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Comprehensive subclass of univalent functions associated with the Gegenbauer polynomial and the Bahalola operator

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Abstract

Our objective in this paper is to introduction a newly-constracted subclass of univalent functions.The Gegenbauer Polynomials are very succesfull in the theory of polynomials because they extend many polynomials such as legender, Chebyshev polynimials etc. Also the Babolala operator for a fixed real number and is used. Furthermore by applying the convolution structures and subordination property, the new subclass is obtaind. For this new subclass firstle the share coefficient bounds are concluded. These coefficient estimates are very important to obtain the geometric properties. So after that we obtain the radii of starlikeness, convexity and close-to-convexity. Also under some restrictions on parameters, we show that the newly defined subclass has convolution preserring property. Finaly the convexity of this subclass is pointed out. Our objective in this paper is to introduction a newly-constracted subclass of univalent functions.The Gegenbauer Polynomials are very succesfull in the theory of polynomials because they extend many polynomials such as legender, Chebyshev polynimials etc. Also the Babolala operator for a fixed real number and is used. Furthermore by applying the convolution structures and subordination property, the new subclass is obtaind. For this new subclass firstle the share coefficient bounds are concluded. These coefficient estimates are very important to obtain the geometric properties. So after that we obtain the radii of starlikeness, convexity and close-to-convexity. Also under some restrictions on parameters, we show that the newly defined subclass has convolution preserring property. Finally, the convexity of this subclass is pointed out.

Keywords: Univalent function, Gegenbauer Polynomial, Bahalola operator, Convolution, Subordination, Convex set, coefficient estimate, radii of starlikeness, Convexity and close-to-convexity
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1 Introduction

Let A denote the class of functions $f(z)$ of the type

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the open unit disk

$$\mathbb{D} = \{z \in C \ : \ |z| < 1\}. \tag{1.2}$$

We denote by S the subclass of A consisting of functions which are univalent in D. Also

$$N = \left\{ f(z) \in A \ : \ f(z) = z - \sum_{n=1}^{+\infty} a_n z^n \cdot a \geq 0 \right\}, \tag{1.3}$$

is a subclass of A consisting of functions with negative coefficients, See [7]. For any real numbers $x \cdot y \in \mathbb{R}$ with $x \geq 0$, $-1 \leq y \leq 1$ and $z \in \mathbb{D}$ the generating function of Gegenbauer Polynomials is given by

$$H_x(z \cdot y) = (z^2 - 2yz + 1)^{-x}. \tag{1.4}$$

Also, for any fixed y, $H_x(z \cdot t)$ is analytic on $\mathbb{D}$ and it's Taylor_Maclaurin seris is introduced by

$$H_x(z \cdot y) = \sum_{n=0}^{+\infty} G_n^x(y) z^n. \tag{1.5}$$

Moreover, the Gegenbauer Polynomials can be defined as follows

$$G_n^x(y) = \frac{2y(n+x-1)G_{n-1}^x(y) - (n+2x-2)G_{n-1}^x(y)}{y}. \tag{1.6}$$

The initial values are

$$G_0^x(y) = 1, \quad G_1^x(y) = 2xy, \quad G_2^x(y) = 2x(x+1)y^2 - x.$$

See [1]-[4]. It is well-known that the special cases of Gegenbauer Polynomials are the Legendre and the Chebyshev polynomials of the second kind. See [6, 8, 10]. The Hadamard product (or convolution) for functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ denoted by $(f * g) * (z)$ is defined by

$$(f * g)(z) = z - \sum_{n=2}^{+\infty} a_n b_n z^n = (g * f)(z), \tag{1.7}$$

where f and g are analytic and univalent in $\mathbb{D}$. Now by using the convolution structure, we introduce the function

$$J(z) = (z(1+2xy) + 1 - H_x(z.y)) * (z(1+2xy) + 1 - H_x(z.y)) * f(z). \tag{1.8}$$

