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Uniqueness of meromorphic functions and nonlinear differential polynomials

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Abstract

In this paper we study the influence of the "normal family" concept on the meromorphic functions $\xi_1(z)$ and $\xi_2(z)$ and the behavior of two nonlinear differential polynomials with a shared polynomial of a specific degree. Our investigation leads to uniqueness results for $\mathcal{P}(\xi_1(z))\mathcal{L}^m(\xi_1(z))$ and $\mathcal{P}(\xi_2(z))\mathcal{L}^m(\xi_2(z))$ under these circumstances. In addition, the results generalize and extend the findings of Cao and Zhang.

Keywords: Differential polynomial, Uniqueness, Meromorphic function, Normal family.
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1 Introduction / Background Information

Meromorphic functions are mathematical functions defined as holomorphic functions outside a set of isolated singularities. We assume that the reader is well-versed in Nevanlinna value distribution theory and its fundamental symbols and results relating to meromorphic functions. Recall the standard notation in the theory of meromorphic functions. Suppose that $\xi_1(z)$ and $\xi_2(z)$ are two non-constant meromorphic functions and $a \in \mathbb{C}\bigcup\{\infty\}$, a is a small meromorphic function, and $T(r) = \max\{T(r,\xi_1), T(r,\xi_2)\}$ in the usual Nevanlinna theory sense. Moreover, we denote the function $S(r)$ as $o(T(r))$ as $r \to \infty$. To denote the counting function (reduced counting function) of those $a-$ points of f whose multiplicities are not less than p, we use the notation $N(r, a : \xi_1(z)| \geq p)(\overline{N}(r, a; \xi_1(z)| \geq p))$, and $N(r, a : \xi_1(z)| \leq p)(\overline{N}(r, a; \xi_1(z)| \leq p))$ to denote the counting function (reduced counting function) of those $a-$ points of $\xi_1(z)$ whose multiplicities are not greater than p, where $p \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$.

Consider two non-constant meromorphic functions $\xi_1(z)$ and $\xi_2(z)$, along with a small function $a(z)$ relative to $\xi_1(z)$ and $\xi_2(z)$. If $\xi_1(z) - a(z)$ and $\xi_2(z) - a(z)$ have the same zeros with the same multiplicities, then we say that $\xi_1(z)$ and $\xi_2(z)$ share $a(z)$.

When a function ξ maps a point z_0 to itself, we call z_0 a fixed point of ξ. Alternatively, if $\xi(z_0) - z_0 = 0$, z_0 is considered a zero of the function $\xi(z) - z$.

A meromorphic function h in the complex plane $\mathbb{C}$ is considered a normal function if there is a positive real number M that satisfies the condition $h^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, where $h^{\#}(z)$ is the spherical derivative of h defined as $\frac{|h'(z)|}{1+|h(z)|^2}$.

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Let a family of meromorphic functions in a domain $D \subset \mathbb{C}$ be denoted as $\mathcal{F}^*$. If every sequence $\{f_n\} \subset \mathcal{F}^*$ contains a subsequence spherically and uniformly convergent on the compact subsets of D, we say that $\mathcal{F}^*$ is normal in D (see [13]).

The proof of a well-known theorem from the value distribution theory, initially proposed by Hayman [5], was nearly simultaneously demonstrated by various authors.

Theorem 1.1. Let ξ be a transcendental meromorphic function, and $n \geq 1$ a positive integer. Then $\xi^n \xi' = 1$ has infinitely many solutions.

To investigate the uniqueness result corresponding to Theorem 1.1, both Fang and Hua [6] and Yang and Hua [15] obtained the following results.

Theorem 1.2. Let $\xi_1(z)$ and $\xi_2(z)$ be two non-constant entire (meromorphic) functions, and let $n \in \mathbb{N}$ with $n \geq 6 (n \geq 11)$. If $\xi_1(z)^n \xi_1'(z)$ and $\xi_2(z)^n \xi_2'(z)$ share 1 CM, then either $\xi_1(z) = c_1 \exp(cz)$ and $\xi_2(z) = c_2 \exp(-cz)$, where $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ satisfy $4\left(c_1 c_2^{(n+1)}\right) c^2 = -1$ or $\xi_1(z) \equiv t\xi_2(z)$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.

Over the years, there has been an increase in research efforts in these areas, establishing them as fields within the realm of uniqueness theory. The results obtained by Fang and Qui's examination of the issue of the uniqueness of entire or meromorphic functions with fixed points were quite noteworthy.

Xu et al. [14] and Li [17] considered the k^{th} derivative instead of the first derivative in Theorem 1.2 and proved the following results:

Theorem 1.3. Let $\xi(z)$ be a transcendental meromorphic function and let $k, n \in \mathbb{N}$ with $n \geq 2$. Then $\xi^n(z)\xi^{(k)}(z)$ takes every finite non-zero value infinitely many times or has infinitely many fixed points.

Recently, the following results were proved by Cao and Zhang [3].

Theorem 1.4. Let $\xi_1(z)$ and $\xi_2(z)$ be two non-constant meromorphic functions with zeros whose multiplicities are of multiplicity at least k, $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n \geq \max\left\{2k-1, k+\frac{4}{k}+4\right\}$. If $\xi_1^n(z)\xi_1^{(k)}(z)$ and $\xi_2^n(z)\xi_2^{(k)}(z)$ share 1 CM, and $\xi_1(z)$ and $\xi_2(z)$ share ∞ IM, then one of the following two conclusions holds:

(i) $\xi_1^n(z)\xi_1^{(k)}(z) \equiv \xi_2^n(z)\xi_2^{(k)}(z)$;

(ii) $\xi_1(z) = c_1 \exp(az^2)$ and $\xi_2(z) = c_2 \exp(-az^2)$, where $a, c_1, c_2 \in \mathbb{C}$ such that $4\,(c_1 c_2)^{(n+1)}\, a^{2k} = 1$.

Theorem 1.5. Let $\xi_1(z)$ and $\xi_2(z)$ be two non-constant meromorphic functions whose zeros are of multiplicity at least $k+1$, $k \in \mathbb{N}$ with $1 \leq k \leq 5$. Let $n \in \mathbb{N}$ such that $n \geq 10$. If $\xi_1^n(z)\xi_1^{(k)}(z)$ and $\xi_2^n(z)\xi_2^{(k)}(z)$ share 1 CM, $\xi_1^{(k)}(z)$ and $\xi_2^{(k)}(z)$ share 0 CM, and $\xi_1(z)$ and $\xi_2(z)$ share ∞ IM, then one of the following two conclusions holds:

(i) $\xi_1(z) \equiv t\xi_2(z)$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$;

(ii) $\xi_1(z) = c_1 \exp(az)$ $\xi_2(z) = c_2 \exp(-az)$, where $a, c_1, c_2 \in \mathbb{C}$ such that $(-1)^k\,(c_1 c_2)^{(n+1)}\, a^{2k} = 1$.

The following questions are unavoidable in relation to the Theorem 1.5.

a). **Question 1.** Can the lower bound of n in the Theorem 1.5 be further reduced?

b). **Question 2.** Can the condition "Let $\xi_1(z)$ and $\xi_2(z)$ be two non-constant meromorphic functions with zeros whose multiplicities are not less than $k+1$, $k \in \mathbb{N}$" in Theorem 1.5 be additionally weakened?

c). **Question 3.** For $k \geq 6$, does the conclusion of Theorem 1.5 hold?

2 Main Results and Some Definitions

Throughout this paper, we always use $\mathcal{F} = \mathcal{L}^m(\xi_1(z))$, $\mathcal{G} = \mathcal{L}^m(\xi_2(z))$ and $\mathcal{P}(z)$ to represent a non-constant polynomial of degree $n \in \mathbb{N}$ which is of the form:

$$\mathcal{P}(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n (z-c_1)^{d_1} a_n (z-c_2)^{d_2} \cdots (z-c_{s_1})^{d_{s_1}},$$

where $a_i(i = 0, 1, \cdots, n-1)$, $a_n \neq 0$ and $c_j(j = 1, 2, \cdots, s_1)$ are distinct finite complex numbers; $d_1, d_2, \cdots, d_{s_1} \in \mathbb{N}\bigcup\{0\}$, $n \in \mathbb{N}$ such that $\sum_{i=1}^{s_1} d_i = n$. Let $d = \max\{d_1, d_2 \cdots, d_{s_1}\}$ and c be the corresponding zero of $\mathcal{P}(z)$ of multiplicity d. We set an arbitrary non-zero polynomial $\mathcal{P}_1(z)$ by

$$\mathcal{P}_1(z) = a_n \prod_{i=1,\ d_i \neq d}^{s_1} (z - c_i)^{d_i} = b_{m_2} z^{m_2} + b_{m_2 - 1} z^{m_2 - 1} + \cdots + b_0,$$

where $a_n = b_{m_1}$ and $m_1 = n - d$. Obviously $\mathcal{P}(z) = (z-c)^d \mathcal{P}_1(z)$. We also use $\mathcal{P}_2(z_1)$ as an arbitrary nonzero polynomial defined by:

$$\mathcal{P}_2(z_1) = a_n \prod_{i=1,\ d_i \neq d}^{s_1} (z_1 + c - c_i)^{d_i} = e_{m_2} z^{m_2} + e_{m_2 - 1} z^{m_2 - 1} + \cdots + e_0,$$

where $z_1 = z - c$ and $deg(\mathcal{P}_2) = m_2 \geq 0$. Obviously

$$\mathcal{P}(z) = z_1^d \mathcal{P}_2(z_1). \tag{2.1}$$

Suppose $\Gamma_1 = m_3 + m_4$ and $\Gamma_2 = m_3 + 2m_4$, where m_3 is the number of simple zeros of $\mathcal{P}_2(z)$ and m_4 is the number of multiple zeros of $\mathcal{P}_2(z)$. We define $k^* \in \mathbb{N}$ as

$$\mathcal{K}^* = \begin{cases} k & if \ \ \mathcal{P}_2(z) \equiv e_i z_1^i \not\equiv 0, \\ k+1 & if \ \ \mathcal{P}_2(z) \not\equiv e_i z_1^i \not\equiv 0 \end{cases} \tag{2.2}$$

for $i \in \{0, 1, 2, \cdots, m_2\}$. Again, we use $p(z)$ to denote a non-zero polynomial defined by

$$p(z) = d_n (z - z_1)^{l_1} (z - z_2)^{l_2} (z - z_3)^{l_3} \cdots (z - z_t)^{l_t}, \tag{2.3}$$

where $d_n \in \mathbb{C}\bigcup\{0\}$, $z_i \in \mathbb{C}$, $i = 1, 2, \cdots, t$ are distinct complex numbers and $l_1, l_2, \cdots, l_t \in \mathbb{N} \bigcup \{0\}$. Here we see that either $\sum_{i=1}^{t} l_i \leq n + m - 1$ or $l_i \leq n - 1$ for all $i = 1, 2, \ldots, t$. We will now recall the definition of weighted sharing, which was mentioned in [9].

Definition 2.1. Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a, f)$ the set of all $a-$points of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, \xi_1(z)) = E_k(a, \xi_2(z))$, we say that $\xi_1(z), \xi_2(z)$ share the value a with weight k.

Definition 2.2. [2] Let k be a positive integer for any constant a in the complex plane $\mathbb{C}$.
We denote

1. By $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function of a-points of $f(z)$ with multiplicity $\leq k$.
2. By $N_{(k}\left(r, \frac{1}{f-a}\right)$ the counting function of a-points of $f(z)$ with multiplicity $\geq k$. Similarly, the reduced counting functions $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ are defined.

We always use $\mathcal{L}(\xi)$ to denote the following differential polynomial throughout this paper:

$$\mathcal{L}(\xi) = a_k \xi^{(k)} + a_{k-1} \xi^{(k-1)} + \cdots + a_1 \xi' + a_0 \xi, \quad a_k,\ a_{k-1} \ldots a_1,\ a_0 \in \mathbb{C} \tag{2.4}$$

In this article, we will investigate the relationship and connection between meromorphic functions and nonlinear differential polynomials through normal families.

