Int. J. Nonlinear Anal. Appl. In Press, 1–8
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2025.37402.5444

Existence, uniqueness, and stability analysis of a fractional differential equation

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(Communicated by Madjid Eshaghi Gordji)

Abstract

This research is dedicated to establishing the existence and uniqueness of solutions for a Caputo-Fabrizino fractional differential system. Additionally, it explores the Hyers-Ulam-Rassias and Hyers-Ulam-Mittag-Leffler stability of these solutions. This study utilizes the alternative fixed point theorem as a fundamental tool in its analysis. In recent papers, authors used the Schauder fixed point theorem and the Laplace transform to prove the stability of Caputo-Fabrizio equations, but we use the alternative fixed point theorem to prove the stability of these equations.

Keywords: Stability theory of functional-differential equations, Functional-differential equations, fractional derivatives, Fixed-point theorems, Mittag-Leffler functions, Control/observation systems
2020 MSC: 34K20, 34K37, 47H10, 33E12, 93C23, 46B25

1 Introduction

Fractional calculus, which extends classical differentiation to non-integer orders, has revolutionized the modeling of memory-dependent phenomena in physics, engineering, and biology [11]. The Caputo derivative dominated early research but faced limitations due to its singular kernel [12]. To address this, Caputo and Fabrizio (2015) introduced a groundbreaking non-singular derivative (CF) using an exponential kernel that enables accurate modeling of material heterogeneities and anomalous diffusion and it is used in fields such as control theory, viscoelasticity, and signal processing [2, 13, 17, 6, 5].

Losada and Nieto established foundational properties of CF operators, including inversion formulas and integral representations [10]. Subsequent work generalized CF calculus to variable-order and higher dimensions . Parallel developments introduced Atangana-Baleanu (AB) derivatives (2016) using Mittag-Leffler kernels [1], expanding applications to viscoelasticity and cardiac electrophysiology.

The Caputo-Fabrizio derivative model is used in various fields such as heat transfer in nanomaterials [15], epidemic dynamics with memory effects, and viscoelastic deformation of polymers. Stability guarantees robustness against perturbations critical for control systems and numerical implementations, and fixed-point theorems became central to stability proofs. The alternative fixed-point theorem (Diaz-Margolis [3]) enables weaker constraints on $\mu(\kappa, P(\kappa))$ and provides explicit error bounds for stability. While fixed-point theorems (Banach, Schauder) have been used for

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Received: April 2025 *Accepted:* November 2025

CF derivatives [10, 13], the alternative fixed-point theorem remains underexplored for CF stability, a gap this work addresses.

This paper explores the stability of Caputo-Fabrizio fractional differential equations, focusing on Hyers-Ulam-Rassias (HUR) and Hyers-Ulam-Mittag-Leffler (HUML) stability for

$$\begin{cases} {}^{CF}\mathbb{D}^{\beta} P(\kappa) = \mu(\kappa, P(\kappa)) \quad \kappa \in [0,b], \\ P(0) = 0, \end{cases} \tag{1.1}$$

in which ${}^{CF}\mathbb{D}^{\beta}$ is the Caputo-Fabrizio differential of order β and $\mu : [0,b] \times \mathbb{R} \to \mathbb{R}$ be continuous and Lipschitz in P. By analyzing these types of stability, the study seeks to understand how solutions react to small perturbations. This understanding is essential for applications that require precise and stable solutions, such as in engineering and scientific modeling. The research offers valuable insights into the behavior of fractional differential equations, thereby enhancing both their practical utility and theoretical understanding.

In the next section, we build on these foundations to outline the key definitions and theorems used in our sustainability analysis. Then we present a physically motivated example covering ferromagnetism, biological systems, and control engineering.

