.11). Equality in the second part of (2.10) holds if and only if we have (2.11), and also

$$\sqrt{r-1}(A|B|^{q-1} - B|A|^{q-1}) = \sqrt{s-1}V|B||A|^{p-1}(|A|^{q-p} - |B|^{q-p}).$$

Since, $r-1 = \frac{r}{s}$ and $s-1 = \frac{s}{r}$, we get (2.12). $\square$

Remark 2.9. By interchanging the roles of operators A and B in (2.10), we get

$$\begin{aligned} |(A|B|^{p-1} - B|A|^{p-1})|A|^{q-p}|^2 &\leq \beta_q^2[A, B] + ||A||B|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 + T + T^* \\ &\leq r\beta_q^2[A, B] + s||A||B|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2, \end{aligned} \quad (2.13)$$

where $T = (|A|^{q-p} - |B|^{q-p})|B|^{p-1}|A|U^*(A|B|^{q-1} - B|A|^{q-1})$. Equality in the first part of (2.13) holds if and only if

$$|A||B|^{p-1}(|A|^{q-p} - |B|^{q-p}) = U^*U|A||B|^{p-1}(|A|^{q-p} - |B|^{q-p}),$$

and equality in the second part of (2.13) holds if and only if

$$|A||B|^{p-1}(|A|^{q-p} - |B|^{q-p}) = U^*U|A||B|^{p-1}(|A|^{q-p} - |B|^{q-p}),$$

and

$$r(A|B|^{q-1} - B|A|^{q-1}) = sU|A||B|^{p-1}(|A|^{q-p} - |B|^{q-p}).$$

Corollary 2.10. Let $A, B \in B(\mathcal{H})$ be such that $|A|$ and $|B|$ are invertible and $p, q \in \mathbb{R}$. Then

$$\beta_p^2[A, B] \leq |B|^{p-q} \left(r\beta_q^2[A, B] + s||B||A|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 \right) |B|^{p-q}, \quad (2.14)$$

$$\beta_p^2[A, B] \leq |A|^{p-q} \left(r\beta_q^2[A, B] + s||A||B|^{p-1}(|A|^{q-p} - |B|^{q-p})|^2 \right) |A|^{p-q}. \quad (2.15)$$

Moreover equality holds in (2.14), if and only if

$$r(A|B|^{q-1} - B|A|^{q-1}) = sB|A|^{p-1}(|A|^{p-q} - |B|^{p-q}),$$

and equality holds in (2.15) if and only if

$$r(A|B|^{p-1} - B|A|^{p-1}) = sA|B|^{q-1}(|A|^{q-p} - |B|^{q-p}).$$

Proof . According to Theorem 2.8, Remark 2.9 and invertibility of $|A|$ and $|B|$, we can easily get the desired results. $\square$

Theorem 2.11. Let $A, B \in B(\mathcal{H})$ be operators with polar decomposition $A = U|A|$ and $B = V|B|$ and let $q \geq 1 \geq p \geq 0$. Then

$$\begin{aligned} |(U|A|^p - V|B|^p)|A|^{1-p}|B|^{q-1}|^2 &\leq \beta_q^2[A, B] + ||B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 - (T + T^*) \\ &\leq r\beta_q^2[A, B] + s||B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2, \end{aligned} \quad (2.16)$$

where $T = (|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1})|B|^pV^*(A|B|^{q-1} - B|A|^{q-1})$. Equality in the first part of (2.16) holds if and only if

$$|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1}) = V^*V|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1}), \quad (2.17)$$

and equality in the second part of (2.16) holds if and only if

$$|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1}) = V^*V|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1}),$$

and

$$r(A|B|^{q-1} - B|A|^{q-1}) = sV|B|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1}). \quad (2.18)$$

Proof . Since $V^*V \leq I$, we get

$$\begin{aligned} |(U|A|^p - V|B|^p)|A|^{1-p}|B|^{q-1}|^2 &= |(U|A||B|^{q-1} - V|B||A|^{q-1}) + V|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 \\ &= |A|B|^{q-1} - B|A|^{q-1}|^2 + |V|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 - (T + T^*) \\ &\leq \beta_q^2[A, B] + ||B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 - (T + T^*), \end{aligned}$$

which is the first inequality. To prove the second one, since $(r-1)(s-1) = 1$ and $V^*V \leq I$, we have

$$\begin{aligned} r\beta_q^2[A, B] + s||B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 &- (\beta_q^2[A, B] + ||B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 - (T + T^*)) \\ &= (r-1)\beta_q^2[A, B] + (s-1)||B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 + T + T^* \\ &\geq (r-1)\beta_q^2[A, B] + (s-1)|V|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 + T + T^* \\ &= |\sqrt{r-1}(A|B|^{q-1} - B|A|^{q-1}) + \sqrt{s-1}V|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 \geq 0. \end{aligned}$$