Theorem 2.3. Let $\xi(z)$ be a transcendental meromorphic function such that zeros of $\xi(z) - c$ are of multiplicities at least $\mathcal{K}^*$, where $\mathcal{K}^*$ is defined in (2.2), and let $\alpha(z)(\not\equiv 0, \infty)$ be a small function of $\xi(z)$. Also let $n, m \in \mathbb{N}$. If $n > \Gamma_1 + m + 1/\mathcal{K}^*$, then $\mathcal{P}(\xi)\left(\mathcal{L}(\xi)\right)^m - \alpha(z)$ has infinitely many zeros, where $\mathcal{P}(z)$ is defined as in (2.1).

Theorem 2.4. Let $\xi_1(z)$ and $\xi_2(z)$ be two transcendental meromorphic functions such that the zeros of $\xi_1(z)-c$ and $\xi_2(z)-c$ are of multiplicities at least k, where $k\in\mathbb{N}$. Let $\mathcal{P}(z)$ and $p(z)$ be defined as in (2.1) and (2.3), respectively, and let $n,\Gamma_2,m\in\mathbb{N}\cup\{0\}$ be such that $n\geq 4\Gamma_2+2m+k+\frac{7}{2k^*}+\frac{3}{2}$. If $\mathcal{P}(\xi_1)\mathcal{F}-p(z)$ and $\mathcal{P}(\xi_2)\mathcal{G}-p(z)$ share $(0,\ k_1)$ where $k_1=\left[\frac{(3+m(k-1))}{(n+m+k(m-2)-1)}\right]+3$, $\xi_1(z)$ and $\xi_2(z)$ share ∞ IM then one of the following conclusions holds:

1. $\xi_1(z)-c\equiv t(\xi_2(z)-c)$ with $t^s=1$, where $\mu=\gcd\{n+m+m_1,\ldots,n+m+m_1-i,\ldots,m+n\}$ with $e_{m-i}\neq 0$.
2. $\mathcal{P}(\xi_1)\mathcal{L}(\xi_1)\equiv\mathcal{P}(\xi_2)\mathcal{L}(\xi_2)$.

Theorem 2.5. Let $\xi_1(z)$ and $\xi_2(z)$ be two transcendental meromorphic functions such that the zeros of $\xi_1(z)-c$ and $\xi_2(z)-c$ are of multiplicities at least k^*, where $k^*\in\mathbb{N}$. Let $\mathcal{P}(z)$ and $p(z)$ be defined as in (2.1) and (2.3) respectively, and let $n,\Gamma_2,m\in\mathbb{N}\cup\{0\}$ be such that $n\geq 4\Gamma_2+2m+k+\frac{7}{2k^*}+\frac{3}{2}$. If $\mathcal{P}(\xi_1)\mathcal{F}-p(z)$ and $\mathcal{P}(\xi_2)\mathcal{G}-p(z)$ share $(0,k_1)$ where $k_1=\left[\frac{(3+m(k-1))}{(n+m+k(m-2)-1)}\right]+3$, $\mathcal{L}(\xi_1(z))$ and $\mathcal{L}(\xi_2(z))$ share $(0,\infty)$ and $\xi_1,\ \ \xi_2$ share $(\infty,0)$ then one of the following cases holds holds:

1. If $\mathcal{P}_2(z_1)\equiv e_iz_1^i\not\equiv 0$, for some $i\in\{0,1,\cdots,m_1\}$, then $\xi_1-c\equiv t(\xi_2-c)$, where t is a constant such that $t^{d+m+i}=1$, for some $i\in\{0,1,\cdots,m_1\}$.
2. If $\mathcal{P}_2(z_1)\not\equiv e_iz_1^i\not\equiv 0$, for some $i\in\{0,1,\cdots,m_1\}$ and $\xi_1,\ \ \xi_2$ share $(c,0)$, then $t^\mu=1$, where $\mu=\gcd\{n+m,\cdots,n+m-i,\cdots,1\}$, $e_{m_1-i}\neq 0$ for some $i=0,1,\cdots,m_1-1$.

Remark 2.6. Considering an example where the condition that $\xi_1(z)$ and $\xi_2(z)$ have zeros of multiplicity not less than $k\in\mathbb{N}$ can not be dropped in Theorem 2.4 is helpful. In this case, it is easy to see that the condition is sharp.

Example 2.7. Let $\xi_1(z)=c_1\exp(az)$ and $\xi_2(z)=c_2\exp(-az)$, where $a,\ c_1,\ c_2\in\mathbb{C}\setminus\{0\}$. Note that

$$\mathcal{L}(\xi_1(z))=a_2\xi_1''(z)+a_1\xi_1'(z)+a_0\xi_1(z)=c_1\exp(az)\left(a_2a^2+a_1a+a_0\right),$$

and

$$\mathcal{L}(\xi_2(z))=a_2\xi_2''(z)+a_1\xi_2'(z)+a_0\xi_2(z)=c_1\exp(-az)\left(a_2a^2-a_1a+a_0\right)$$

where $a_2(\neq 0)$ a_1 $a_0\in\mathbb{C}$ are such that

$$c_1^{n+m}\left(a_2a^2+a_1a+a_0\right)^m=c_2^{n+m}\left(a_2a^2-a_1a+a_0\right)^m\quad ,m,n\in\mathbb{N}.$$

As $\xi_1(z)$ and $\xi_2(z)$ have no zeroes, the condition which states "Let $\xi_1(z)$ and $\xi_2(z)$ be two transcendental meromorphic functions with zeroes whose multiplicities are not less than $k\in\mathbb{N}$" is not met. Notably, both $\xi_1(z)$ and $\xi_2(z)$ share ∞ CM, while $\mathcal{L}(\xi_1)$ and $\mathcal{L}(\xi_2)$ share 0 CM.

On the other hand, we can show that

$$\xi_1^n(z)\left(a_2\xi_1''(z)+a_1\xi_1'(z)+a_0\xi_1(z)\right)^m-p(z)=c_1^{n+m}\left(a_2a^2+a_1a+a_0\right)^m[\exp(az)-1]$$

and

$$\xi_2^n(z)\left(a_2\xi_2''(z)+a_1\xi_2'(z)+a_0\xi_2(z)\right)^m-p(z)=c_2^{n+m}\left(a_2a^2-a_1a+a_0\right)^m[\exp(-az)-1],$$

where $p(z)=c_1^{n+m}\left(a_2a^2+a_1a+a_0\right)^m$. Clearly, $\xi_1^n\left(a_2\xi_1''(z)+a_1\xi_1'(z)+a_0\xi_1(z)\right)^m-p(z)$ and $\xi_2^n(z)\left(a_2\xi_2''(z)+a_1\xi_2'(z)+a_0\xi_2(z)\right.$ $p(z)$ share $(0,\infty)$ but $\xi_1(z)\not\equiv t\xi_2(z)$, where $t\in\mathbb{C}\setminus\{0\}$ with $t^{n+m}=1$

Remark 2.8. The accuracy of the conditions outlined in Theorem 2.5 for $\mathcal{L}(\xi_1(z))$ and $\mathcal{L}(\xi_2(z))$ sharing $(0,\infty)$, as well as the sharing $(c,0)$ for $\xi_1(z)$ and $\xi_2(z)$ can be demonstrated with the given example.

Example 2.9. Let $\mathcal{P}(z)=z^{n+1}\left((n+3)z(n+2)\right)$, $\xi_1(z)=\frac{1-h^{n+2}}{1-h^{n+3}}$ and $\xi_2(z)=h\left(\frac{1-h^{n+2}}{1-h^{n+3}}\right)$, where $h(z)=\exp(z)-1$ and $n\in\mathbb{N}$ with $n\geq 10$. Observe that $\xi_1(z)$ and $\xi_2(z)$ share (∞,∞) but $\xi_1(z)$ and $\xi_2(z)$ do not share the value 0.

Note that $\xi_1'(z)=\frac{h^{n+1}h'\left[(n+3)h-h^{n+3}-(n+2)\right]}{(1-h^{n+3})^2}$ and $\xi_2'(z)=\frac{h'\left(1+(n+2)h^{n+3}-(n+3)h^{n+2}\right)}{(1-h^{n+3})^2}$. This shows that $\xi_1'(z)$ and $\xi_2'(z)$ do not share the value 0. Also we observe that $\xi_1^{n+2}(z)(\xi_1(z)-1)\equiv\xi_2^{n+2}(z)(\xi_2(z)-1)$, that is $\xi_1^{n+1}(z)((n+3)\xi_1(z)-(n+2))\xi_1'\equiv\xi_2^{n+1}(z)((n+3)\xi_2(z)-(n+2))\xi_2'$. Therefore $\xi_1^{n+1}(z)((n+3)\xi_1(z)-(n+2))\xi_1'$ and $\xi_2^{n+1}(z)((n+3)\xi_2(z)-(n+2))\xi_2'$ share $(1,\infty)$, but $\xi_1(z)\equiv t\xi_2(z)$, where $t\in\mathbb{C}\setminus\{0\}$ with $t^{n+3}=1$.

Remark 2.10. The outcomes extend the scope of Theorem 1.4, 1.5 towards diverse directions. Specifically, in Theorem 1.4, 1.5, the variables $\xi^n(z)$ and $\xi^{(k)}(z)$ have been substituted with $\mathcal{P}(\xi(z))$ and $\mathcal{L}^m(\xi(z))$ respectively.

Remark 2.11. The authors have established the assertion in Theorem 1.5 that the conclusions are valid solely for $k \in \{1,2,3,4,5\}$. However, the outcomes obtained in Theorem 2.5 are applicable for $k \in \mathbb{N}$, which is an enhancement over Theorem 1.5.

Now, we explain some of the necessary definitions and notations used in this present paper.

Definition 2.12. Let $q \in \mathbb{N}$ and $a \in \mathbb{C}\cup\{\infty\}$, $N(r,a;\xi| \geq q)$ $\left(\overline{N}(r,a;\xi| \geq q)\right)$ denotes the counting function (reduced counting function) of those $a-$points of ξ whose multiplicities are not less than q. $N(r,a;\xi| \leq q)$ $\left(\overline{N}(r,a;\xi| \geq q)\right)$ denotes the counting function (reduced counting function) of those $a-$points of ξ whose multiplicities are not smaller than q.

Definition 2.13. Let ξ_1 and ξ_2 share the value a IM. We denote by $\overline{N}_*(r,a;\xi_1,\xi_2)$ the reduced counting function of the $a-$points of ξ_1 whose multiplicities of the corresponding $a-$ points of ξ_2. Clearly,

$$\overline{N}_*(r,a;\xi_1,\xi_2) = \overline{N}_L(r,a;\xi_1) + \overline{N}_L(r,a;\xi_2).$$

Definition 2.14. Let ξ_1 and ξ_2 be two non-constant meromorphic function such that ξ_1 and ξ_2 share 1 IM. Let z_0 be 1$-$point of ξ_1 with multiplicity q_1 and 1$-$points of ξ_2 with multipilicity q_2. By $\overline{N}_L(r,1;\xi_1)$, $\overline{N}_E^{1)}(r,1;\xi_1)$ and $\overline{N}_E^{(2}(r,1;\xi_1)$ denote the counting function of the 1$-$points of ξ_1 and ξ_2 with $q_1 \geq q_2$, $q_1 = q_2 = 1$ and $q_1 = q_2 \geq 2$ respectively, each point in these counting functions is counted only once. In the same way, we can define $\overline{N}_L(r,1;\xi_2)$, $\overline{N}_E^{1)}(r,1;\xi_2)$, $\overline{N}_E^{(2}(r,1;\xi_2)$.

