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FURTHER GROWTH OF ITERATED ENTIRE FUNCTIONS IN TERMS OF ITS MAXIMUM TERM

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ABSTRACT. In this article we consider relative iteration of entire functions and study comparative growth of the maximum term of iterated entire functions with that of the maximum term of the related functions.

1. INTRODUCTION

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function defined in the open complex plane C. Then $M(r, f) = \max_{|z|=r} |f(z)|$ and $\mu(r, f) = \max_n |a_n| r^n$ are respectively called the maximum modulus and maximum term of $f(z)$ on $|z| = r$. The following definition are well known.

Definition 1.1. The order ρ_f and lower order λ_f of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

Notation 1.2. [4] *$log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer m, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.*

A simple but useful relation between $M(r, f)$ and $\mu(r, f)$ is the following theorem.

Theorem 1.3. [5] *For $0 \le r < R$,*

$$\mu(r, f) \le M(r, f) \le \frac{R}{R-r}\mu(R, f).$$

Taking $R = 2r$, for all sufficiently large values of r,

$$\mu(r, f) \le M(r, f) \le 2\mu(2r, f). \tag{1.1}$$

Taking two times logarithms in (1.1) it is easy to verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f)}{\log r}$$

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and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log r}.$$

In 1997 Lahiri and Banerjee [3] form the iterations of $f(z)$ with respect to $g(z)$ as follows:

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z)) = f(g_1(z))\\
f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))\\
&\quad .... \qquad\qquad .... \qquad\qquad ....\\
f_n(z) &= f(g(f........(f(z) \text{ or } g(z))........)),\\
&\quad \text{according as } n \text{ is odd or even,}
\end{aligned}$$

and so

$$\begin{aligned}
g_1(z) &= g(z)\\
g_2(z) &= g(f(z)) = g(f_1(z))\\
g_3(z) &= g(f_2(z)) = g(f(g(z)))\\
&\quad .... \qquad\qquad ....\\
g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))).
\end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

In this paper we study growth properties of the maximum term of iterated entire functions as compared to the growth of the maximum term of the related function to generalize some earlier results. Throughout the paper we denote by $f(z)$, $g(z)$ etc. non-constant entire functions of order (lower order) $\rho_f(\lambda_f)$, $\rho_g(\lambda_g)$ etc. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [2], [6] and [7].

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1. [1] *If f and g are any two entire functions, for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|,f\right) \le M(r,f_o g) \le M(M(r,g),f)$$

Lemma 2.2. *If ρ_f and ρ_g are finite, then for any $\varepsilon > 0$,*

$$\log^{[n]} \mu(r,f_n) \le \begin{cases} (\rho_f+\varepsilon)\log M(r,g) + O(1) & \text{when } n \text{ is even}\\ (\rho_g+\varepsilon)\log M(r,f) + O(1) & \text{when } n \text{ is odd}\end{cases}$$

for all sufficiently large values of r.

Proof. First suppose that n is even. Then in view of (1.1) and by Lemma 2.1 it follows that for all sufficiently large values of r,

$$\begin{aligned}
\mu(r, f_n) &\leq M(r, f_n) \\
&\leq M(M(r, g_{n-1}), f) \\
\text{i.e., } \log \mu(r, f_n) &\leq \log M(M(r, g_{n-1}), f) \\
&\leq [M(r, g_{n-1})]^{\rho_f+\varepsilon}. \\
\text{So, } \log^{[2]} \mu(r, f_n) &\leq (\rho_f + \varepsilon) \log M(r, g(f_{n-2})) \\
&\leq (\rho_f + \varepsilon)[M(r, f_{n-2})]^{\rho_g+\varepsilon}. \\
\text{i.e., } \log^{[3]} \mu(r, f_n) &\leq (\rho_g + \varepsilon) \log M(r, f_{n-2}) + O(1). \\
\dots\dots \quad \dots\dots &\quad \dots\dots \quad \dots\dots \\
\dots\dots \quad \dots\dots &\quad \dots\dots \quad \dots\dots \\
\text{Therefore } \log^{[n]} \mu(r, f_n) &\leq (\rho_f + \varepsilon) \log M(r, g) + O(1).
\end{aligned}$$