According to the above proof, equality in the first part of (2.16) holds if and only if

$$|V|B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 = ||B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2,$$

which according to the Lemma 2.3, it is equivalent to (2.17). Equality in the second part of (2.16) holds if and only if we have (2.17), and also

$$\sqrt{r-1}(A|B|^{q-1} - B|A|^{q-1}) = \sqrt{s-1}V|B|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1}).$$

Since, $r-1 = \frac{r}{s}$ and $s-1 = \frac{s}{r}$, it is equivalent to (2.18). $\square$

Remark 2.12. By interchanging the roles of operators A and B in (2.16), we get

$$\begin{aligned} |(U|A|^p - V|B|^p)|B|^{1-p}|A|^{q-1}|^2 &\leq \beta_q^2[A, B] + ||A|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1})|^2 - (T + T^*) \\ &\leq r\beta_q^2[A, B] + s||A|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1})|^2, \end{aligned} \quad (2.19)$$

where $T = (|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1})|A|^p U^*(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1})$. Equality in the first part of (2.19) holds if and only if

$$|A|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1}) = U^*U|A|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1}),$$

and equality in the second part of (2.19) holds if and only if

$$|A|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1}) = U^*U|A|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1}),$$

and

$$r(A|B|^{q-1} - B|A|^{q-1}) = sU|A|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1}).$$

Corollary 2.13. Let $A, B \in B(\mathcal{H})$ be such that $|A|$ and $|B|$ are invertible, and $p, q \in \mathbb{R}$. Then

$$\alpha_p^2[A, B] \leq |A|^{p-1}|B|^{1-q} \left(r\beta_q^2[A, B] + s||B|^p(|B|^{1-p}|A|^{q-1} - |A|^{1-p}|B|^{q-1})|^2 \right) |B|^{1-q}|A|^{p-1}, \quad (2.20)$$

$$\alpha_p^2[A, B] \leq |B|^{p-1}|A|^{1-q} \left(r\beta_q^2[A, B] + s||A|^p(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1})|^2 \right) |A|^{1-q}|B|^{p-1}. \quad (2.21)$$

Moreover equality holds in (2.20) if and only if

$$r(A|B|^{q-1} - B|A|^{q-1}) = sB|B|^{p-1}(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1})$$

and equality holds in (2.21) if and only if

$$r(A|B|^{q-1} - B|A|^{q-1}) = sA|A|^{p-1}(|A|^{1-p}|B|^{q-1} - |B|^{1-p}|A|^{q-1}).$$

Proof . According to Theorem 2.11, Remark 2.12 and invertibility of $|A|$ and $|B|$, we can easily get the desired results. $\square$

3 Some other operator inequalities

In this section, we prove some inequalities for absolute value operators which are generalization of some main results of [13]. As a consequence of it, we also present inequalities associated with p -angular and q -angular distances which are a refinements of the results already obtained in previous sections. We start with the following lemma.

Lemma 3.1. [13] Let $A, B \in B(\mathcal{H})$. If $1 < r \leq 2$, then

$$|A - B|^2 + \frac{2}{r}|(r-1)A + B|^2 \leq r|A|^2 + s|B|^2 \leq |A - B|^2 + \frac{2}{s}|A + (s-1)B|^2. \quad (3.1)$$

If $r > 2$, then

$$|A - B|^2 + \frac{2}{s}|A + (s-1)B|^2 \leq r|A|^2 + s|B|^2 \leq |A - B|^2 + \frac{2}{r}|(r-1)A + B|^2. \quad (3.2)$$

Theorem 3.2. Let $A, B \in B(\mathcal{H})$ be operators with polar decomposition $A = U|A|$ and $B = V|B|$, and $q \geq p \geq 0$.

i). If $1 < r \leq 2$, then

$$\begin{aligned} |(U|A|^p - V|B|^p)|A|^{q-p}|^2 + \frac{2}{r}|(r-1)(U|A|^q - V|B|^q) + V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ \leq r|U|A|^q - V|B|^q|^2 + s||B|^p(|A|^{q-p} - |B|^{q-p})|^2. \end{aligned} \quad (3.3)$$

ii). If $r > 2$, then

$$\begin{aligned} & |(U|A|^p - V|B|^p)|A|^{q-p}|^2 + \frac{2}{s}|U|A|^q - V|B|^q + (s-1)V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \leq r|U|A|^q - V|B|^q|^2 + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2. \end{aligned} \quad (3.4)$$