3 Some Useful Lemmas

We will introduce several necessary lemmas for the following content within this segment. Assuming F and G are two non-trivial meromorphic functions, we introduce two supplementary functions denoted as $\mathcal{H}$ and $\mathcal{V}$, respectively.

$$\mathcal{H} = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{3.1}$$

$$\mathcal{V} = \left(\frac{F'}{F-1} - \frac{2F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{2G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)} \tag{3.2}$$

Lemma 3.1. [10] If $N\left(r,0;\xi^{(k)}|\xi \neq 0\right)$ is the counting function of those zeros of $\xi^{(k)}$ that are not zeros of ξ where a zero of $\xi^{(k)}$ is counted according to its multiplicity, then

$$N\left(r,0;\xi^{(k)}|\xi \neq 0\right) \leq k\overline{N}(r,\infty;\xi) + N(r,0;\xi| < k) + k\overline{N}(r,0;\xi| \geq k) + S(r,\xi).$$

Lemma 3.2. [16] Let ξ be a non-constant meromorphic function and $P(z) = a_0 + a_1\xi + \cdots + a_n\xi^n$ where $a_0, a_1 \cdots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(\xi)) = nT(r,\xi) + O(1)$.

Lemma 3.3. [16] Let $\xi_j,\ j = 1,2,3$ be a meromorphic function and let ξ_1 be a non constant function. Suppose that $\sum\limits_{j=1}^{3} \xi_j \equiv 1$ and

$$\sum_{j=1}^{3} N(r,0;\xi_j) + 2\sum_{j=1}^{3} \overline{N}(r,\infty;\xi_j) < (\lambda + o(1))T_1(r),$$

as $r \to \infty,\ r \in I,\ \lambda < 1$ and $T_1(r) = \max\limits_{1\leq j\leq 3} T(r,\xi_j)$, where I is a set of infinite linear measure. Then either $\xi_2 \equiv 1$ or $\xi_3 \equiv 1$.

Lemma 3.4. [18] Let ξ be a non-constant meromorphic function and let $\mathcal{L}(\xi)$ be a differential polynomial defined as in 2.4. If $\mathcal{L}(\xi) \not\equiv 0$ and $q \in \mathbb{N}$, then

$$N_q(r, 0; \mathcal{L}(\xi)) \leq k\overline{N}(r, \infty; \xi) + N_{q+k}(r, 0; \xi) + S(r, \xi).$$

Lemma 3.5. [16] Let ξ be a non-constant meromorphic function and let $k \in \mathbb{N}$. Suppose that $\xi^{(k)} \not\equiv 0$. Then

$$N(r, 0; \xi^{(k)}) \leq N(r, 0; \xi) + k\overline{N}(r, 0; \xi) + S(r, \xi).$$

Lemma 3.6. Let ξ be a transcendental meromorphic function $n, m \in \mathbb{N}$. Then $\Psi(z) = \mathcal{P}(\xi)\mathcal{F} \notin \mathbb{C}$, where $\mathcal{P}(z)$ is defined as in 2.4.

Proof . If possible, let $\Psi(z) \in \mathbb{C}$. Then $\overline{N}(r, 0; \mathcal{P}(\xi)) = S(r, \xi)$ and $\overline{N}(r, \infty; \mathcal{P}(\xi)) = S(r, \xi)$. If $\mathcal{P}_1(z)$ is a non-constant polynomial, by the Second fundamental theorem, we arrive at a contradiction.
Next we suppose $\mathcal{P}(z) = a_n(z - c)^n$. Let $\varepsilon(z) = \xi(z) - c$, therefore $\Psi(z) = a_n\varepsilon^n(z)\mathcal{F}_1(z)$, where $\mathcal{F}_1(z) = \mathcal{L}(\xi - c)$. Also, we note that

$$\frac{1}{\varepsilon(z)}^{n+m} = a_n \frac{\mathcal{F}_1}{\varepsilon^s(z)} \frac{1}{\Psi(z)}.$$

Using Lemma 3.2, we see that

$$\begin{aligned}(n+m)T(r, \varepsilon(z)) &\leq T\left(r, \frac{\mathcal{F}_1}{\varepsilon^m(z)}\right) + T\left(r, \frac{1}{\Psi}\right) + O(1) \leq N\left(r, \infty; \frac{\mathcal{L}^m(\varepsilon)}{\varepsilon^m(z)}\right) + S(r, \varepsilon) \\ &\leq m\left\{N_k(r, 0; \varepsilon) + k\overline{N}(r, \infty; \varepsilon(z))\right\} + S(r, \varepsilon(z)) = S(r, \varepsilon(z)),\end{aligned}$$

which is impossible. Hence $\Psi(z) \notin \mathbb{C}$. □

Lemma 3.7. Let $\xi(z)$ be a non constant meromorphic function and let $\Psi(z) = \mathcal{P}(\xi)\mathcal{F}(z)$, where $n, m, k \in \mathbb{N}$ are such that $n > m$. Then

$$(n - m)T(r, \xi) \leq T(r, \Psi) - mN(r, \infty; \xi) - N(r, 0; \mathcal{F}) + S(r, \xi).$$

Proof . Note that $N(r, \infty; \Psi) = N(r, \infty; \mathcal{P}(\xi)) + N(r, \infty; (\mathcal{L}(\xi))^m)$,
which implies that, $N(r, \infty; \mathcal{P}(\xi)) = N(r, \infty; \Psi) - mN(r, \infty; \xi) - m\overline{N}(r, \infty; \xi)$.
In addition,

$$\begin{aligned}m(r, \mathcal{P}(\xi)) = m\left(r, \frac{\Psi}{\mathcal{F}}\right) &\leq m(r, \Psi) + m\left(r, \frac{1}{\mathcal{F}}\right) + S(r, \xi) = m(r, \Psi) + T(r, \mathcal{F}) - N(r, 0; \mathcal{F}) + S(r, \xi), \\ &\leq m(r, \Psi) + mN(r, \infty; \xi) + mk\overline{N}(r, \infty; \xi) - N(r, 0; \mathcal{F}) + S(r, \xi).\end{aligned}$$

Now

$$nT(r, \xi) = N(r, \infty; \mathcal{P}(\xi)) + m(r, \mathcal{P}(\xi)),$$

that is

$$(n - m)T(r, \xi) \leq T(r, \Psi) - mN(r, \infty; \xi) - N(r, 0; \mathcal{F}) + S(r, \xi).$$

This completes the proof. □

Lemma 3.8. Let $\xi_1(z)$ and $\xi_2(z)$ be two non constant meromorphic functions such that zeros of $\xi_1 - c$ and $\xi_2 - c$ are of multiplicities at least $\mathcal{K}^*$ where $\mathcal{K}^*$ is defined as in (2.2). Let $F(z) = \mathcal{P}(\xi_1)\mathcal{F}(z)/p(z)$ and $G(z) = \mathcal{P}(\xi_2)\mathcal{G}(z)/p(z)$, where $p(z)(\neq 0)$ is a polynomial and let $m, n \in \mathbb{N}$ be such that $n > m + k + 2\Gamma_1 + \frac{2}{\mathcal{K}^*}$. If $\xi_1,\ \xi_2$ share $(\infty, 0)$ and $\mathcal{H} \equiv 0$, then one of the following three cases holds:

1. $\xi_1 - c \equiv t(\xi_2 - c)$, $t^s = 1$, where $\mu = \gcd\{n + m + m_1, \cdots, n + m + m_1 - i, \cdots, m + n\}$ with $e_{m_1 - i} \neq 0$ for some $i \in \{0, 1, \cdots, m_1\}$.
2. $\mathcal{P}(\xi_1)\mathcal{F}.\mathcal{P}(\xi_2)\mathcal{G} \equiv p^2(z)$, where $\mathcal{P}(\xi_1)\mathcal{F} - p(z)$ and $\mathcal{P}(\xi_2)\mathcal{G} - p(z)$ share (o, ∞).

3. $\mathcal{P}(\xi_1)\mathcal{F} \equiv \mathcal{P}(\xi_2)\mathcal{G}$.

Proof . Since $\mathcal{H} \equiv 0$, by integration we get

$$\frac{F'(z)}{(F(z)-1)^2} \equiv \mathcal{A}\frac{G'(z)}{(G(z)-1)^2},$$

where $\mathcal{A}$ is non zero constant. This shows that $\left(\frac{\mathcal{P}(\xi_1)\mathcal{F}-p(z)}{p(z)}\right)$ and $\left(\frac{\mathcal{P}(\xi_2)\mathcal{G}-p(z)}{p(z)}\right)$ share $(0,\infty)$. Therefore $\mathcal{P}(\xi_1)\mathcal{F}-p(z)$ and $\mathcal{P}(\xi_2)\mathcal{G}-p(z)$ share $(0,\infty)$. By integration, we obtain

$$\frac{1}{F(z)-1} \equiv \frac{bG(z)+a-b}{G(z)-1}, \tag{3.3}$$

where $a, b \in \mathbb{C}\setminus\{0\}$ and $a \neq 0$. We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$. If $b=-1$, then from (3.3) we have $F(z) \equiv \frac{-a}{(G(z)-a-1)}$. Therefore

$$\overline{N}(r, a+1; G) = \overline{N}(r,\infty;F) \leq \overline{N}(r,\infty;\xi_1) + S(r,\xi_1).$$

Thus, in view of Lemma 3.7 and the Second Fundamental theorem, we get

$$\begin{aligned}(n-m)T(r,\xi_2) &\leq T(r,G) - mN(r,\infty;\xi_2 - N(r,0;\mathcal{G})) + S(r,\xi_2),\\ &\leq \overline{N}(r,0;\mathcal{P}_1(\xi_2)) + \overline{N}(r,\infty;\xi_2) + \overline{N}(r,0;G) + \overline{N}(r,0;\xi_2-c) - N(r,0;G) + S(r,\xi_2),\\ &\leq \overline{N}(r,\infty;\xi_2) + \left(\Gamma_1 + \frac{1}{\mathcal{K}^*}\right)T(r,\xi_2) + S(r,\xi_2),\\ &\leq \left(\Gamma_1 + 1 + \frac{1}{\mathcal{K}^*}\right)T(r,\xi_2) + S(r,\xi_2).\end{aligned}$$

which leads to a contradiction as $n > \Gamma_1 + m + 1 + \frac{1}{\mathcal{K}^*}$. If $b \neq -1$, from (3.3) we obtain that $F(z) - \left(1+\frac{1}{b}\right) \equiv \frac{-a}{[b^2(G(z)+(a-b))/b]}$. So $\overline{N}\left(r,(b-a)G\right) = \overline{N}(r,\infty;F) \leq \overline{N}(r,\infty;\xi_1) + S(r,\xi_1)$. Using Lemma 3.7 and the same arguments as used in the case when $b=-1$, we get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b=-1$, from (3.3), we get $F(z)G(z) \equiv 1$,
that is, $\mathcal{P}(\xi_1)\mathcal{F}(z)\mathcal{P}(\xi_2)\mathcal{G}(z) \equiv p^2(z)$. If $b \neq -1$, from (3.3) we have $\frac{1}{F(z)} = b\left(\frac{G(z)}{(1+b)G(z)-1}\right)$. Therefore $\overline{N}\left(r, \frac{1}{b+1}; G\right) = \overline{N}(r,o;F)$. So in view of Lemma 3.7 and 3.4, and the Second fundamental theorem, we get

$$\begin{aligned}(n-m)T(r,\xi_2) &\leq T(r,G) - N(r,0;G) - mN(r,\infty;\xi_2) + S(r,\xi_2)\\ &\leq \overline{N}(r,0;\mathcal{P}(\xi_2)) + \overline{N}(r,0;G) - N(r,0;G) + \overline{N}(r,0;F) + S(r,\xi_2)\\ &\leq \left(\Gamma_1 + \frac{1}{\mathcal{K}^*}\right)(T(r,\xi_1) + T(r,\xi_2)) + kT(r,\xi_1) + S(r,\xi_1) + S(r,\xi_2),\end{aligned}$$

we now suppose that there exist a set I of infinite measure such that $T(r,\xi_1) \leq T(r,\xi_2)$ for $r \in I$. Hence for $r \in I$, we get

$$(n-m)T(r,\xi_2) \leq \left(k + 2\Gamma_1 + \frac{2}{\mathcal{K}^*}\right)T(r,\xi_2) + S(r,\xi_2),$$

which is a contradiction because $N > k + m + 2\Gamma_1 + \frac{2}{\mathcal{K}^*}$.