Similarly if n is odd then for all sufficiently large values of r

$$\log^{[n]} \mu(r, f_n) \leq (\rho_g + \varepsilon) \log M(r, f) + O(1).$$

This proves the lemma. □

Lemma 2.3. *If λ_f, λ_g are non-zero finite, then*

$$\log^{[n]} \mu(r, f_n) > \begin{cases} (\lambda_f - \varepsilon) \log M(r, g) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g - \varepsilon) \log M(r, f) + O(1) & \text{when } n \text{ is odd.} \end{cases}$$

Proof. First suppose that n is even. Let $\epsilon (> 0)$ be such that $\epsilon < min\{\lambda_f, \lambda_g\}$. Now we have from {[5], p-113} for all sufficiently large values of r,

$$\begin{aligned}
\mu(r, f_o g) &> e^{[M(r,g)]^{\lambda_f-\varepsilon}}. \\
\text{So, } \quad \log \mu(r, f_o g) &> [M(r, g)]^{\lambda_f-\varepsilon}. \qquad (2.1)
\end{aligned}$$

Now

$$\begin{aligned}
\log \mu(r, f_n) &= \log \mu(r, f(g_{n-1})) \\
&> [M(r, g_{n-1})]^{\lambda_f-\varepsilon} \quad \text{using (2.1)} \\
&\geq [\mu(r, g_{n-1})]^{\lambda_f-\varepsilon} \quad \text{from (1.1).} \\
\therefore \quad \log^{[2]} \mu(r, f_n) &> (\lambda_f - \varepsilon) \log \mu(r, g(f_{n-2})) \\
&> (\lambda_f - \varepsilon)[M(r, f_{n-2})]^{\lambda_g-\varepsilon} \quad \text{using (2.1).}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \log^{[3]} \mu(r, f_n) &> (\lambda_g - \varepsilon) \log[\mu(r, f_{n-2})] + O(1) \\
&> (\lambda_g - \varepsilon)[M(r, g_{n-3})]^{\lambda_f-\varepsilon} + O(1).
\end{aligned}$$

Taking repeated logarithms

$$\begin{aligned}
log^{[n-1]} \mu(r, f_n) &\geq (\lambda_g - \varepsilon)[M(r, g)]^{\lambda_f-\varepsilon} + O(1). \\
\therefore \quad \log^{[n]} \mu(r, f_n) &\geq (\lambda_f - \varepsilon) \log M(r, g) + O(1).
\end{aligned}$$

Similarly,

$$\log^{[n]} \mu(r, f_n) \geq (\lambda_g - \varepsilon) \log M(r, f) + O(1) \quad \text{when n is odd.}$$

This proves the lemma. □

3. Theorems

Theorem 3.1. *Let f and g be two non constant entire functions such that $0 < \lambda_f \le \rho_f < \infty$ and $0 < \lambda_g \le \rho_g < \infty$. Then for any positive number A and every real number α*

$$(i) \quad \lim_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\{\log\log \mu(r^A, f)\}^{1+\alpha}} = \infty,$$

and

$$(ii) \quad \lim_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\{\log\log \mu(r^A, g)\}^{1+\alpha}} = \infty.$$

Proof. If $\alpha \le -1$ then the theorem is trivial. So we suppose that $\alpha > -1$ and n is even. Then from Lemma 2.3 we get for all sufficiently large values of r and any ε $(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$

$$\begin{aligned} \log^{[n]} \mu(r, f_n) &\ge (\lambda_f - \varepsilon) \log M(r, g) + O(1) \\ &\ge (\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1). \end{aligned} \tag{3.1}$$

Again from Definition 1.1 it follows that for any $\varepsilon > 0$ and for all large values of r,

$$\{\log\log \mu(r^A, f)\}^{1+\alpha} < (\rho_f + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}. \tag{3.2}$$

From (3.1) and (3.2) we have for all large values of r and any ε $(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$

$$\begin{aligned} \frac{\log^{[n]} \mu(r, f_n)}{\{\log\log \mu(r^A, f)\}^{1+\alpha}} &\ge \frac{(\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1)}{(\rho_f + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}} \\ &\ge \frac{(\lambda_f - \varepsilon)}{(\rho_f + \varepsilon)^{1+\alpha} A^{1+\alpha}} \frac{r^{\lambda_g - \varepsilon}}{(\log r)^{1+\alpha}} + o(1). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \quad \lim_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\{\log\log \mu(r^A, f)\}^{1+\alpha}} = \infty. \tag{3.3}$$

