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On the algebraic and topological structure of temporal graphs: A unified framework for (f,g)-homomorphisms and Spatio-Temporal flows

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Abstract

Modelling complex systems where connections are transient requires a departure from static graph theory. This paper proposes a comprehensive mathematical framework for Temporal Graphs, extending classical concepts to capture dynamic interactions and causality. We identify fundamental limitations in static representations, specifically the "Path Existence Fallacy" and "Flow Indistinguishability." To resolve these, we introduce the novel concept of (f,g)-homomorphism, a structural mapping f coupled with a temporal mapping g, which allows for the modelling of time dilation, contraction, and reversal within network flows. Furthermore, we define hierarchical temporal graphs (HTGs) for multi-scale analysis. Finally, we present a rigorous list of open problems in pure mathematics—ranging from temporal fixed point theory to persistent homology—and future applications in quantum computing and blockchain dynamics.

Keywords: Temporal Graphs, (f,g)-Homomorphism, Dynamic Networks, Time-Varying Topology, Causal Paths, Fixed Point Theory
2020 MSC: Primary 05C20, 05C85; Secondary 37B99, 47H10, 94C15

1 Introduction

Network science has traditionally relied on static graphs $G = (V, E)$ to model systems ranging from social networks to biological pathways. However, in reality, edges are rarely permanent; they represent events that occur at specific timestamps or intervals. A static aggregation of these events creates a "flattened" view that destroys temporal information, leading to erroneous conclusions about connectivity and reachability.

The primary contribution of this paper is a rigorous algebraic framework that treats time as a first-class citizen of the topology. We propose:

1. A formal definition of Temporal Graphs avoiding static aggregation.
2. The (f,g)-Homomorphism theory to compare networks operating at different timescales.
3. A discussion on the intersection of nonlinear analysis, fixed point theory, and temporal graph dynamics.

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2 Historical Background and Related Work

The transition from static to dynamic graph theory has evolved over several decades.

2.1 Early Foundations

The concept of time-varying graphs was arguably touched upon by Harary and Gupta [1] in the context of dynamic connectivity. However, the computational complexity of these structures was formally established by Kempe, Kleinberg, and Kumar [2], who proved that many trivial problems in static graphs (like finding the connected components) become NP-complete in temporal settings due to the time-ordering constraint.

2.2 Modern Era

Holme and Saramäki [3] provided the definitive survey of temporal networks, categorizing them into contact sequences and interval graphs. Simultaneously, Masuda and Lambiotte [4] explored random walks on temporal networks, highlighting how "burstiness" slows down diffusion—a phenomenon invisible to static analysis.

2.3 Algebraic Gaps

While algorithmic properties have been studied (e.g., by Michail [5]), a unified *algebraic* theory—specifically regarding homomorphisms and category theory of temporal graphs—remains significantly underdeveloped. This paper aims to fill this gap using tools from nonlinear analysis.

3 Problem Formulation: The Failure of Static Graphs

Using a static graph to model dynamic processes leads to two major fallacies.

3.1 Problem 1: The Path Existence Fallacy

In a static graph, transitivity implies if $u \to v$ and $v \to w$, then $u \to w$. In a temporal graph, this holds only if the edge (u, v) occurs *before* (v, w). Static graphs create "Ghost Paths"—paths that exist in the topology but are impossible in time.

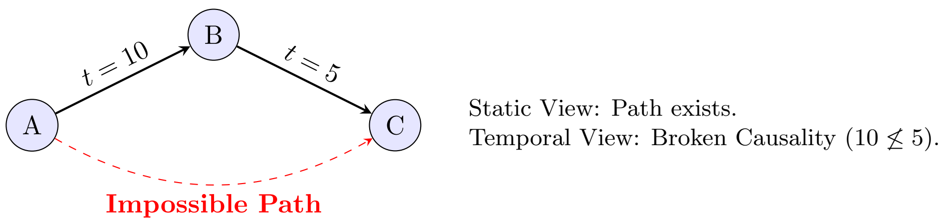


Figure 1: Illustration of the Path Existence Fallacy. Information arriving at B at $t = 10$ cannot travel back in time to $t = 5$ to reach C.

3.2 Problem 2: Flow Indistinguishability

A static weighted graph cannot distinguish between a "Burst" (many events happening at once) and a "Round-Robin" flow (events happening sequentially). This distinction is crucial for analyzing congestion and server loads.