Case 3:- Let $b=0$. From (3.3) we obtain $F(z) = \frac{G(z)+a-1}{a}$. If $a \neq 1$, then we obtain $\overline{N}(r,1-a;G) = \overline{N}(r,0;F)$. We can deduce a contradiction similarly as in case 2. Therefore, $a=1$ and so we have $F(z) \equiv G(z)$. This gives

$$\varepsilon_1^d(z)\left(\sum_{i=0}^{m_2} e_i\varepsilon_1^i(z)\right)\mathcal{F}_1 \equiv \varepsilon_2^d(z)\left(\sum_{i=0}^{m_2} e_i\varepsilon_2^i(z)\right)\mathcal{G}_1 \tag{3.4}$$

Let $h(z) = \frac{\varepsilon_1}{\varepsilon_2}$. If $h(z)$ is a constant by putting $\varepsilon_1 = h\varepsilon_2$ in (3.4), we obtain

$$e_{m_2}\varepsilon_2^{d+m_2}\left[h^{d+m+m_2}-1\right] + e_{m_2-1}\varepsilon_2^{d+m_2-1}\left[h^{d+m+m_2-1}-1\right] + \cdots + e_0\varepsilon_2^d\left[h^{d+m}-1\right] \equiv 0,$$

which gives $h^s = 1$ where $\mu = \gcd\{d+m+m_2, \cdots, n+m+m_2-i, \cdots, d+m\}$ with $e_{m_1}-i \neq 0$ for some $i \in \{0,1,\cdots,m_1\}$. Thus $\varepsilon_1(z) = t\varepsilon_2(z)$, that is $\xi_1 - c \equiv t(\xi_2 - c)$. If h is not constant, then we must have $\mathcal{P}(\xi_1)\mathcal{F}(z) \equiv \mathcal{P}(\xi_2)\mathcal{G}(z)$. □

Lemma 3.9. [4] Let ξ be a meromorphic function on $\mathbb{C}$ with finitely many poles. If ξ has a bounded spherical derivative on $\mathbb{C}$, then the order of ξ is not greater than 1.

Lemma 3.10. [12, Zalcman's Lemma] Let F be a family of meromorphic functions in the unit disc Δ and let α be a real number satisfying the inequality $-1 < \alpha < 1$. If F is not normal at a point $z_0 \in \Delta$, then for each α, $-1 < \alpha < 1$, there exist

i) Points $z_n \in \Delta$, $z_n \to z_0$,

ii) Positive numbers ρ_n, $\rho_n \to 0^+$,

iii) Functions $f_n \in F$, such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically uniformly on a compact subset of $\mathbb{C}$, where g is a non constant meromorphic function. The function g can be choosen to satisfy the normalization $g^{\#}(\zeta) \leq g^{\#}(0) = 1$, $\zeta \in \mathbb{C}$.

Lemma 3.11. [7] Suppose that $F(z)$ is meromorphic in a domian D and set $f(z) = \frac{F'(z)}{F(z)}$. We have

$$\frac{F^{(n)}(z)}{F(z)} = f^{(n)}(z) + \frac{n(n-1)}{2} f^{n-2} f'(z) + a_n f^{n-3} f''(z) + b_n f^{n-4} \left(f'(z)\right)^2 + P_{n-3}(f(z)),$$

where $a_n = \left(\frac{1}{6}\right) n(n-1)(n-2)$, $b_n = \left(\frac{1}{8}\right) n(n-1)(n-2)(n-3)$ and $P_{n-3}(f(z))$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

Lemma 3.12. Let ξ_1 and ξ_2 be two transcendental meromorphic function such that the zeros of $\xi_1 - c$ and $\xi_2 - c$ are of multiplicities at least $k \in \mathbb{N}$ and let $\mathcal{P}(\xi_1)\mathcal{F}(z) - p(z)$ and $\mathcal{P}(\xi_2)\mathcal{G}(z) - p(z)$ share 0 CM and in addition, ξ_1, ξ_2 share ∞ IM, where $\mathcal{P}(z)$ and $p(z)$ are defined in (2.1) and (2.3) respectively. Then $\mathcal{P}(\xi_1)\mathcal{F}(z)\mathcal{P}(\xi_2)\mathcal{G}(z) \not\equiv p^2(z)$

Proof . Suppose that

$$\mathcal{P}(\xi_1)\mathcal{F}(z)\mathcal{P}(\xi_2)\mathcal{G}(z) \equiv p^2(z). \tag{3.5}$$

Since ξ_1 and ξ_2 share ∞ IM, from (3.5), one can easily say that ξ_1 and ξ_2 are transcendental entire functions. Suppose that $\mathcal{P}(z)$ is a non-constant polynomial. For the sake of simplicity we may assume that $\mathcal{P}_1(z) = a_n(z - c_{m_2})^{m_2}$, where $d + m_2 = n$. Obviously, $c \neq c_{m_2}$. By (3.5) we have $N(r, c; \xi_1) = O(\log r)$ and $N(r, c_{m_2}; \xi_1) = O(\log r)$. So by second fundamental theorem we obtain

$$T(r, \xi_1) \leq \overline{N}(r, c; \xi_1) + \overline{N}(r, c_{m_2}; \xi_1)\overline{N}(r, \infty; \xi_1) + S(r, \xi_1) = S(r, \xi).$$

Therefore $\mathcal{P}(z)$ must be of the form $a_n(z-c)^n$ and so 93.5 reduces to the form

$$\varepsilon_1^n(z)\mathcal{F}_1(z)\varepsilon_2^n(z)\mathcal{G}_1(z) \equiv p_1^2(z) \tag{3.6}$$

where $p_1(z) = \frac{p(z)}{a_n}$. We now consider the following two cases.

Case 1. Let $\deg(p_1(z)) \in \mathbb{N}$. Then from (3.6) we see that $N\left(r, 0; \varepsilon_1^n\right) = O(\log r)$ and $N\left(r, 0; \varepsilon_2^n\right) = O(\log r)$. Let

$$F_1 = \frac{\varepsilon_1^n(z)\mathcal{F}_1}{p_1(z)}, \qquad G_1 = \frac{\varepsilon_2^n(z)\mathcal{G}_1}{p_1(z)} \tag{3.7}$$

Then (3.6) reduces to

$$F_1 G_1 \equiv 1. \tag{3.8}$$

If $F_1(z) \equiv eG_1$, where $e \in \mathbb{C} \setminus \{0\}$, then $F_1(z)$ must be a constant, which is not possible by Lemma 3.6. Hence $F_1(z) \not\equiv eG_1(z)$. Let

$$\Phi(z) = \frac{\varepsilon_1^n(z)\left(\mathcal{L}(\varepsilon_1(z))\right)^m - p_1(z)}{\varepsilon_2^n(z)\left(\mathcal{L}(\varepsilon_2(z))\right)^m - p_1(z)}. \tag{3.9}$$

Since $\varepsilon_1(z)$ and $\varepsilon_2(z)$ are transcendental entire functions, it follows that $\varepsilon_1^n\left(\mathcal{L}(\varepsilon_1(z))\right)^m - p_1(z) \neq \infty$ and $\varepsilon_2^n\left(\mathcal{L}(\varepsilon_2(z))\right)^m - p_1(z) \neq \infty$. Moreover $\varepsilon_1^n\left(\mathcal{L}(\varepsilon_1(z))\right)^m - p_1(z)$ and $\varepsilon_1^n\left(\mathcal{L}(\varepsilon_1(z))\right)^m - p_1(z)$ share 0 CM, it follows from (3.9) that

$$\Phi(z) = e^{\beta(z)}, \tag{3.10}$$

where β is an entire function. Let $\xi_{11}(z) = F_1(z)$, $\xi_{12} = -e^{\beta(z)}G_1(z)$ and $\xi_{13} = e^{\beta(z)}$, where ξ_{11} is transcendental. Thus from (3.10), we obtain $\xi_{11}(z) + \xi_{12}(z) + \xi_{13}(z) \equiv 1$. Also, by Lemma 3.2, we get

$$\sum_{j=1}^{3} N(r,0;\xi_{1j}) + 2\sum_{j=1}^{3} \overline{N}(r,\infty;\xi_{1j}) \leq N(r,0;F_1) + N(r,0;e^{\beta}G_1) + O(\log r) \leq (\lambda + o(1))T_1(r),$$

as $r \to \infty$, $r \in I$, $\lambda < 1$, where $T_1(r) = \max T(r,\xi_{1j})$. Therefore by Lemma 3.3, we get either $e^{\beta}G \equiv -1$ or $e^{\beta} \equiv 1$. However in this case the only possibility is $e^{\beta}G \equiv -1$. Hence from (3.5), we obtain $F \equiv e^{\gamma_1}G$ i.e., $\varepsilon_1^n(z)\left(\mathcal{L}(\varepsilon_1(z))\right)^m \equiv \varepsilon_2^n(z)\left(\mathcal{L}(\varepsilon_2(z))\right)^m$, where γ_1 is a non constant entire function. Thus it follows from (3.5) that

$$\varepsilon_1^n(z)\left(\mathcal{L}(\varepsilon_1(z))\right)^m \equiv d_2 e^{\gamma_1/2}p(z) \qquad and \qquad \varepsilon_2^n(z)\left(\mathcal{L}(\varepsilon_2(z))\right)^m \equiv d_2 e^{-\gamma_1/2}p(z), \tag{3.11}$$

where $d_2 = \pm 1$. This means that $\varepsilon_1^n(z)\left(\mathcal{L}(\varepsilon_1(z))\right)^m$ and $\varepsilon_2^n(z)\left(\mathcal{L}(\varepsilon_2(z))\right)^m$ share 0 CM. Clearly in view of (3.11), we see that F_1 and G_1 are entire functions without zeros. Let z_p be a zero of $\varepsilon_1(z)$ with multiplicity p and let z_q be a zero of $\varepsilon_2(z)$ with multiplicity q. Clearly z_p be a zero of $\varepsilon_1^n(z)\left(\mathcal{L}(\varepsilon_1(z))\right)^m$ of multiplicity $(n+1)p-k$ and z_q be a zero of $\varepsilon_2^n(z)\left(\mathcal{L}(\varepsilon_2(z))\right)^m$ with multiplicity $(n+1)q-k$. Since $\varepsilon_1^n(z)\left(\mathcal{L}(\varepsilon_1(z))\right)^m$ and $\varepsilon_2^n(z)\left(\mathcal{L}(\varepsilon_2(z))\right)^m$ share 0 CM, we conclude that $z_p = z_q$ and $p = q$. Consequently, $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share 0 CM. Since $N(r,0;\varepsilon_1(z)) = O(\log r) = N(r,0;\varepsilon_2(z))$, we can also take

$$\varepsilon_1(z) = h(z)e^{\alpha_{11}(z)} \qquad and \qquad \varepsilon_2(z) = h(z)e^{\alpha_{12}(z)} \tag{3.12}$$

where $h(z)$ is a non constant polynomial and α_{11} and α_{12} are two non constant entire functions. It follows from (3.12) that

$$\varepsilon_1^n(z)\left(\mathcal{L}(\varepsilon_1(z))\right)^m \equiv P_1\left(h,h',\ldots,h^{(k)},\alpha_{11}',\ldots,\alpha_{11}^{(k)}\right)e^{(n+m)\alpha_{11}(z)} \tag{3.13}$$

where $P_1\left(h,h',\ldots,h^{(k)},\alpha_{11}',\ldots,\alpha_{11}^{(k)}\right) = h^n\left(\sum\limits_{i=0}^{k} a_i P_{1i}(h,h',\ldots,h^{(i)},\alpha_{11}',\ldots,\alpha_{11}^{(i)})\right)^m$, $P_{1i}(h,h',\ldots,h^{(i)},\alpha_{11}',\ldots,\alpha_{11}^{(i)})$ is a differential polynomial in $h,h',\ldots,h^{(i)},\alpha_{11}',\ldots,\alpha_{11}^{(i)}$, $i = 1,2,\cdots,k$ $\;P_{10} = a_0 h$ and

$$\varepsilon_2^n(z)\left(\mathcal{L}(\varepsilon_2(z))\right)^m \equiv P_2\left(h,h',\ldots,h^{(k)},\alpha_{12}',\ldots,\alpha_{12}^{(k)}\right)e^{(n+m)\alpha_{12}(z)} \tag{3.14}$$

where $P_2\left(h,h',\ldots,h^{(k)},\alpha_{12}',\ldots,\alpha_{12}^{(k)}\right) = h^n\left(\sum\limits_{i=0}^{k} a_i P_{2i}(h,h',\ldots,h^{(i)},\alpha_{12}',\ldots,\alpha_{12}^{(i)})\right)^m$, $P_{2i}(h,h',\ldots,h^{(i)},\alpha_{12}',\ldots,\alpha_{12}^{(i)})$ is a differential polynomial in $h,h',\ldots,h^{(i)},\alpha_{12}',\ldots,\alpha_{12}^{(i)}$, $i = 1,2,\ldots,k$ $\;P_{20} = a_0 h$. Let $\wp_1 = \{F_{1w}\}$ and $\wp_2 = \{G_{1w}\}$, where $F_{1w}(z) = F_1(z+c)$ and $G_{1w}(z) = G_1(z+c)$, $z \in \mathbb{C}$. Clearly $\wp_1$ and $\wp_2$ are two families of entire functions defined on $\mathbb{C}$. We now consider the following two subcases.