Babalola [5] for a fixed real number σ and $k \in \mathbb{N}$ defined the operator $\mathcal{L}_k^\sigma : \mathcal{N} \longrightarrow \mathcal{N}$ as follows

$$\mathcal{L}_k^\sigma f(z) = (\mathrm{T}_\sigma * \mathcal{T}_{\sigma \cdot n}^{-1} * f)(z),$$

where $\mathcal{T}_{\sigma.k}(z) = \dfrac{z}{(1-z)^{\sigma-(k-1)}}$, $\sigma - (k-1) > 0 \cdot \mathcal{T}_\sigma = \mathcal{T}_{\sigma \cdot 0}$ and $\mathcal{T}_{\sigma \cdot k}^{(-1)}$ is such that $(\mathcal{T}_{\sigma \cdot k} * \mathcal{T}_{\sigma \cdot k}^{(-1)})(z) = \dfrac{z}{1-z}$. Hence, we get

$$\mathcal{L}_k^\sigma J(z) = z - \sum_{n=2}^{+\infty} \left\{ \frac{(\sigma+n-1)!}{\sigma!} \cdot \frac{(\sigma-k)!}{(\sigma+n-k-1)!} \right\} (G_n^x(y))^2 a_n z^n. \tag{1.9}$$

See also [12]. For tow analytic function f and F in $\mathbb{D}$, we say that f is subordinate to F in $\mathbb{D}$, and write $f(z) \prec F(z)$, $(z) \in \mathbb{D}$. If there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$), such that $f(z) = F(w(z)), (z \in \mathbb{D}$ See [9]. Now we consider $Q_k^\sigma(A \cdot B)$ as a subclass of $\mathcal{N}$ for which

$$\frac{z(\mathcal{L}_k^\sigma J(z))'}{f_t(z)} \prec \frac{1+Az}{1+Bz} \tag{1.10}$$

or equivalently

$$\left| \frac{\frac{z(\mathcal{L}_k^\sigma J(z))'}{f_t(z)} - 1}{A - B\frac{(z(\mathcal{L}_k^\sigma J(z))'}{f_t(z)}} \right| < 1, \tag{1.11}$$

where $A = B + (\alpha - \beta)(1 - \eta)$. $-1 \leq \beta < \alpha \leq 1$. $0 < \eta < 1$. $0 \leq t \leq 1$. $f(z) \in \mathcal{N} \cdot \mathcal{L}_k^\sigma J(z)$ is given in $(z \in D)$ and

$$f_t(z) = (1 - t)z + tf(z). \tag{1.12}$$

Such as this class of univalent functions was considered in recent works of many authors. See [11].

2 Main Results

In this section we obtain the sharp coefficient estimates for functions belong to $Q_k^\sigma(A \cdot B)$. Also the convexity of this class is proved.

Theorem 2.1. Let $f(z) = z - \sum_{n=2}^{+\infty} a_n z^n$ be analytic in $\mathbb{D}$ and $\mathcal{L}_k^\sigma J(z)$ is given by (1.9).

Then $f(z) = Q_k^\sigma(A \cdot B)$ if and only if

$$\sum_{n=2}^{+\infty} \left[n \left(\left(\frac{(\sigma + n -)!}{\sigma!} \cdot \frac{(\sigma - k)!}{(\sigma + n - k - 1)!} \right) (G_n^x(y))^2 - t \right) (1 - \beta) + t(\alpha - \beta)(1 - \eta) \right] a_n. \tag{2.1}$$