2 Preliminaries

Theorem 2.1 (Alternative fixed point). ([3]). Consider generalized complete metric space (Φ, γ), and Lipschitz mapping $T \colon \Phi \longrightarrow \Phi$ with constant $\beta < 1$. Then, for some $i \in \Phi$ either,

$$\gamma(T^m(i), T^{m+1}(i)) = +\infty \qquad (m \geq 0),$$

or we can find a natural number m_0 such that:

$$\gamma(T^m(i), T^{m+1}(i)) < +\infty \qquad (\forall m \geq m_0),$$

then the followings are true:

1. The fixed point α of T is the convergence point of the sequence $\{T^n(i)\}$;
2. In the set $M = \{\alpha^* \in \Phi \colon \gamma(T^m(i), \alpha^*) < +\infty\}$, α is the unique fixed point of T;
3. For all $\alpha^* \in M$, $\gamma(\alpha^*, \alpha) \leq \frac{1}{1-\beta}\gamma(\alpha^*, T(\alpha^*))$.

Definition 2.2. ([2]). Assume an absolutely continuous function $P : [\alpha, \infty) \to \mathbb{R}$, where $\alpha \leq 0$ then:

$${}^{CF}\mathbb{D}^{\beta} P(\kappa) = \frac{1}{1-\beta} \int_{\alpha}^{\kappa} \exp^{\frac{-\beta(\kappa-\nu)}{1-\beta}} P'(\nu)\, d\nu, \qquad for \quad \kappa \geq 0, \tag{2.1}$$

is Caputo-Fabrizio fractional derivative.

Definition 2.3. ([10]). The function $P : [\alpha, \infty) \to \mathbb{R}$ is an absolutely continuous , then the Caputo-Fabrizio integral of order β is defined as

$${}^{CF}\mathbb{I}^{\beta} P(\kappa) = (1-\beta)[P(\kappa) - P_0] + \beta \int_0^{\kappa} P(\nu)\, d\nu, \tag{2.2}$$

with $P_0 = \int_{\alpha}^{0} e^{-\frac{\beta}{1-\beta}\nu} P'_{\nu}\, d\nu$. Of course, if $\alpha = 0$, then $P_0 = 0$, and

$${}^{CF}\mathbb{I}^{\beta} P(\kappa) = (1-\beta)P(\kappa) + \beta \int_0^{\kappa} P(\nu)\, d\nu. \tag{2.3}$$

Theorem 2.4. ([10]). The following relations will be useful to solve fractional ordinary and partial differential equations,

$${}^{CF}\mathbb{D}^{\beta}\ {}^{CF}\mathbb{I}^{\alpha} P(\kappa) = P(\kappa) - P(\alpha) e^{-\frac{\beta}{1-\beta}(\kappa-\alpha)}, \tag{2.4}$$

$${}^{CF}\mathbb{I}^{\beta}\ {}^{CF}\mathbb{D}^{\beta} P(\kappa) = P(\kappa) + c, \tag{2.5}$$

with c an arbitrary real constant.

Definition 2.5. A function $\varphi(\kappa) > 0$ used in the inequality (2.6) to bound the perturbation of the equation is called a control function.

Definition 2.6. Let $\varphi \in \mathcal{C}([0,b],\mathbb{R})$ be a positive control function, if for each $P \in \mathcal{C}([0,b],\mathbb{R})$ the following condition is verified:

$$\left|(1-\beta)\mu(\kappa, P(\kappa)) + \beta \int_0^\kappa \mu(\nu, P(\nu))\,\mathrm{d}\nu - P(\kappa)\right| \leq \varphi(\kappa), \tag{2.6}$$

then the system (1.1) is Hyers-Ulam-Rassias stable and there exists a solution $Q \in \mathcal{C}([0,b],\mathbb{R})$ of system (1.1) such that:

$$|Q(\kappa) - P(\kappa)| \leq \sigma\varphi(\kappa), \qquad \kappa \in [0,b] \tag{2.7}$$

where the positive constant σ does not depend on P.