Subcase 1.1. Let us assume that one of two families, denoted as $\wp_1$ and $\wp_2$, specifically $\wp_1$, is classified as normal on $\mathbb{C}$. Then by Mary's theorem, $F_1^{\#}(w) = F_{1w}^{\#}(0) \leq M$ for some $M > 0$ and all $w \in \mathbb{C}$. Hence, by Lemma 3.9, It is deduced that the order of F_1 does not exceed 1. Hence from (3.8), we obtain

$$\rho\left[\varepsilon_1^n(z)(\mathcal{L}(\varepsilon_1(z)))^m\right] = \rho(F_1) = \rho(G_!) = \rho\left[\varepsilon_2^n(z)(\mathcal{L}(\varepsilon_2(z)))^m\right] \leq 1. \tag{3.15}$$

Consequently, we get

$$\varepsilon_1^n(z)(\mathcal{L}(\varepsilon_1(z)))^m = d_3 p e^{az} \quad and \quad \varepsilon_2^n(z)(\mathcal{L}(\varepsilon_2(z)))^m = d_4 p e^{bz}, \tag{3.16}$$

where $a,b,d_3,d_4 \in \mathbb{C}\backslash\{0\}$. From (3.15) we see that $a+b=0$. We claim that $(n+m)\alpha_{11} - az \in \mathbb{C}$ and $(n+m)\alpha_{12} - az \in \mathbb{C}$. On the contrary, suppose that$(n+m)\alpha_{11} - az \notin \mathbb{C}$ and $(n+m)\alpha_{12} - az \notin \mathbb{C}$. Assume that $\alpha_{21} = (n+m)\alpha_{11} - az$ and $\alpha_{22} = (n+m)\alpha_{12} - bz$. Note that

$$\begin{aligned} T(r,\alpha_{11}') &= m(r,\alpha_{11}') \leq m(r,(n+m)\alpha_{11}') + O(1) \\ &\leq m(r,\alpha_{21}') + O(1) = m\left(\frac{(e^{\alpha_{21}})'}{e^{\alpha_{21}}}\right) + +O(1) = S(r,e^{\alpha_{21}}). \end{aligned}$$

Clearly, $\alpha_{11}^{(i)} \in S(\alpha_{21})$ for $i \in \mathbb{N}$. Therefore $P_1 \in S(\alpha_{21})$ and hence $\frac{p}{P_1} \in S(\alpha_{21})$. Similarly, we find $\frac{p}{P_2} \in S(\alpha_{22})$. Further, from (3.13),(3.14) and (3.16), we conclude that $e^{\alpha_{21}} \in S(\alpha_{21})$ and $e^{\alpha_{22}} \in S(\alpha_{22})$ which is contradiction. Hence $\alpha_{21},\ \alpha_{22} \in \mathbb{C}$ and therefore, both α_{11} and α_{12} are polynomials of degree 1. Finally, we take

$$\varepsilon_1(z) = d_5 h(z)e^{az} \qquad and \qquad \varepsilon_2(z) = d_6 h(z)e^{-az} \tag{3.17}$$

$d_5, d_6 \in \mathbb{C}\setminus\{0\}$. Hence from (3.17), we get

$$\varepsilon_1^n(z)(\mathcal{L}(\varepsilon_1(z)))^m = d_5^{m+n} h^n(z)\left(a_0 h(z) + \sum_{j=1}^{k} a_j\left(\sum_{i=0}^{j} C_i^j a^{j-i} h^{(i)}(z)\right)\right)^m e^{(n+m)az}$$

where $h^{(0)}(z) = h(z)$. Similarly, we obtain

$$\varepsilon_2^n(z)(\mathcal{L}(\varepsilon_2(z)))^m = d_6^{m+n} h^n(z)\left(a_0 h(z) + \sum_{j=1}^{k} a_j\left(\sum_{i=0}^{j} C_i^j (-1)^{j-i} a^{j-i} h^{(i)}(z)\right)\right)^m e^{-(n+m)az}.$$

Given that $\varepsilon_1^n(z)(\mathcal{L}(\varepsilon_1(z)))^m$ and $\varepsilon_2^n(z)(\mathcal{L}(\varepsilon_2(z)))^m$share 0 CM, it can be deduced that

$$a_0 h(z) + \sum_{j=1}^{k} a_j\left(\sum_{i=0}^{j} C_i^j a^{j-i} h^{(i)}(z)\right) \equiv d_7\left(a_0 h(z) + \sum_{j=1}^{k} a_j\left(\sum_{i=0}^{j} C_i^j (-1)^{j-i} a^{j-i} h^{(i)}(z)\right)\right) \tag{3.18}$$

where $d_7 \in \mathbb{C}\setminus\{0\}$. However, the relation (3.18) is not true.

Subcase 1.2. Suppose that one of the families $\wp_1$ and $\wp_2$, say $\wp_1$ is normal on $\mathbb{C}$. In this case by Marty's theorem, there exists a sequence of meromorphic functions $\{F_1(z+w_j)\}\subset\wp_1$, where $z\in\{z:|z|<1\}$ and $\{w_j\}\subset\mathbb{C}$ is a sequence such that $F_1^{\#}(w_j)\to\infty$ as $|w_j|\to\infty$. Thus by Lemma 3.10 there exist,

- Points $z_j,\ |z_j|<1$,
- Positive numbers $\rho_j,\ \rho_j\to 0^+$,
- a subsequence $\{F_1(w_j+z_j+\rho_j\zeta)\}$ of $\{F_1(w_j+z)\}$ such that

$$h_j^*(\zeta) = \rho^{-1/2} F_1(w_j+z_j+\rho_j\zeta)\to h^*(\zeta) \tag{3.19}$$

Spherically uniformly on a compact subset of $\mathbb{C}$, where $h^*(\zeta)$ is a non constant holomorphic function such that $h^{*\#}(\zeta)\le h^{*\#}(0)=1$. Further, from Lemma 3.9, we see that $\rho(h^*)\le 1$. By Hurwitz's theorem, we can show that $h^*(\zeta)\ne 0$. In the proof of Zalcman's Lemma, we see that

$$\rho_j = \frac{1}{F_1^{\#}(b_j)}, \tag{3.20}$$

and

$$F_1^{\#}(b_j)\ge F_1^{\#}(w_j), \tag{3.21}$$

where $b_j = w_j + z_j$. Let

$$h_j^{**}(\zeta) = \rho_j^{1/2} G_1(w_j+z_j+\rho_j\zeta), \tag{3.22}$$

from (3.8),

$$F_1(w_j+z_j+\rho_j\zeta)G_1(w_j+z_j+\rho_j\zeta)\equiv 1.$$

Hence from (3.19) and (3.22), we get

$$h_j^*(\zeta)h_j^{**}(\zeta)\equiv 1. \tag{3.23}$$

Further, from (3.19) and (3.23) we can deduce that

$$h_j^*(\zeta)\to h_j^{**}(\zeta), \tag{3.24}$$

Spherically uniformly on a compact subset of $\mathbb{C}$, where $h_j^{**}(\zeta)$ is a non-constant holomorphic function in the complex plane. By Hurwitz's theorem, we can conclude that $h_j^{**}(\zeta)\ne 0$. Further, it follows from (3.19), (3.23) and (3.24), that

$$h_j^*(\zeta)h_j^{**}(\zeta)\equiv 1. \tag{3.25}$$

Thus in view of (3.25) and $\rho(h^*) \leq 1$, we can show that

$$\rho(h^*) = \rho(h^{**}) \leq 1 \tag{3.26}$$

Note that h^* and h^{**} are transcendental entire functions without zeros. Thus it follows from (3.25) that

$$h^*(z) = d_8 e^{cz} \quad and \quad h^{**}(z) = d_9 e^{-cz} \tag{3.27}$$

where $c, d_8, d_9 \in \mathbb{C} \setminus \{0\}$ are such that $d_8 d_9 = 1$. From (3.27), we obtain

$$\frac{h_j^{*'}(\zeta)}{h_j^*(\zeta)} = \rho_j \frac{F_1'(w_j + z_j + \rho_j\zeta)}{F_1(w_j + z_j + \rho_j\zeta)} \to \frac{h^{*'}(\zeta)}{h^*(\zeta)} = C \tag{3.28}$$

Spherically uniformly on a compact subset of $\mathbb{C}$. Thus it follows from (3.20) and (3.28), that

$$\left|\frac{h_j^{*'}(0)}{h_j^*(0)}\right| = \rho_j \left|\frac{F_1'(w_j + z_j)}{F_1(w_j + z_j)}\right| = \frac{1 + |F_1'(w_j + z_j)|^2}{|F_1'(w_j + z_j)|} \frac{|F_1'(w_j + z_j)|}{|F_1(w_j + z_j)|}$$

$$\left|\frac{h_j^{*'}(0)}{h_j^*(0)}\right| = \frac{1 + |F_1'(w_j + z_j)|^2}{|F_1(w_j + z_j)|} \to \left|\frac{h_j^{*'}(0)}{h_j^*(0)}\right| = |C|. \tag{3.29}$$

This implies that

$$\lim_{j\to\infty} F_1(w_j + z_j) \neq 0, \ \infty \tag{3.30}$$

By using (3.19), (3.30), we find

$$h_j^*(0) = \rho_j^{-1/2} F_1(w_j + z_j) \to \infty. \tag{3.31}$$

In addition, by virtue of (3.19) and (3.27), we get

$$h_j^*(0) \to h^*(0) = C. \tag{3.32}$$

Thus, in view of (3.31) and (3.32), we arrive at a contradiction.