Proof . Let $|z| = 1$. Then

$$\begin{aligned}
&|z(\mathcal{L}_k^\sigma J(z))' - f_t(z)| - |Af_t(z) - Bz(\mathcal{L}_k^\sigma J(z))'| \\
&\quad = \left| z \left(1 - \sum_{n=2}^{+\infty} \left\{ \frac{(\sigma + n - 1)!}{\sigma!} \cdot \frac{(\sigma - k)!}{(\sigma + n - k - 1)!} \right\} (G_n^z(y))^2 a_n z^{n-1} \right) - (1 - t)z + tf(z) \right| \\
&\quad - \left| A((1 - t)Z + tf(z) - Bz \left(1 - \sum_{n=2}^{+\infty} \left(\frac{(\sigma + n - 1)!}{\sigma!} \cdot \frac{(\sigma - k)!}{(\sigma + n - k - 1)!} \right) (G_n^z(y))^2 a_n z^n \right) \right| \\
&\quad = \left| - \sum_{n=2}^{+\infty} \left[n \left(\left(\frac{(\sigma + n - 1)!}{\sigma!} \cdot \frac{(\sigma - k)!}{(\sigma + n - k - 1)!} \right) (G_n^z(y))^2 - t \right) a_n z^n \right] \right| \\
&\quad - \left| (A - B)z - \sum_{n=2}^{+\infty} (At - Bn \left(\frac{(\sigma + n - 1)!}{\sigma!} \cdot \frac{(\sigma - k)!}{(\sigma + n - k - 1)!} \right) (G_n^z(y))^2 a_n z^n \right|
\end{aligned}$$

By putting

$$\begin{aligned}
&At - Bn \left(\frac{(\sigma + n - 1)!}{\sigma!} \cdot \frac{(\sigma - k)!}{(\sigma + n - k - 1)!} \right) (G_n^z(y))^2 \\
&\quad = t(A - B) - \left(n \left(\frac{(\sigma + n - 1)!}{\sigma!} \cdot \frac{(\sigma - k)!}{(\sigma + n - k - 1)!} \right) (G_n^z(y))^2 - t \right) B.
\end{aligned}$$

The above expression reduces to

$$\leq \left| \sum_{n=2}^{+\infty} \left(n \left(\frac{(\sigma + n - 1)!}{\sigma!} \cdot \frac{(\sigma - k)!}{(\sigma + n - k - 1)!} \right) (G_n^z(y))^2 - t \right) (1 - B) + t(A - B) - (A - B) \right|.$$

So by (2.1) we conclude that $f(z) \in Q_k^\sigma(A\cdot B)$ To prove the converse, let $f(z) \in Q_k^\sigma(A\cdot B)$, the for $z \in \mathbb{D}$, we have

$$\left|\frac{\frac{z(\mathcal{L}_k^\sigma J(z))'}{f_t(z)}-1}{A-B\frac{(z(\mathcal{L}_k^\sigma J(z))'}{f_t(z)}}\right|$$

$$=\left|\frac{z\left(1-\sum\limits_{n=2}^{+\infty} n\left\{\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right\}(G_n^z(y))^2 a_n z^n\right)-(1-t)z-tf(z)}{A\left((1-t)z+tf(z)-Bzz\left(1-\sum\limits_{n=2}^{+\infty}\left\{\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right\}(G_n^z(y))^2 a_n z^{n-1}\right)-(1-t)z+ft(z)\right)}\right|.$$

By letting $\mathrm{Re}(z) \le |z|$ for all z, we get

$$\mathrm{Re}\left\{\frac{\sum\limits_{n=2}^{+\infty}\left[n\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right]a_n z^n}{(A-B)z-\sum\limits_{n=2}^{+\infty}\left[At-Bz\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2\right]a_n z^n}\right\}<1.$$

By letting $z \to 1$ through positive real values and choose the values of z such that $\frac{z(\mathcal{L}_k^\sigma J(z))'}{f_t(z)}$ is real, we conclude the required result. So the proof is complete. We rote that the function

$$F(z)=z-\sum_{n=2}^{+\infty}\frac{(\alpha-\beta)(1-\eta)z^n}{\left[n\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right](1-\beta)+t(\alpha-\beta)(1-\eta)}$$

Shows the sharpness of (2.1). □

Corollary 2.2. If $f(z) \in Q_k^\sigma(A\cdot B)$, then

$$a_n \le \frac{(\alpha-\beta)(1-\eta)}{\left[n\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right](1-\beta)+t(\alpha-\beta)(1-\eta)}\cdot n\ge 2$$

Theorem 2.3. $Q_k^\sigma(A\cdot B)$ is a convex set.