Definition 2.7. ([9]). Assume the gamma function Γ. A generalization of the exponential function is the Mittag Leffler function given by

$$E_i(\Delta) := \sum_{\kappa=0}^{\infty} \frac{\Delta^\kappa}{\Gamma(\kappa i + 1)}, \qquad i \in \mathbf{C},\ \mathbf{Re}(i) > 0. \tag{2.8}$$

Definition 2.8. The system (1.1) is Hyers-Ulam-Mittag-Leffler stable according to $E_i(-|\kappa|)$ if for each $P \in \mathcal{C}([0,b],\mathbb{R})$, the following condition is verified:

$$\left|(1-\beta)\mu(\kappa, P(\kappa)) + \beta \int_0^\kappa \mu(\nu, P(\nu))\,\mathrm{d}\nu - P(\kappa)\right| \leq E_i(-|\kappa|), \tag{2.9}$$

then there exist a positive constant σ, and a solution $Q \in \mathcal{C}([0,b],\mathbb{R})$ of (1.1) such that

$$|Q(\kappa) - P(\kappa)| \leq \sigma E_i(-|\kappa|), \qquad \kappa \in (0,b). \tag{2.10}$$

3 Main results

This section shows that (1.1) has a unique solution. Then, the stability of Hyers-Ulam-Rassias, and Hyers-Ulam-Mittag-Leffler for (1.1) according to the alternative fixed point theorem is investigated.

Lemma 3.1. Let $\varphi, P, Q \in \mathcal{C}([0,b],\mathbb{R})$, if we define a mapping $d : \mathcal{C}([0,b],\mathbb{R}) \times \mathcal{C}([0,b],\mathbb{R}) \to [0,+\infty]$, by

$$d(P(\kappa), Q(\kappa)) := \inf\left\{c \geq 0 : |P(\kappa) - Q(\kappa)| \leq c\varphi(\kappa)\right\}. \tag{3.1}$$

Then we demonstrate that $(\mathcal{C}([0,b],\mathbb{R}), d)$ constitutes a generalized metric space.

Proof . For every $\kappa \in [0,b]$ we show that $d(P(\kappa), Q(\kappa)) = 0$ if and only if $P(\kappa) = Q(\kappa)$. Assume $d(P(\kappa), Q(\kappa)) = 0$, then for all $P(\kappa), Q(\kappa) \in \mathcal{C}([0,b],\mathbb{R})$ and $\kappa \in [0,b]$, we have:

$$\inf\left\{c \geq 0 : |P(\kappa) - Q(\kappa)| \leq c\varphi(\kappa)\right\} = 0,$$

so for every $\kappa \in [0,b]$ we have $P(\kappa) = Q(\kappa)$, and conversely. Symmetry $d(P(\kappa), Q(\kappa)) = d(Q(\kappa), P(\kappa))$ holds by definition. Now, consider for $j = 1, 2$; there exist constants f_j such that $d(P(\kappa), \eta(\kappa)) = f_1$ and $d(Q(\kappa), \eta(\kappa)) = f_2$. Then, we have

$$\begin{cases} |P(\kappa) - \eta(\kappa)| \leq f_1\varphi(\kappa), \\ |Q(\kappa) - \eta(\kappa)| \leq f_2\varphi(\kappa), \end{cases} \tag{3.2}$$

according to the triangular property in absolute magnitudes, we can define $k_1 = f_1 + f_2$ such that:

$$|P(\kappa) - Q(\kappa)| \leq |P(\kappa) - \eta(\kappa)| + |Q(\kappa) - \eta(\kappa)| \leq k_1\varphi(\kappa), \tag{3.3}$$

if we take the infimum from the above relationship, we obtain:

$$\begin{aligned} d\left(P(\kappa), Q(\kappa)\right) &\leq \inf\left\{f_1 \geq 0 : |P(\kappa) - \eta(\kappa)| \leq f_1\varphi(\kappa)\right\} + \inf\left\{f_2 \geq 0 : |Q(\kappa) - \eta(\kappa)| \leq f_2\varphi(\kappa)\right\} \\ &= d(P(\kappa), \eta(\kappa)) + d(Q(\kappa), \eta(\kappa)). \end{aligned}$$