Proof . For $1 < r \leq 2$, applying the first part of (3.1) on the operators $U|A|^q - V|B|^q$ and $V|B|^p(|A|^{q-p} - |B|^{q-p})$, we have

$$\begin{aligned} & r|U|A|^q - V|B|^q|^2 + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \geq r|U|A|^q - V|B|^q|^2 + s|V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \geq |(U|A|^p - V|B|^p)|A|^{q-p}|^2 + \frac{2}{r}(r-1)(U|A|^q - V|B|^q) + V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \end{aligned}$$

For $r > 2$, applying the first part of (3.2) on the operators $U|A|^q - V|B|^q$ and $V|B|^p(|A|^{q-p} - |B|^{q-p})$, we have

$$\begin{aligned} & r|U|A|^q - V|B|^q|^2 + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \geq r|U|A|^q - V|B|^q|^2 + s|V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \geq |(U|A|^p - V|B|^p)|A|^{q-p}|^2 + \frac{2}{s}|U|A|^q - V|B|^q + (s-1)V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \end{aligned}$$

□

Remark 3.3. Interchanging the operators A and B in the inequalities (3.3) and (3.4), we have the following:
If $1 < r \leq 2$, then

$$\begin{aligned} & |(U|A|^p - V|B|^p)|B|^{q-p}|^2 + \frac{2}{r}|(1-r)(U|A|^q - V|B|^q) - V|A|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \leq r|U|A|^q - V|B|^q|^2 + s|A|^p(|A|^{q-p} - |B|^{q-p})|^2. \end{aligned}$$

If $r > 2$, then

$$\begin{aligned} & |(U|A|^p - V|B|^p)|B|^{q-p}|^2 + \frac{2}{s}|U|A|^q - V|B|^q + (s-1)V|A|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \leq r|U|A|^q - V|B|^q|^2 + s|A|^p(|A|^{q-p} - |B|^{q-p})|^2. \end{aligned}$$

The following corollary gives a refinement of (2.7) and (2.8).

Corollary 3.4. Let $A, B \in B(H)$ be such that $|A|$ and $|B|$ are invertible, and $p, q \in \mathbb{R}$. If $1 < r \leq 2$, then

$$\begin{aligned} \alpha_p^2[A, B] & \leq |A|^{p-q}(r\alpha_q^2[A, B] + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \quad - \frac{2}{r}(r-1)(A|A|^{q-1} - B|B|^{q-1}) + V|B|^p(|A|^{q-p} - |B|^{q-p})|^2)|A|^{p-q}, \end{aligned}$$

and

$$\begin{aligned} \alpha_p^2[A, B] & \leq |B|^{p-q}(r\alpha_q^2[A, B] + s|A|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \quad - \frac{2}{r}(r-1)(A|A|^{q-1} - B|B|^{q-1}) + V|A|^p(|A|^{q-p} - |B|^{q-p})|^2)|B|^{p-q}. \end{aligned}$$

If $r > 2$, then

$$\begin{aligned} \alpha_p^2[A, B] & \leq |A|^{p-q}(r\alpha_q^2[A, B] + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \quad - \frac{2}{s}|A|A|^{q-1} - B|B|^{q-1} + (s-1)V|B|^p(|A|^{q-p} - |B|^{q-p})|^2)|A|^{p-q}, \end{aligned}$$

and

$$\begin{aligned} \alpha_p^2[A, B] & \leq |B|^{p-q}(r\alpha_q^2[A, B] + s|A|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ & \quad - \frac{2}{s}|A|A|^{q-1} - B|B|^{q-1} + (s-1)V|A|^p(|A|^{q-p} - |B|^{q-p})|^2)|B|^{p-q}. \end{aligned}$$

Proof . According to Theorem 3.2, Remark 3.3 and invertibility of $|A|$ and $|B|$, we get the desired inequalities. $\square$

Theorem 3.5. Let $A, B \in B(\mathcal{H})$ be operators with polar decomposition $A = U|A|$ and $B = V|B|$ and $q \geq p \geq 0$. If $1 < r \leq 2$, then

$$\begin{aligned} |(U|A|^p - V|B|^p)|A|^{q-p}|^2 &\geq r|U|A|^q - V|B|^q|^2 + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ &\quad - \frac{2}{s}|U|A|^q - V|B|^q + (s-1)V|B|^p(|A|^{q-p} - |B|^{q-p})|^2. \end{aligned} \quad (3.5)$$