Proof . It is enough to show that if the functions $f_j(z)\cdot(j=1,2,\cdots,s)$ be in the class $Q_k^\sigma(A\cdot B)$, then the function $h(z)=\sum\limits_{j=1}^{s}\gamma_j f_j(z)$ is also in the same class, where $\gamma_j>0$ and $\sum\limits_{j=1}^{s}\gamma_j=1$. By definition of $h(z)$ we obtain

$$h(z)=\sum_{j=1}^{s}\gamma_j(z-\sum_{n=2}^{+\infty}a_{n\cdot j}z^n=z-\sum_{n=2}^{+\infty}\left(\sum_{j=1}^{s}\gamma_j a_{n\cdot j}\right)z^n.$$

But by Theorem 2.1, we have

$$\sum_{n=2}^{+\infty}\left[\left(n\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-\beta)+t(\alpha-\beta)(1-\eta)\right]\sum_{j=1}^{s}\gamma_j a_{n\cdot j}$$
$$\sum_{j=1}^{s}\gamma_j\left\{\sum_{n=2}^{+\infty}n\left(\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-\beta)+t(\alpha-\beta)(1-\eta)\right\}a_{n\cdot j}.$$
$$=\sum_{j=1}^{s}\gamma_j(\alpha-\beta)(1-\eta)=(\alpha-\beta)(1-\eta).$$

Which completes the proof. □

3 Convolution and radii properties

In the section, we show that the class $Q_k^\sigma(A\cdot B)$ with some special parameter is closed under convolution structure. Also the radii of starlikeness, convexity and close_to_convexity are obtained.

Theorem 3.1. Let $f(z)=z-\sum\limits_{n=2}^{+\infty}a_nz^n$ and $g(z)=z-\sum\limits_{n=2}^{+\infty}b_nz^n$ be in the class $Q_k^\sigma(A\cdot B)$ then $(f*g)(z)$ belongs to $Q_k^\sigma(A_0\cdot B)$, where

$$\begin{aligned} A_0 &= B+(\alpha-\beta)(1-\eta_0) \qquad (3.1)\\ \eta_0 &\le \frac{(\alpha-\beta)(1-\eta)^2V}{[V+t(\alpha-\beta)(1-\eta)^2-t(\alpha-\beta)^2(1-\eta)^2} \end{aligned}$$

and

$$V=n\left(\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B). \qquad (3.2)$$

Proof . It is sufficient to show that

$$\sum_{n=2}^{+\infty}\left\{\frac{n\left(\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)}{(\alpha-\beta)(1-\eta)}+t\right\}a_nb_n\le 1.$$

By using the Cauchy_Schwarz inequation, from (2.1) we obtain

$$\sum_{n=2}^{+\infty}\left\{\frac{n\left(\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)}{(\alpha-\beta)(1-\eta)}+t\right\}\sqrt{a_nb_n}\le 1.$$

Hence, we find the largest η_0, such that

$$\sum_{n=2}^{+\infty}\left\{\frac{n\left(\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)}{(\alpha-\beta)(1-\eta)}+t\right\}a_nb_n\le$$

$$\sum_{n=2}^{+\infty}\left\{\frac{n\left(\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)}{(\alpha-\beta)(1-\eta)}+t\right\}\sqrt{a_nb_n}\le 1.$$

or equivalently for $n\ge 2$

$$\sqrt{a_nb_n}\le\frac{\left[\left(n\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)+t(\alpha-\beta)(1-\eta)\right](1-\eta_0)}{\left[\left(n\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)+t(\alpha-\beta)(1-\eta_0)\right](1-\eta)}$$

The above inequality holds if

$$\frac{(\alpha-\beta)(1-\eta)}{n\left(\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)+t(\alpha-\beta)(1-\eta)}\le$$

$$\frac{\left[\left(n\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)+t(\alpha-\beta)(1-\eta)\right](1-\eta_0)}{\left[\left(n\left(\frac{(\sigma+n-1)!}{\sigma!}\cdot\frac{(\sigma-k)!}{(\sigma+n-k-1)!}\right)(G_n^x(y))^2-t\right)(1-B)+t(\alpha-\beta)(1-\eta_0)\right](1-\eta)}$$

or equivalently

$$\eta_0 \le 1 - \frac{(\alpha-\beta)(1-\eta)^2 n \left(\left(\frac{(\sigma+n-1)!}{\sigma!} \cdot \frac{(\sigma-k)!}{(\sigma+n-k-1)!} \right) (G_n^x(y))^2 - t \right) (1-B)}{\left[\left(n \left(\frac{(\sigma+n-1)!}{\sigma!} \cdot \frac{(\sigma-k)!}{(\sigma+n-k-1)!} \right) (G_n^x(y))^2 - t \right) (1-B) + t(\alpha-\beta)(1-\eta) \right]^2 - t(\alpha-\beta)^2(1-\eta)^2}.$$