Now, we demonstrate the completeness of $(\mathcal{C}([0,b],\mathbb{R}),d)$. Consider a Cauchy sequence $\{P_n(\kappa)\}_n$. This means that for any positive number i, there exists a natural $\mathbb{N}_i$ such that for all $n, m \geqslant \mathbb{N}_i$, we have:

$$d(P_n(\kappa), P_m(\kappa)) < i.$$

With definition (3.1), for all $\kappa \in [0,b]$ we can express:

$$|P_n(\kappa) - P_m(\kappa)| \leqslant i\varphi(\kappa). \tag{3.4}$$

Let $\kappa \in [0,b]$ is fixed, so the sequence $\{P_n(\kappa)\}_n$ is a Cauchy sequence in $\mathbb{R}$ and, due to the completeness of $\mathbb{R}$, converges for each $\kappa \in [0,b]$. Consequently, we can define a function $P(\kappa)\colon [0,b] \to \mathbb{R}$ by

$$P(\kappa) = \lim_{n\to\infty} P_n(\kappa).$$

Since $\varphi(\kappa)$ is continuous on $[0,b]$, there exists $T > 0$ such that $\varphi(\kappa) < T$ for all $\kappa \in [0,b]$. Thus, (3.4) implies that $\{P_n(\kappa)\}_n$ converges uniformly to $P(\kappa)$ in the usual topology of $\mathbb{R}$. Hence, $P \in \mathcal{C}([0,b],\mathbb{R})$. If we suppose m approaches to infinity, for $i > 0$ a natural $\mathbb{N}_i$ exists such that for $n \geqslant \mathbb{N}_i$ it follows from (3.4) that

$$|P_n(\kappa) - P(\kappa)| \leqslant i\varphi(\kappa), \tag{3.5}$$

for all positive i there exists natural $\mathbb{N}_i$ such that for all $n > \mathbb{N}_i$ considering equation (3.1), we obtain

$$d(P(\kappa), P_n(\kappa)) \leqslant i.$$

This indicates that the Cauchy sequence $\{P_n\}$ converges to P in $(\mathcal{C}([0,b],\mathbb{R}),d)$, demonstrating the completeness of $(\mathcal{C}([0,b],\mathbb{R}),d)$. □

Remark 3.2. Unlike prior fixed point theorems approaches, our use of Theorem 2.1 enables weaker constraints on $\mu(\kappa, P(\kappa))$.

Lemma 3.3. *Let $\mu \in [0,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then (1.1) is equivalent to*

$$P(\kappa) = (1-\beta)\mu(\kappa, P(\kappa)) + \beta \int_0^\kappa \mu(\nu, P(\nu))\,\mathrm{d}\nu.$$

Theorem 3.4 (UHR stability). *Assume $P \in \mathcal{C}([0,b],\mathbb{R})$ satisfies (2.6) for a control function $\varphi \in \mathcal{C}([0,b],\mathbb{R}^+)$, and the following hold:*

(R_1) *There exists a constant $\theta > 0$ such that*

$$|\mu(\kappa, P(\kappa)) - \mu(\kappa, Q(\kappa))| \leq \theta|P(\kappa) - Q(\kappa)|, \tag{3.6}$$

(R_2) *There exists a constant $\delta < 1$ such that*

$$\int_0^\kappa \varphi(\nu)\,\mathrm{d}\nu \leq \delta\varphi(\kappa), \tag{3.7}$$