If $r > 2$, then

$$\begin{aligned} |(U|A|^p - V|B|^p)|A|^{q-p}|^2 &\geq r|U|A|^q - V|B|^q|^2 + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ &\quad - \frac{2}{r}[(r-1)(U|A|^q - V|B|^q) + V|B|^p(|A|^{q-p} - |B|^{q-p})]^2. \end{aligned} \quad (3.6)$$

Proof . For $1 < r \leq 2$, applying the second part of (3.1) on the operators $U|A|^q - V|B|^q$ and $V|B|^p(|A|^{q-p} - |B|^{q-p})$, we have

$$\begin{aligned} r|U|A|^q - V|B|^q|^2 + s|V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ \leq |(U|A|^p - V|B|^p)|A|^{q-p}|^2 + \frac{2}{s}|U|A|^q - V|B|^q + (s-1)V|B|^p(|A|^{q-p} - |B|^{q-p})|^2. \end{aligned}$$

For $r > 2$, applying the second part of (3.2) on the operators $U|A|^q - V|B|^q$ and $V|B|^p(|A|^{q-p} - |B|^{q-p})$, we have

$$\begin{aligned} r|U|A|^q - V|B|^q|^2 + s|V|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ \leq |(U|A|^p - V|B|^p)|A|^{q-p}|^2 + \frac{2}{r}[(r-1)(U|A|^q - V|B|^q) + V|B|^p(|A|^{q-p} - |B|^{q-p})]^2. \end{aligned}$$

$\square$

Remark 3.6. Interchanging the operators A and B in the inequalities (3.5) and (3.6), we have the following: If $1 < r \leq 2$, then

$$\begin{aligned} |(U|A|^p - V|B|^p)|B|^{q-p}|^2 &\geq r|U|A|^q - V|B|^q|^2 + s|A|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ &\quad - \frac{2}{s}|U|A|^q - V|B|^q + (s-1)V|A|^p(|A|^{q-p} - |B|^{q-p})|^2. \end{aligned} \quad (3.7)$$

If $r > 2$, then

$$\begin{aligned} |(U|A|^p - V|B|^p)|B|^{q-p}|^2 &\geq r|U|A|^q - V|B|^q|^2 + s|A|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ &\quad - \frac{2}{r}[(r-1)(U|A|^q - V|B|^q) + V|A|^p(|A|^{q-p} - |B|^{q-p})]^2. \end{aligned} \quad (3.8)$$

Corollary 3.7. Let $A, B \in B(H)$ be such that $|A|$ and $|B|$ are invertible, and $p, q \in \mathbb{R}$. If $1 < r \leq 2$, then

$$\begin{aligned} \alpha_p^2[A, B] &\geq |A|^{p-q}(r\alpha_q^2[A, B] + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ &\quad - \frac{2}{s}[(A|A|^{q-1} - B|B|^{q-1}) + (s-1)V|B|^p(|A|^{q-p} - |B|^{q-p})]^2)|A|^{p-q}. \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \alpha_p^2[A, B] &\geq |B|^{p-q}(r\alpha_q^2[A, B] + s|A|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ &\quad - \frac{2}{s}[(A|A|^{q-1} - B|B|^{q-1}) + (s-1)V|A|^p(|A|^{q-p} - |B|^{q-p})]^2)|B|^{p-q}. \end{aligned} \quad (3.10)$$

If $r > 2$, then

$$\begin{aligned} \alpha_p^2[A, B] &\geq |A|^{p-q}(r\alpha_q^2[A, B] + s|B|^p(|A|^{q-p} - |B|^{q-p})|^2 \\ &\quad - \frac{2}{r}[(r-1)(A|A|^{q-1} - B|B|^{q-1}) + V|B|^p(|A|^{q-p} - |B|^{q-p})]^2)|A|^{p-q}. \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \alpha_p^2[A, B] &\geq |B|^{p-q} \left(r\alpha_q^2[A, B] + s \left| |A|^p(|A|^{q-p} - |B|^{q-p}) \right|^2 \right. \\ &\quad \left. - \frac{2}{r} \left| (r-1)(A|A|^{q-1} - B|B|^{q-1}) + V|A|^p(|A|^{q-p} - |B|^{q-p}) \right|^2 \right) |B|^{p-q}. \end{aligned} \quad (3.12)$$

Proof . According to Theorem 3.5, Remark 3.6 and invertibility of $|A|$ and $|B|$, we get the desired inequalities. $\square$

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