Therefore

$$\eta_0 \le 1 - \frac{(\alpha-\beta)(1-\eta)^2 V}{[V + t(\alpha-\beta)(1-\eta)]^2 - t(\alpha-\beta)^2(1-\eta)^2}.$$

where V is given in (3.2). So the proof is complete. □

Theorem 3.2. Let $f(z) \in Q_k^\sigma(A \cdot B)$, then

1. $f(z)$ is starlike of order $\theta(0 \le \theta \le 1)$ in $|z| < R_1$ where

$$R_1 = \inf_n \left\{ \frac{V(1-B) + t(\alpha-\beta)(1-\eta)}{(\alpha-\beta)(1-\eta)(n-\theta)} \right\}^{\frac{1}{n-1}} \tag{3.3}$$

2. $f(z)$ is convex of order $\theta(0 \le \theta \le 1)$ in $|z| < R_2$ where

$$R_2 = \inf_n \left\{ \frac{V(1-B) + t(\alpha-\beta)(1-\eta)}{n(\alpha-\beta)(1-\eta)(n-\theta)} (1-\theta) \right\}^{\frac{1}{n-1}} \tag{3.4}$$

3. $f(z)$ is close_to_convex of order $\theta(0 \le \theta \le 1)$ in $|z| < R_2$ where

$$R_3 = \inf_n \left\{ \frac{V(1-B) + t(\alpha-\beta)(1-\eta)}{n(\alpha-\beta)(1-\eta)} (1-\theta) \right\}^{\frac{1}{n-1}} \tag{3.5}$$

In the above relation of R_1, R_2 and R_3, V is given in (3.2).

Proof .

1. for $0 \le \theta \le 1$, we need to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \theta.$$

In the other words, it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{z - \sum_{n=2}^{+\infty} n a_n z^n}{z - \sum_{n=2}^{+\infty} a_n z^n} - 1 \right| = \left| \frac{\sum_{n=2}^{+\infty} (n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{+\infty} a_n z^{n-1}} \right|$$
$$\le \frac{\sum_{n=2}^{+\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{+\infty} a_n |z|^n} < 1 - \theta.$$

or equivalently

$$\sum_{n=2}^{+\infty} \left(\frac{n-\theta}{1-\theta} \right) a_n |z|^{n-1} < 1.$$

By (2.1), it is easy to see that, the above inequality holds if

$$|z|^{n-1} \le \frac{V(1-B) + t(\alpha-\beta)(1-\eta)}{(\alpha-\beta)(1-\eta)(n-\theta)} (1-\theta),$$

where V is given in (3.2). This complete the proof of (1).

2. Since f is convex if and only if $zf'(z)$ is starlike, we obtain (2).
3. We must show that $|f'(z)| \leq 1-\theta$. For $|z| < R_3$ where R_3 is given by (3.5). But

$$|f'(z)-1| = \left|\sum_{n=2}^{+\infty} na_n z^{n-1}\right| \leq \sum_{n=2}^{+\infty} na_n |z|^{n-1}.$$

Thus $|f'(z)-1| < 1-\theta$ if

$$\sum_{n=2}^{+\infty} \frac{na_n}{1-\theta}|z|^{n-1} \leq 1,$$

But by Theorem (2.1), the above inequality holds true if

$$|z|^{n-1} \leq \frac{V(1-B)+t(\alpha-\beta)(1-\eta)}{n(\alpha-\beta)(1-\eta)}(1-\theta).$$

This complete the proof of (3).

□

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