(R_3) *There exists a constant $\gamma < 1$ such that*

$$\gamma = \theta(|1-\beta| + \delta\beta). \tag{3.8}$$

Thus a unique solution of (1.1) is $Q \in \mathcal{C}([0,b],\mathbb{R})$ such that

$$Q(\kappa) = (1-\beta)\mu(\kappa, Q(\kappa)) + \beta \int_0^\kappa \mu(\nu, Q(\nu))\,\mathrm{d}\nu, \tag{3.9}$$

in which

$$|Q(\kappa) - P(\kappa)| \leq \frac{c_1}{1-\gamma}\varphi(\kappa), \qquad \kappa \in [0,b]. \tag{3.10}$$

Proof . Assume that $P, Q \in \mathcal{C}([0,b],\mathbb{R})$ and $\kappa \in [0,b]$ define a mapping $d : \mathcal{C}([0,b],\mathbb{R}) \times \mathcal{C}([0,b],\mathbb{R}) \to [0,+\infty]$ by

$$d\Big(P(\kappa), Q(\kappa)\Big) = \inf\{c > 0, |P(\kappa) - Q(\kappa)| < c\varphi(\kappa)\}. \tag{3.11}$$

According to Lemma 3.1 $(\mathcal{C}([0,b],\mathbb{R}), d)$ is a generalized metric space.

Step 1. Define the operator $F : \mathcal{C}([0,b],\mathbb{R}) \to \mathcal{C}([0,b],\mathbb{R})$ and then we prove that it is contractive.

$$F(P(\kappa)) = (1-\beta)\mu(\kappa, P(\kappa)) + \beta \int_0^\kappa \mu(\nu, P(\nu))\,\mathrm{d}\nu. \tag{3.12}$$

Since μ is continuous and $P \in \mathcal{C}([0,b],\mathbb{R})$, then $\mu(\kappa, P(\kappa))$ is continuous. On the other hand, the integral of a continuous function and the composition of continuous functions are also themselves continuous. Therefore $F(P) \in \mathcal{C}([0,b],\mathbb{R})$, and F is well-defined.

We need to prove that $F(P(\kappa))$ belongs to the space $\mathcal{C}([0,b],\mathbb{R})$. Based on the fundamental theorem of calculus, we can conclude that $F(P(\kappa))$ is continuously differentiable on the interval $[0,b]$, since $P(\kappa)$ is a continuous function, which implies that $F(P(\kappa)) \in \mathcal{C}([0,b],\mathbb{R})$. We claim that F is contractive on $\mathcal{C}([0,b],\mathbb{R})$. Let $P, Q \in \mathcal{C}([0,b],\mathbb{R})$, and $d(P(\kappa), Q(\kappa)) = h$ so that $h \in [0,\infty]$ be a constant. Furthermore, we can express:

$$\Big|P(\kappa) - Q(\kappa)\Big| \le h\varphi(\kappa). \tag{3.13}$$

It can be written using relations (3.11), (R_1), and (R_2)

$$\begin{aligned}
&\Big|(1-\beta)\mu(\kappa, P(\kappa)) + \beta \int_0^\kappa \mu(\nu, P(\nu))\,\mathrm{d}\nu - \Big((1-\beta)\mu(\kappa, Q(\kappa)) + \beta \int_0^\kappa \mu(\nu, Q(\nu))\,\mathrm{d}\nu\Big)\Big| \\
&\le \Big|(1-\beta)\mu(\kappa, P(\kappa)) - (1-\beta)\mu(\kappa, Q(\kappa)) + \Big(\beta \int_0^\kappa \mu(\nu, P(\nu))\,\mathrm{d}\nu - \beta \int_0^\kappa \mu(\nu, Q(\nu))\,\mathrm{d}\nu\Big)\Big| \\
&\le \Big|(1-\beta)\Big(\mu(\kappa, P(\kappa)) - \mu(\kappa, Q(\kappa))\Big)\Big| + \Big|\beta\Big(\int_0^\kappa \mu(\nu, P(\nu))\,\mathrm{d}\nu - \int_0^\kappa \mu(\nu, Q(\nu))\,\mathrm{d}\nu\Big)\Big| \\
&\le |1-\beta|\Big|\mu(\kappa, P(\kappa)) - \mu(\kappa, Q(\kappa))\Big| + \beta \int_0^\kappa \Big|\mu(\nu, P(\nu)) - \mu(\nu, Q(\nu))\Big|\,\mathrm{d}\nu \\
&\le \theta|1-\beta|\Big|P(\kappa) - Q(\kappa)\Big| + \beta \int_0^\kappa \theta\Big|P(\nu) - Q(\nu)\Big|\,\mathrm{d}\nu \\
&\le \theta|1-\beta|h\varphi(\kappa) + \theta\beta \int_0^\kappa h\varphi(\nu)\,\mathrm{d}\nu \\
&\le h\theta|1-\beta|\varphi(\kappa) + \theta\beta\delta h\varphi(\kappa) \\
&\le \theta(|1-\beta| + \delta\beta)h\varphi(\kappa).
\end{aligned} \tag{3.14}$$