Case 2:- Let $p_1(z) \equiv b \in \mathbb{C} \setminus \{0\}$. Then (3.6) reduces to $\varepsilon_1^n(z)\mathcal{F}_1(z)\varepsilon_2^n(z)\mathcal{G}_1(z) \equiv b^2$. This shows that both $\varepsilon_1(z)$ and $\varepsilon_2(z)$ have no zeros. But this is not possible as zeros of $\varepsilon_1(z)$ and $\varepsilon_2(z)$ are of multiplicities at least $k(\geq 1)$. This completes the proof. □

Lemma 3.13. [8] Let $\xi_1(z)$ and $\xi_2(z)$ be two non-constant meromorphic functions. Suppose that $\xi_1(z)$ and $\xi_2(z)$ share $(0, \infty)$, (∞, ∞) $\xi_2^{(k)}(z)$ and $\xi_1^{(k)}(z)$ share $(0, \infty)$ for $k = 1, 2, \ldots, 6$. Then $\xi_1(z)$ and $\xi_2(z)$ satisfy one of the following cases:

(i) $\xi_1(z) \equiv t\xi_2(z)$, where $t \in \mathbb{C} \setminus \{0\}$,
(ii) $\xi_1(z) = e^{az+b}$ and $\xi_2(z) = e^{cz+d}$, where $a, b, c, d \in \mathbb{C} \setminus \{0\}$ such that $ac \neq 0$,
(iii) $\xi_1(z) = a/(1 - be^{\alpha(z)})$ and $\xi_2(z) = a/(e^{-\alpha(z)-b})$, where $a, b \in \mathbb{C} \setminus \{0\}$ and α is a non constant entire function;
(iv) $\xi_1(z) = a(1 - be^{cz})$ and $\xi_2(z) = \xi_1(z) = a(e^{-cz-b})$, where $a, b, c, d \in \mathbb{C} \setminus \{0\}$.

Lemma 3.14. let $\xi_1(z)$ and $\xi_2(z)$ be two transcendental meromorphic functions such that the zeros of $\xi_1(z) - c$ and $\xi_2(z) - c$ are of multiplicities at least k, where $k, n, m \in \mathbb{N}$. Assume that $\mathcal{L}(\xi_1(z))$ and $\mathcal{L}(\xi_2(z))$ share 0 CM and that $\xi_1(z)$ and $\xi_2(z)$ share ∞ IM. If $(\xi_1 - c)^n \mathcal{L}^m(\xi_1) \equiv (\xi_2 - c)^n \mathcal{L}^m(\xi_2)$ then $(\xi_1 - c) \equiv t(\xi_t - c)$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+m} = 1$.

Proof . Suppose that

$$\left(\frac{\varepsilon_1(z)}{\varepsilon_2(z)}\right)^n \equiv \left(\frac{\mathcal{L}(\varepsilon_1(z))}{\mathcal{L}(\varepsilon_2(z))}\right)^m. \tag{3.33}$$

Since $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share $(\infty, 0)$, it follows from (3.33) that $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share (∞, ∞) so $\mathcal{L}(\varepsilon_1(z))$ and $\mathcal{L}(\varepsilon_2(z))$ share ∞ CM. Moreover, since $\mathcal{L}(\xi_1(z))$ and $\mathcal{L}(\xi_2(z))$ share 0 CM, it follows that $\mathcal{L}(\varepsilon_1(z))$ and $\mathcal{L}(\varepsilon_2(z))$ share

0 CM. We conclude that $\varepsilon_1(z)$ and $\varepsilon_2(z)$ also share 0 CM. Let $H_1(z)=\varepsilon_1(z)/\varepsilon_2(z)$ and $H_2(z)=\mathcal{L}(\varepsilon_1(z))/\mathcal{L}(\varepsilon_2(z))$. Then $H_1\neq 0,\ \infty$ and $H_2\neq 0,\ \infty$. From (3.33), we see that

$$H_1^n H_2^m \equiv 1 \tag{3.34}$$

Let us assume that $H_1(z)$ is an entire function that is not constant. $H_2(z)$ is also an entire function which is not constant. Let $F_1(z)=H_1^n(z)$ and $G_1(z)=H_2^m(z)$. Further, from (3.34) we get

$$F_1(z)G_1(z)\equiv 1. \tag{3.35}$$

Clearly, $F_1(z)\equiv d_0G_1$, where $d_0\in\mathbb{C}\setminus\{0\}$. Otherwise, $F_1(z)\in\mathbb{C}$ and, therefore $H_1(z)$ is constant. Since $F_1(z)\neq 0,\ \infty$ and $G_1(z)\neq 0,\ \infty$, there exist two non constant entire functions α_1 and α_2 such that $F_1(z)=e^{\alpha_1(z)}$ and $G_1(z)=e^{\alpha_2(z)}$. Further from (3.35) we see that $\alpha_1+\alpha_2\in\mathbb{C}$ and so $\alpha_1'=-\alpha_2'$. Note that $F_1'=\alpha_1'e^{\alpha_1}$ and $G_1'=\alpha_2'e^{\alpha_2}$. This shows that F_1' and G_1' share 0 CM.

Thus in view of Lemma 3.13, we get $F_1(z)=c_1e^{az}$ and $G_1(z)=c_2e^{-az}$ where $a,\ c_1,\ c_2\in\mathbb{C}\setminus\{0\}$ with $c_1c_2=1$. Since $\left(\frac{\varepsilon_1(z)}{\varepsilon_2(z)}\right)^n=c_1e^{az}$ and $\left(\frac{\mathcal{L}(\varepsilon_1(z))}{\mathcal{L}(\varepsilon_2(z))}\right)$ it follows that

$$\begin{cases}\frac{\varepsilon_1(z)}{\varepsilon_2(z)}=t_1e^{\frac{a}{n}z}=t_1e^{cz},\\ \frac{\varepsilon_1(z)}{\varepsilon_2(z)}=t_1e^{\frac{a}{n}z}=t_1e^{cz},\end{cases} \tag{3.36}$$

where $c,\ d,\ t_1\ t_2\in\mathbb{C}\setminus\{0\}$ are such that $t_1^n=c_1,\ t_2^m=c_2,\ c=a/n$ and $d=-a/m$. Let

$$\Phi_1(z)=\frac{\mathcal{L}'(\varepsilon_1(z))}{\mathcal{L}(\varepsilon_1(z))}-\frac{\mathcal{L}'(\varepsilon_2(z))}{\mathcal{L}(\varepsilon_2(z))}, \tag{3.37}$$

it follows from (3.36) that

$$\Phi_1(z)=d\left(\frac{-a}{m}\right). \tag{3.38}$$

Further, by using (3.36), we see that

$$\varepsilon_1^{(j)}(z)=t_1\sum_{i=0}^{j}C_i^j(e^{cz})^{(i)}\varepsilon_2^{(j-i)}(z)$$

i.e.,

$$\varepsilon_1^{(j)}(z)=t_1e^{cz}\left(\varepsilon_2^{(j)}(z)+jc\varepsilon_2^{(j-1)}(z)+\frac{j(j-1)}{2}c^2\varepsilon_2^{(j-2)}(z)+\cdots+c^j\varepsilon_2(z)\right).$$

Consequently,

$$\mathcal{L}(\varepsilon_1(z))=t_1e^{cz}\left(a_k\varepsilon_2^{(k)}(z)+(kca_k+a_{k-1})\xi_2^{(k-1)}(z)+\left(\frac{k(k-1)}{2}c^2a_k+(k-1)ca_{k-1}+a_{k-2}\right)\varepsilon_2^{(k-2)}+\cdots\right), \tag{3.39}$$

and

$$\mathcal{L}'(\varepsilon_1(z))=t_1e^{cz}\left(a_k\varepsilon_2^{(k+1)}(z)+((k+1)ca_k+a_{k-1})\xi_2^{(k)}(z)+\left(\frac{k(k+1)}{2}c^2a_k+(k)ca_{k-1}+a_{k-2}\right)\varepsilon_2^{(k-1)}+\cdots\right). \tag{3.40}$$

Thus from (3.37), (3.39) and (3.40), we obtain

$$\Phi_1(z)=\frac{G_2(z)+(k+1)c\varepsilon_2^{(k)^2}-kc\varepsilon_2^{(k-1)}\varepsilon_2^{(k+1)}}{G_3+\left(\varepsilon_2^{(k)}(z)\right)^2}, \tag{3.41}$$

where $G_2(z)=\sum A_{i,j}\varepsilon_2^{(i)}(z)\varepsilon_2^{(j)}(z)$ and $G_3(z)=\sum B_{i,j}\varepsilon_2^{(i)}(z)\varepsilon_2^{(j)}(z)$ where $0\le i\le k+1,\ 0\le j\le k,\ 0\le i+j\le 2k-1$. $A_{i,j},\ B_{i,j}\in\mathbb{C}$. Let z_p be a zero of $\varepsilon_2(z)$ with multiplicity $p(\ge k)$. Then the Taylor expansion of $\varepsilon_2(z)$ about z_p is

$$\varepsilon_2(z)=b_p(z-z_p)^p+b_{p+1}(z-z_p)^{p+1}+b_{p+2}(z-z_p)^{p+2}+\cdots,\quad b_p\neq 0 \tag{3.42}$$

We now consider the following two cases:

Case 1. Suppose that $p = k$. Then

$$\varepsilon_2^{(k)}(z) = k!b_k + (k+1)!b_{k+1}(z - z_k) + \cdots \tag{3.43}$$

and

$$\varepsilon_2^{(k+1)}(z) = (k+1)!b_{k+1} + (k+2)!b_{k+2}(z - z_k) + \cdots \tag{3.44}$$

Thus, by using (3.41), (3.43) and (3.44), we obtain

$$\Phi_1(z_k) = c\frac{(k+1)(k!)^2 b_k^2}{(k!)^2 b_k^2} = c(k+1) \tag{3.45}$$

Therefore, we arrive at a contradiction from (3.38) and (3.45).

Case 2:- Suppose that $p \geq k+1$. Then

$$\begin{aligned} \varepsilon_1^{(k-1)}(z) &= p(p-1)\cdots(p-k+2)b_p(z-z_p)^{(p-k+1)} + \cdots \\ \varepsilon_1^{(k)}(z) &= p(p-1)\cdots(p-k+1)b_p(z-z_p)^{(p-k)} + \cdots \\ \varepsilon_1^{(k+1)}(z) &= p(p-1)\cdots(p-k)b_p(z-z_p)^{(p-k-1)} + \cdots . \end{aligned}$$

Therefore

$$\left(\varepsilon_2^{(k)}(z)\right)^2 = \mathcal{K}b_p^2(z-z_p)^{2(2-k)} + \cdots \tag{3.46}$$

$$\varepsilon_2^{(k-1)}(z)\varepsilon_2^{(k+1)}(z) = \frac{p-k}{p-k+1}\mathcal{K}b_p^2(z-z_p)^{2(2-k)} + \cdots \tag{3.47}$$

where $\mathcal{K} = [p(p-1)\cdots(p-k+1)]^2$. In addition $G_2(z) = O\left((z-z_p)^{2p-i-j}\right)$ and $G_3(z) = O\left((z-z_p)^{2p-i-j}\right)$ where $2(p-k)+2 \leq 2p-i-j \leq 2p$. Thus, it follows from (3.41), (3.46) and (3.47) that

$$\Phi_1(z_p) = \frac{(k+1)c\mathcal{K}b_p^2 - kc\left\{(p-k)/(p-k+1)\right\}\mathcal{K}b_p^2}{\mathcal{K}b_p^2} = c\frac{p+1}{p-k+1}. \tag{3.48}$$

Therefore, we arrive at a contradiction between (3.38) and (3.48). The non-existence of zeros in $\varepsilon_2(z)$ can be confidently stated in each instance. Given that $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share 0 CM, it can be concluded that neither of them has zeros. This conclusion is problematic because the multiplicities of zeros in both functions are at least $k \in \mathbb{N}$. Hence $H_1 \in \mathbb{C}\setminus\{0\}$. Thus it follows from (3.33) that $H_1^{n+m} = 1$. Therefore we get $\varepsilon_1(z) \equiv t\varepsilon_2(z)$, where $t \in \mathbb{C}\setminus\{0\}$ is such that $t^{n+m} = 1$, i.e., $(\xi_1(z) - c) \equiv t\,(\xi_2(z) - c)$, $t \in \mathbb{C}\setminus\{0\}$ with $t^{n+m} = 1$. □