Similarly for odd n we get

$$\log^{[n]} \mu(r, f_n) \ge (\lambda_g - \varepsilon) r^{\lambda_f - \varepsilon} + O(1). \tag{3.4}$$

So from (3.2) and (3.4) we have the equation (3.3) for odd n.
Therefore for all n the statement (i) follows.

Second part of this theorem follows similarly by using the following inequality instead of (3.2)

$$\{\log\log \mu(r^A, g)\}^{1+\alpha} < (\rho_g + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}$$

for all large values of r and arbitrary $\varepsilon > 0$.
This proves the theorem. □

Theorem 3.2. *Let f and g be two entire functions of finite orders and $\lambda_f, \lambda_g > 0$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$*

$$\begin{aligned}(i)\quad \lim_{r\to\infty} \frac{\{\log^{[n]} \mu(r, f_n)\}^{1+\alpha}}{\log\log \mu(\exp(r^p), f)} &= 0 \text{ if } p > (1+\alpha)\rho_g \text{ and } n \text{ is even,}\\ (ii)\quad \lim_{r\to\infty} \frac{\{\log^{[n]} \mu(r, f_n)\}^{1+\alpha}}{\log\log \mu(\exp(r^p), f)} &= 0 \text{ if } p > (1+\alpha)\rho_f \text{ and } n \text{ is odd.}\end{aligned}$$

Proof. If $\alpha \leq -1$ then the theorem is trivial. So we suppose that $\alpha > -1$ and n is even. Then from Lemma 2.2 we get for all sufficiently large values of r and any $\varepsilon > 0$

$$\begin{aligned}\log^{[n]} \mu(r, f_n) &\leq (\rho_f + \varepsilon) \log M(r, g) + O(1)\\ &\leq (\rho_f + \varepsilon) r^{\rho_g + \varepsilon} + O(1). \end{aligned} \tag{3.5}$$

Again from Definition 1.1 it follows that for any $0 < \varepsilon < \lambda_f$ and for all large values of r,

$$\log\log \mu(\exp(r^p), f) > (\lambda_f - \varepsilon) r^p. \tag{3.6}$$

So from (3.5) and (3.6) we have for all large vales of r and any ε $(0 < \varepsilon < \lambda_f)$

$$\frac{\{\log^{[n]} \mu(r, f_n)\}^{1+\alpha}}{\log\log \mu(\exp(r^p), f)} \leq \frac{(\rho_f + \varepsilon)^{1+\alpha} r^{(1+\alpha)(\rho_g+\varepsilon)}}{(\lambda_f - \varepsilon) r^p} + o(1).$$

Since $\varepsilon > 0$ is arbitrary, we can choose ε such that $0 < \varepsilon <\min\{\lambda_f, \frac{p}{1+\alpha} - \rho_g\}$,

$$\therefore \qquad \lim_{r\to\infty} \frac{\{\log^{[n]} \mu(r, f_n)\}^{1+\alpha}}{\log\log \mu(\exp(r^p), f)} = 0.$$

Similarly when n is odd then we get the second part of this theorem.
This proves the theorem. □

Theorem 3.3. *Let f and g be two entire functions of finite orders and $\lambda_f, \lambda_g > 0$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$*

$$\begin{aligned}(i)\quad \lim_{r\to\infty} \frac{\{\log^{[n]} \mu(r, f_n)\}^{1+\alpha}}{\log\log \mu(\exp(r^p), g)} &= 0 \text{ if } p > (1+\alpha)\rho_g \text{ and } n \text{ is even,}\\ (ii)\quad \lim_{r\to\infty} \frac{\{\log^{[n]} \mu(r, f_n)\}^{1+\alpha}}{\log\log \mu(\exp(r^p), g)} &= 0 \text{ if } p > (1+\alpha)\rho_f \text{ and } n \text{ is odd.}\end{aligned}$$

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