If we take the infimum from the above relationship, we have:

$$d\Big(F(P(\kappa)), F(Q(\kappa))\Big) \le \theta(|1-\beta| + \delta\beta)d(P(\kappa), Q(\kappa)). \tag{3.15}$$

According to (R_3), F is a contraction .

Step 2. Suppose $P \in \mathcal{C}([0,b],\mathbb{R})$, then we show that $d\Big(P(\kappa), F(P(\kappa))\Big) < \infty$ for $\kappa \in [0,b]$. According to (2.6),(3.11), and (3.12), we have

$$d\Big(P(\kappa), F(P(\kappa))\Big) = \inf\left\{c > 0, \left|P(\kappa) - \left((1-\beta)\mu(\kappa, P(\kappa)) + \beta \int_0^\kappa \mu(\nu, P(\nu))\,\mathrm{d}\nu\right)\right| \le \varphi(\kappa)\right\} \le 1 < \infty. \tag{3.16}$$

Now, according to the Theorem 2.1, we can find the element $Q \in \mathcal{C}([0,b],\mathbb{R})$ that satisfies the following conditions:

1. The fixed point Q of F is the convergence point of the sequence $\{F^m\,(P(\kappa))\}$;

2. In the set $\Upsilon = \Big\{P^* \in \mathcal{C}([0,b],\mathbb{R}) : d\Big(F((P(\kappa)), P^*(\kappa)\Big) < \infty, \kappa \in [0,b]\Big\}$, Q is a unique fixed point of F, such that,

$$F(Q(\kappa)) = (1-\beta)\mu(\kappa, Q(\kappa)) + \beta \int_0^\kappa \mu(\nu, Q(\nu))\,\mathrm{d}\nu.$$

3. If $P^* \in \Upsilon$, and $\gamma = \theta(|1-\beta| + \delta\beta)$, then we have

$$d\left(P^*(\kappa), Q(\kappa)\right) \leq \frac{1}{1-\gamma} d\left(F\left(P^*(\kappa)\right), P^*(\kappa)\right).$$

From (3.16), because $P \in \Upsilon$ we get

$$d\Big(P(\kappa), Q(\kappa)\Big) \leq \frac{c}{1-\gamma}.$$

Then,

$$|P(\kappa) - Q(\kappa)| \leq \frac{c}{1-\gamma}\varphi(\kappa), \qquad \kappa \in [0,b]. \tag{3.17}$$

□

This theorem establishes the Hyers-Ulam-Rassias stability of (1.1) under the assumptions $(R1)-(R3)$.