Lemma 3.15. Let ξ_1 and ξ_2 be two transcendental meromorphic functions such that the zeros of $\xi_1(z)-c$ and $\xi_2(z)-c$ are of multiplicities at least $\mathcal{K}^*$, where $\mathcal{K}^*$ is defined as in (2.2) and $F(z) = \mathcal{P}(\xi_1)\mathcal{F}(z)/p(z)$ and $G(z) = \mathcal{P}(\xi_2)\mathcal{G}(z)/p(z)$, where $p(z)(\neq 0)$ is a polynomial and $m, n \in \mathbb{N}$ are such that $n + m(k+1) > 2k+1$. Suppose $\mathcal{H} \not\equiv 0$. If $F(z)$ and $G(z)$ share $(1, k_1)$ except for the zeros of $p(z)$ and $\xi_1(z)$ and $\xi_2(z)$ share $(\infty, 0)$ where $0 \leq k_1 \leq \infty$, then

$$\overline{N}(r,\infty;\xi_1) \leq \frac{(\Gamma_1+1)\mathcal{K}^*+1}{\mathcal{K}^*(n+m+(m-2)k-1)}\left(T(r,\xi_1)+T(r,\xi_2)\right) + \frac{\overline{N}_*(r,1;F,G)}{n+m+(m-2)k-1} + S(r,\xi_1) + S(r,\xi_2)$$

Proof . Since $\mathcal{H}(z) \neq 0$, it follows that $F \neq G$. We observe that if ∞ is Picard's exceptional value of $\xi_1(z)$, then the Lemma follows immediately. Next, we suppose that ∞ is not Picard's exceptional $\xi_1(z)$ value. Since $\xi_1(z)$ and $\xi_2(z)$share $(\infty, 0)$, it follows that ∞ is not a Picard's exceptional value of $\xi_2(z)$. We claim that $\mathcal{V} \not\equiv 0$. Indeed, suppose that $\mathcal{V} \equiv 0$. Then, as a result of integration we obtain

$$1 - \frac{1}{F} \equiv A\left(1 - \frac{1}{G}\right),\ A \in \mathbb{C}\setminus\{0\}$$

it is clear that if z_0 is a pole of $\xi_1(z)$, then it is a pole of $\xi_2(z)$. Hence, it follows from the definition of F and G that $\frac{1}{F(z_0)} = 0 = \frac{1}{G(z_0)}$. Therefore $A = 1$ and hence $F \equiv G$. Since $\mathcal{H} \not\equiv 0$, we conclude that $F(z) \not\equiv G(z)$. Therefore, we arrive at a contradiction. Hence $\mathcal{V} \not\equiv 0$. Let z_0 be a pole of $\xi_1(z)$ with multiplicity q and a pole of $\xi_2(z)$ with multiplicity r such that $p(z_0) \neq 0$. Clearly z_0 is a pole of $F(z)$ with multiplicity $(n+m)q + mk$ and a pole of $G(z)$ with multiplicity $(n+m)r + mk$. Therefore

$$\frac{F'(z)}{F(z)(F(z)-1)} = o\left((z-z_0)^{(n+m)q+mk-1}\right)$$

and

$$\frac{G'(z)}{G(z)(G(z)-1)} = o\left((z-z_0)^{(n+m)r+mk-1}\right).$$

Consequently, $\mathcal{V}(z) = o\left((z-z_0)^{(n+m)t+mk-1}\right)$, where $t = \max\{q, r\}$. Since $\xi_1(z)$ and $\xi_2(z)$ share $(\infty, 0)$, from the definition of $\mathcal{V}$, it is clear that z_0 is a zero of $\mathcal{V}$ with mulitiplicity of at least $n + m(k+1) - 1$. Also, $m(r, \mathcal{V}) = S(r, \xi_1) + S(r, \xi_2)$. Thus by using the definition of $\mathcal{V}$ and Lemma 3.2 and 3.1, we obtain

$$\begin{aligned}(n+m(k+1)-1)\overline{N}(r,\infty;\xi_1(z)) &\leq N(r,0'\mathcal{V}) + O(\log r) + T(r,\mathcal{V}) + S(r,\xi_1) + S(r,\xi_2)\\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,1;F,G) + S(r,\xi_1) + S(r,\xi_2)\\ &\leq \left[\frac{\mathcal{K}^*(\Gamma_1+1)+1}{\mathcal{K}^*}\right](T(r,\xi_1)+T(r,\xi_2)) + 2k\overline{N}(r,\infty;\xi_1) + \overline{N}_*(r,1;F,G)\\ &\quad + S(r,\xi_1) + S(r,\xi_2)\end{aligned}$$

Thus, the Lemma follows. □

Lemma 3.16. [1] Let $\xi_1(z)$ and $\xi_2(z)$ be two non-constant meromorphic functions sharing $(1, k_1)$, where $2 \leq k_1 \leq \infty$. Then

$$\begin{aligned}N(r,1;\xi_2) - \overline{N}(r,1;\xi_2) &\geq \overline{N}(r,1;\xi_1(z)|=2) + 2\overline{N}(r,1;\xi_1(z)|=3) + \cdots + k_1\overline{N}_L(r,1;\xi_1(z))\\ &+ (k_1+1)\overline{N}_L(r,1;\xi_2(z)) + k_1\overline{N}_L^{(k_1+1}(r,1;\xi_2(z)) \leq N(r,1;\xi_2(z)) - \overline{N}(r,1;\xi_2(z)).\end{aligned}$$

Lemma 3.17. Let ξ_1 and ξ_2 be two transcendental meromorphic functions such that the zeros of $\xi_1(z)-c$ and $\xi_2(z)-c$ are of multiplicities at least $\mathcal{K}^*$, where $\mathcal{K}^*$ is defined as in (2.2). Let $n,\ m \in \mathbb{N}$, suppose $\mathcal{L}(\xi_1(z))$, $\mathcal{L}(\xi_2(z))$ share $(0,\infty)$ also we assume that $\xi_1(z)$ and $\xi_2(z)$ share $(\infty, 0)$. If $\varepsilon_1^d(z)\mathcal{P}_2(\varepsilon_1(z))\mathcal{L}^m(\varepsilon_1(z)) \equiv \varepsilon_2^d(z)\mathcal{P}_2(\varepsilon_2(z))\mathcal{L}^m(\varepsilon_2(z))$, where $\varepsilon_1(z) = \xi_1(z) - c$ and $\varepsilon_2(z) = \xi_2(z) - c$, then one of the following cases holds:

1. If $\mathcal{P}_2(z_1) \equiv e_i z_1^i \not\equiv 0$ for some $i \in \{0,1,2,\cdots,m_1\}$ and $\mathcal{L}(\varepsilon_1(z))$, $\mathcal{L}(\varepsilon_2(z))$ share $(0,\infty)$ then $\xi_1 - c \equiv t(\xi_2 - c)$, where $t \in \mathbb{C}\setminus\{0\}$ such that $t^{d+m+i} = 1$, for some $i \in \{0,1,2,\cdots,m_1\}$.
2. If $\mathcal{P}_2(z_1) \not\equiv e_i z_1^i \not\equiv 0$ for some $i \in \{0,1,2,\cdots,m_1\}$ and $\mathcal{L}(\varepsilon_1(z))$, $\mathcal{L}(\varepsilon_2(z))$ share $(0,\infty)$ and $\xi_1(z)$, $\xi_2(z)$ share $(c, 0)$, then $\xi_1(z) - c \equiv t(\xi_2 - c)$ for a constant t such that $t^s = 1$, where $\mu = \gcd(n+m,\cdots,n+m-i,\cdots,1)$ $e_{m_1-i} \neq 0$, for some $i = 0,1,2,\ldots,m_1-1$

Proof . Suppose

$$\varepsilon_1^d(z)\mathcal{P}_2(\varepsilon_1(z))\mathcal{L}^m(\varepsilon_1(z)) \equiv \varepsilon_2^d(z)\mathcal{P}_2(\varepsilon_2(z))\mathcal{L}^m(\varepsilon_2(z)), \tag{3.49}$$

$$\frac{\mathcal{P}_2(\varepsilon_1(z))}{\mathcal{P}_2(\varepsilon_2(z))} \equiv \frac{\varepsilon_1^d(z)\mathcal{L}^m(\varepsilon_1(z))}{\varepsilon_2^d(z)\mathcal{L}^m(\varepsilon_2(z))} \tag{3.50}$$

We now consider the following two cases.

Case 1. Suppose $\mathcal{P}_2(z_1) \equiv e_i z_1^i \not\equiv 0$ for some $i \in \{0,1,2,\cdots,m_1\}$. Then, the results follow from the Lemma 3.14.

Case 2. Suppose $\mathcal{P}_2(z_1) \not\equiv e_i z_1^i \not\equiv 0$. For the sake of simplicity we assume that $\mathcal{P}_2(z_1) = e_{m_1}z_1^{m_1} + e_{m_1-1}z_1^{m_1-1} + \cdots + e_1 z_1 + e_0$, e_{m_1}, $e_0 \neq 0$. Since $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share $(\infty, 0)$, from (3.49), we see that $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share (∞,∞). Now we prove that $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share $(0,\infty)$. Note that $\mathcal{P}_2(0) \neq 0$. Let z_{12} be a zero of $\varepsilon_1(z)$ of multiplicity $r_{12}(\geq k+1)$. Since $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share $(0,0)$, z_{12} is a zero of $\varepsilon_2(z)$ of multiplicity $q_{12}(\geq k+1)$. Clearly z_{12} is a zero of $\mathcal{L}(\varepsilon_1(z))$ of multiplicity $\mathcal{L}(\varepsilon_2(z))$ $r_{12} - k$ and a zero of multiplicity $q_{12} - k$. Since $\mathcal{L}(\varepsilon_1(z))$ and $\mathcal{L}(\varepsilon_2(z))$ share $(0,\infty)$, we have $r_{12} = q_{12}$. Therefore $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share $(0,\infty)$. Since $\varepsilon_1(z)$ and $\varepsilon_2(z)$ share $(0,\infty)$ and (∞,∞), it follows that $\varepsilon_1(z) = e^{\gamma(z)}\varepsilon_2(z)$, where $\gamma(z)$ is an entire function. Let $H_1^*(z) = \frac{\mathcal{P}_2(\varepsilon_1(z_1))}{\mathcal{P}_2(\varepsilon_2(z_1))}$ and

$H_2^*(z) = \frac{\varepsilon_1^d(z)\mathcal{L}^m(\varepsilon_1)}{\varepsilon_2^d(z)\mathcal{L}^m(\varepsilon_2)}$. Since $\mathcal{L}^m(\varepsilon_1)$ and $\mathcal{L}^m(\varepsilon_2)$ share $(0,\infty)$, we have $H_2^*(z) \neq 0,\infty$. Also from (3.49) we see that $H_2^*(z) \neq 0$ and

$$H_1^*(z)H_2^*(z) \equiv 1. \tag{3.51}$$

Now, we consider the following two subcases.

Subcase 2.1. Suppose $H_1^*(z) = b \in \mathbb{C} \setminus \{0\}$. Let $b = 1$. Then from (3.49), we have $\varepsilon_1^d(z)\mathcal{L}^m(\varepsilon_1(z)) \equiv \varepsilon_2^d(z)\mathcal{L}^m(\varepsilon_2(z))$. Then, the results follow from Lemma 3.14. Let $b \neq 1$. Then we have

$$\sum_{i=0}^{m_1} e_i\, \varepsilon_1^i(z) \;\equiv\; b\sum_{i=0}^{m_1} e_i\, \varepsilon_2^i(z) \tag{3.52}$$

Since $\varepsilon_1(z) = e^{\gamma(z)}\varepsilon_2(z)$, from (3.52), we have

$$e_{m_1}\varepsilon_2^{m_1}(z)\left(e^{m_1\gamma(z)} - b\right) + \cdots + e_1\varepsilon_2(z)\left(e^{\gamma(z)} - b\right) \equiv e_0(b-1) \tag{3.53}$$

Note that $\varepsilon_2(z) \not\equiv d \in \mathbb{C}$. Then from (3.53) we see that $\varepsilon_2(z)$ has no zero. But this is impossible because zeros of $\varepsilon_2(z)$ are of multiplicities at least $k+1$.