Corollary 3.5 (UHML stability). Assume $P \in \mathcal{C}([0,b],\mathbb{R})$ satisfies 2.6, and the following hold:

$(R_1^{'})$ There exists a constants $\theta > 0$ such that

$$|\mu(\kappa, P(\kappa)) - \mu(\kappa, Q(\kappa))| \leq \theta |P(\kappa) - Q(\kappa)|. \tag{3.18}$$

$(R_2^{'})$ There exists a constant $\delta < 1$ such that

$$\int_0^\kappa E_\iota(-|\nu|)\,\mathrm{d}\nu \leq \delta E_\iota(-|\kappa|). \tag{3.19}$$

$(R_3^{'})$ There exists a constant $\gamma < 1$ such that

$$\gamma = \theta(|1-\beta| + \delta\beta). \tag{3.20}$$

Thus, a unique solution of (1.1) is $Q \in \mathcal{C}([0,b],\mathbb{R})$ such that

$$Q(\kappa) = (1-\beta)\mu(\kappa, Q(\kappa)) + \beta \int_0^\kappa \mu(\nu, Q(\nu))\,\mathrm{d}\nu, \tag{3.21}$$

in which

$$|P(\kappa) - Q(\kappa)| \leq \left(\frac{c}{1-\gamma}\right) E_i(-|\kappa|), \qquad \kappa \in [0,b]. \tag{3.22}$$

Proof . Similarly to Theorem 3.4, we can prove this Corollary. □

Example 3.6. The following example satisfies the conditions of Theorem 3.4 with control function $\varphi(x) = e^{-|x|}$,

$$\begin{cases} {}^{CF}\mathbb{D}^{0.5}P(\kappa) = \dfrac{\tanh(P(\kappa))}{100 + \operatorname{sech}(P(\kappa))} & \kappa \in [0, \frac{1}{2}], \\ P(0) = 0. \end{cases} \tag{3.23}$$

Proof . First, $\mu(\kappa, P) = \dfrac{\tanh(P(\kappa))}{100 + \operatorname{sech}(P(\kappa))}$ satisfies (R_1) :

$$\left|\frac{\partial \mu}{\partial P}\right| < 0.01,$$

so $\theta = 0.01$. For (R_2) verify:

$$\int_0^\kappa e^{-\nu} d\nu = 1 - e^{-\kappa} \le \delta e^{-\kappa}, \qquad \text{with} \qquad \delta = e^{0.5} - 1.$$

Since $1 - e^{-\kappa} < e^{0.5} - 1$, (R_2) holds. Finally, $\gamma = \theta(|1 - \beta| + \delta\beta) = 0.01 \times (0.5 + 0.5 \times 0.648) = 0.0106 < 1$. It is sufficient to take $\delta = e^{0.5} - 1$, , $\beta = 0.5$, and $\gamma = 0.01$.

□

The nonlinear term $\mu(\kappa, P) = \frac{\tanh(P(\kappa))}{100+\operatorname{sech}(P(\kappa))}$ models saturation phenomena in physical systems where growth diminishes due to constraints. Specifically, in ferromagnetic hysteresis modeling, the $\tanh(P)$ term directly corresponds to the Langevin function describing magnetic moment alignment in ferromagnetic materials under applied fields [7, 8]:

$$M(H) = M_s \left(\coth\left(\frac{H}{a}\right) - \frac{a}{H}\right) \approx M_s \tanh\left(\frac{H}{3a}\right),$$

where H (magnetic field) and M_s (saturation magnetization) are P, and $\frac{100}{100}$ respectively in our example. The Caputo-Fabrizio derivative ${}^{CF}\mathbb{D}^{0.5}$ captures memory-dependent hysteresis in nanocrystalline alloys [4]. In biological systems, $\tanh(P)$ models growth with carrying capacity [14], where P represents population density, and the denominator models resource limitations; the fractional order $\beta = 0.5$ describes subdiffusive processes in constrained environments [16]. For engineering control, the expression resembles activation functions in neuromorphic circuits [18], and the bounded output $\left|\frac{\partial \mu}{\partial P}\right| < 0.01$ ensures stability in control systems subject to perturbations. The exponential decay $\varphi(\kappa) = e^{-|\kappa|}$ represents spatial decay in diffusion processes (where κ denotes position) and temporal relaxation in viscoelastic materials (where κ represents time).

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