Subcase 2.2. Suppose H_1^* is non constant.

Therefore, H_2^* is a non-constant entire function. Note that $H_1^*(z) \not\equiv d_0^* H_2^*(z)$, where d_0^* is a non zero constant. Since $H_1^*(z)$ and $H_2^*(z)$ both are $\neq 0,\infty$, then there exist two non constant entire functions $\alpha^*(z)$ and $\beta^*(z)$ such that $H_1^*(z) = e^{\alpha^*(z)}$ and $H_2^*(z) = e^{\beta^*(z)}$. Now from (3.51) we see that $(\alpha^*(z))' = -(\beta^*(z))'$. Therefore $H_1^*(z)$ and $H_2^*(z)$ share $(0,\infty)$. Now in view of Lemma 3.13 we get $H_1^*(z) = c_1^* e^{az}$ and $H_2^*(z) = c_2^* e^{-az}$, where $a,\ c_1^*,\ c_2^* \in \mathbb{C}\setminus\{0\}$ are such that $c_1^*c_2^* = 1$. Therefore we have

$$e_{m_1}\varepsilon_2^{m_1}(z)\left(e^{m_1\gamma(z)} - c_1^*e^{az}\right) + \cdots + e_1\varepsilon_2(z)\left(e^{\gamma(z)} - c_1^*e^{az}\right) \equiv e_0(c_1^*e^{az} - 1) \tag{3.54}$$

Note that $c_1^*e^{az} - 1$ has only simple zeros. Also from (3.54), we see that zeros of $\varepsilon_2(z)$ are also the zero of $c_1^*e^{az} - 1$. Since all the zeros of $\varepsilon_2(z)$ are of multiplicities at least $k+1$, from (3.54), we arrive at contradiction. This completes the Lemma. □

4 Proof of Main Theorems

Proof .[Proof of Theorem 2.3]

Assume that $F = \mathcal{P}(\xi(z))\mathcal{L}^m(\xi(z))$. Now in view of Lemma 3.7

$$\begin{aligned}(n-m)T(r,\xi) &\leq T(r,F) - mN(r,\infty;\xi) - N(r,0;\mathcal{F}) + S(r,\xi),\\ &\leq \overline{N}(r,0;\mathcal{P}_1(\xi)) + \overline{N}(r,0;\xi - c) + \overline{N}(r,a;\xi) + (\epsilon + o(1))\,T(r,\xi)\\ &\leq \left(\Gamma_1 + \frac{1}{\mathcal{K}^*}\right)T(r,\xi) + \overline{N}(r,a;\xi) + (\epsilon + o(1))\,T(r,\xi),\end{aligned}$$

For all $\epsilon > 0$. Take $\epsilon < n - m - \Gamma_1 - \frac{1}{\mathcal{K}^*}$. Since $n > m + \Gamma_1 + \frac{1}{\mathcal{K}^*}$, one can easily say that $F - a$ has infinitely many zeros. □

Proof .[Proof of Theorem 2.4] Let $F(z) = \frac{\mathcal{P}(\xi_1(z))\mathcal{L}^m(\xi_1(z))}{p(z)}$ and $G(z) = \frac{\mathcal{P}(\xi_2(z))\mathcal{L}^m(\xi_2(z))}{p(z)}$. Then $F(z)$ and $G(z)$ share $(1,k_1)$ except for the zeros of $p(z)$ and $\xi_1(z)$ and $\xi_2(z)$ share $(\infty,0)$.

Case 1:- Let $\mathcal{H} \not\equiv 0$. Now from (3.1), we observe that

$$\begin{aligned}N(r,\infty;\mathcal{H}) \leq &\overline{N}_*(r,\infty;\xi_1,\xi_2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2)\\ &+ \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,\xi_1) + S(r,\xi_2).\end{aligned} \tag{4.1}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(z)(F(z)-1)$ and $\overline{N}_0(r,0;G')$ is defined similarly. Let z_0 be a simple zero of $F-1$ but $p(z_0) \neq 0$. Then z_0 is a simple zero of

$G(z)-1$ and a zero of $\mathcal{H}(z)$. Therefore $N(r,1;F|=1)\leq N(r,0;\mathcal{H})\leq N(r,\infty;\mathcal{H})+S(r,\xi_1)+S(r,\xi_2)$. and so from (4.1) we get

$$\begin{aligned}\overline{N}(r,1;F)&\leq N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2)\leq N(r,\infty;\xi_1)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,1;G|\geq 2)+\overline{N}_*(r,1;F,G)\\&\quad+\overline{N}(r,1;F|\geq 2)++\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,\xi_1)+S(r,\xi_2).\end{aligned}\tag{4.2}$$

Now in view of Lemma 3.16, (3.1) we get

$$\begin{aligned}\overline{N}_0(r,0;G')+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F|\geq 2)&\leq\overline{N}(r,1;F|=2)+\overline{N}(r,1;F|=2)+\cdots+\overline{N}(r,1;F|=k_1)\\&\quad+\overline{N}_E^{(K_1+1}(r,1;F)+\overline{N}_0(r,0;G')+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G)\\&\leq\overline{N}(r,0;G)+\overline{N}(r,\infty;\xi_2)-(k_1-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G).\end{aligned}\tag{4.3}$$

Hence, by using (4.3), (4.2) and Lemma 3.4, from Second Fundamental Theorem, we get

$$\begin{aligned}T(r,F)&\leq\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-\overline{N}_0(r,1;F')+S(r,\xi_1),\\&\leq 2\overline{N}(r,\infty;\xi_1(z))+N_2(r,0;F)+\overline{N}(r,0;G\geq 2)+\overline{N}(r,1;F\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,1;F')\\&\quad+S(r,\xi_1)+S(r,\xi_2),\\&\leq 3\overline{N}(r,\infty;\xi_1(z))+N_2(r,0;F)+N_2(r,0;\mathcal{P}_1(\xi_1))+N_2(r,0;\mathcal{P}_1(\xi_2))+N_2(r,0;\mathcal{L}^m(\xi_1))+N_2(r,0;\mathcal{L}^m(\xi_2))\\&\quad+\overline{N}_2(r,0;\xi_2)-(k_1-2)\overline{N}_*(r,1;F,G)+S(r,\xi_1)+S(r,\xi_2)\\&\leq(3+mk)\overline{N}(r,\infty;\xi_1)+mN(r,0;\xi_2)+N(r,0;\mathcal{L}^m(\xi_1))-(k_1-2)\overline{N}_*(r,1;F,G)+S(r,\xi_1)+S(r,\xi_2).\end{aligned}\tag{4.4}$$

Further, in view of Lemma 3.15, 3.7, we get from (4.4)

$$\begin{aligned}(n-m)T(r,\xi_1)&\leq T(r,F)-mN(r,\infty;\xi_1)-N(r,0;\mathcal{F}(\xi_1))+S(r,\xi_1)\\&\leq(3+(k-1)m)\,\overline{N}(r,\infty;\xi_1)+(\Gamma_2+2/\mathcal{K}^*)\,(T(r,\xi_1)+T(r,\xi_2))+mN(r,0;\xi_2)\\&\quad-(k_1-2)\overline{N}_*(r,1;F,G)+S(r,\xi_1)+S(r,\xi_2)\\&\leq\left\{\frac{((\Gamma_1+1)\mathcal{K}^*+1)\,((k-1)m+3)}{\mathcal{K}^*(n+m(k+1)-2k-1)}+(\Gamma_1+2/\mathcal{K}^*+m/2)\right\}T(r)+S(r)\end{aligned}\tag{4.5}$$

we obtain a similar inequality for $\xi_2(z)$. Combining this inequality, we obtain,

$$(n-m)T(r)\leq\left\{\frac{2\,((\Gamma_1+1)\mathcal{K}^*+1)\,((k-1)m+3)}{\mathcal{K}^*(n+m(k+1)-2k-1)}+(2\Gamma_1+4/\mathcal{K}^*+m)\right\}T(r)+S(r)\tag{4.6}$$

i.e.,

$$\left[n^2\mathcal{K}^*-((2\Gamma_2+m(1-k)+1+2k)\mathcal{K}^*+4)\,n+\mathcal{B}\right]T(r)\leq S(r),$$

where

$$\begin{aligned}\mathcal{B}=&2mk\mathcal{K}^*+4m\mathcal{K}^*+8k+2m\Gamma_1\mathcal{K}^*-2m+4\Gamma_2k\mathcal{K}^*+4\Gamma_2\mathcal{K}^*-2m^2\mathcal{K}^*-2m^2k\mathcal{K}^*\\&-2-6\mathcal{K}^*(\Gamma_1+1)-2mk\mathcal{K}^*(\Gamma_1+\Gamma_2)-6mk-2m\Gamma_2\mathcal{K}^*\end{aligned}$$

Therefore

$$(n-K_1)(n-K_2)T(r)\leq S(r),\tag{4.7}$$

where $K_1=\frac{2\Gamma_2+m(1-k)+1+2k)\mathcal{K}^*+4+\sqrt{\mathcal{D}}}{2\mathcal{K}^*}$ and $K_2=\frac{2\Gamma_2+m(1-k)+1+2k)\mathcal{K}^*+4-\sqrt{\mathcal{D}}}{2\mathcal{K}^*}$. So that

$$\mathcal{D}=((2\Gamma_2+m+1-mk+2k)\mathcal{K}^*+4)^2-4\mathcal{K}^*\mathcal{B},$$

which implies that

$$\begin{aligned}\mathcal{D}&=((2\Gamma_2+m+1-mk+2k)\mathcal{K}^*+4)^2-4\mathcal{K}^*\{2mk\mathcal{K}^*+4m\mathcal{K}^*+8k+2m\Gamma_1\mathcal{K}^*-2m\\&\quad+4\Gamma_2k\mathcal{K}^*+4\Gamma_2\mathcal{K}^*-2m^2\mathcal{K}^*-2m^2k\mathcal{K}^*-2-6\mathcal{K}^*(\Gamma_1+1)-2mk\mathcal{K}^*(\Gamma_1+\Gamma_2)-6mk-2m\Gamma_2\mathcal{K}^*\}\\&\leq 4(\mathcal{K}^*\Gamma_2)^2+(9m^2+1)(\mathcal{K}^*)^2+16+(mk\mathcal{K}^*)^2+6k(m\mathcal{K}^*)^2-14(\mathcal{K}^*)^2mk+16(\mathcal{K}^*)^2\Gamma_2\\&\quad+16\mathcal{K}^*(mk+1)+12(\mathcal{K}^*)^2m\Gamma_2+8(\mathcal{K}^*)^2mk\Gamma_1-2(\mathcal{K}^*)^2m-48k\mathcal{K}^*-24(\mathcal{K}^*)^2k\Gamma_2\\&\quad-12(\mathcal{K}^*)^2\Gamma_2+4(k\mathcal{K}^*)^2(1+m)-4(\mathcal{K}^*)^2k+24(\mathcal{K}^*)^2-12(\mathcal{K}^*)^2m+24(\mathcal{K}^*)^2\Gamma_1-8(\mathcal{K}^*)^2m\Gamma_1\\&\leq\left[\mathcal{K}^*(mk+3m+6\Gamma_2+2)+3\right]^2.\end{aligned}$$

Therefore $K_1 < k + 2(m + 2\Gamma_2) + \frac{3\mathcal{K}^*+7}{2\mathcal{K}^*}$. Since $n \geq 4\Gamma_2 + 2m + k + \frac{7}{2\mathcal{K}^*} + \frac{3}{2}$, (4.7) leads to a contradiction.

Case 2:- Let $\mathcal{H}(z) \equiv 0$. Now the theorem follows from Lemma 3.8, 3.17 and 3.12. This completes the proof. □

Proof .[**Proof of Theorem 2.5**] Using Lemma 3.8 and 3.12, the theorem can be proved in the line of the proof of Theorem 2.4. So, we omit the details. □

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