4 Mathematical Preliminaries

Definition 4.1 (Temporal Graph). A temporal graph is a tuple $\mathcal{TG} = (V, E, \mathcal{T}, \rho, w)$, where:

1. V is a set of vertices.
2. $E \subseteq V \times V$ is the underlying set of edges.
3. $\mathcal{T} \subseteq \mathbb{R}^+$ is the time domain.
4. $\rho : E \times \mathcal{T} \to \{0, 1\}$ is the **presence function**. $\rho(e, t) = 1$ iff edge e is active at time t.

5. $w : E \times \mathcal{T} \to \mathbb{R}$ is the **weight function**, defined for all (e, t) where $\rho(e, t) = 1$. For unweighted graphs, we set $w(e, t) = 1$ for all active edges.

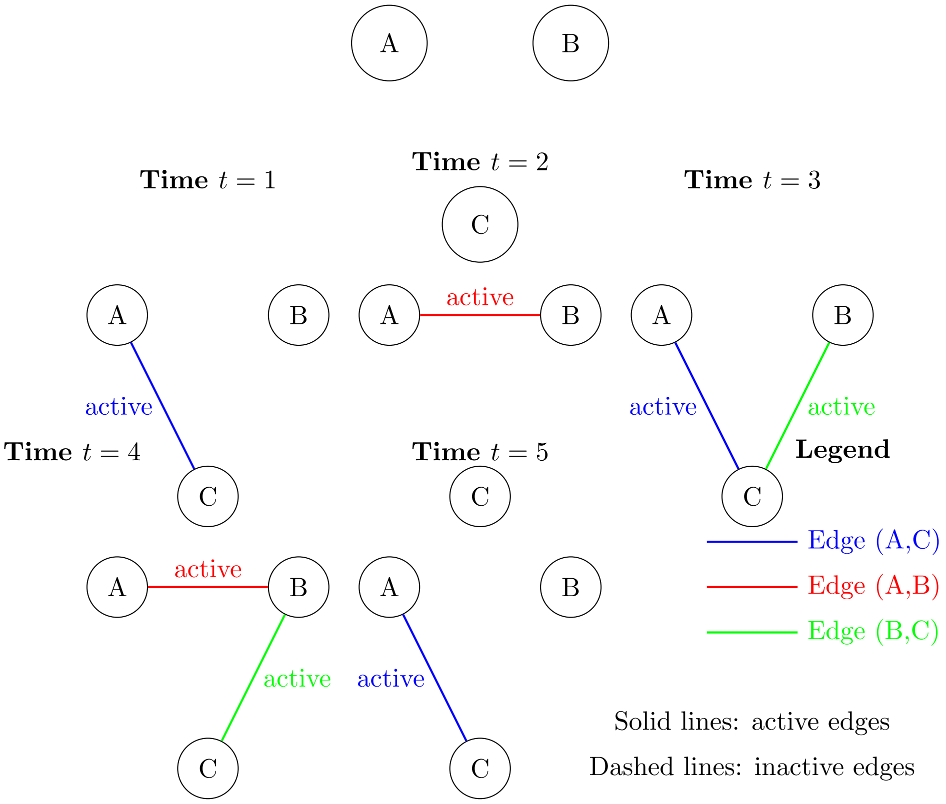
Example 4.2 (Communication Network). Consider $\mathcal{TG} = (V, E, \mathcal{T}, \rho, w)$ where:

1. $V = \{A, B, C\}$
2. $E = \{(A, B), (A, C), (B, C)\}$ (underlying edges)
3. $\mathcal{T} = \{1, 2, 3, 4, 5\}$ (discrete time steps)
4. ρ defined by:

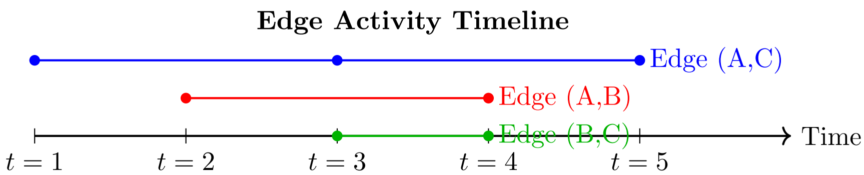
t	$\rho((A, B), t)$	$\rho((A, C), t)$	$\rho((B, C), t)$
1	0	1	0
2	1	0	0
3	0	1	1
4	1	0	1
5	0	1	0

For simplicity, we define the weight function $w(e, t) = 1$ for all $e \in E$ and $t \in \mathcal{T}$ where $\rho(e, t) = 1$.

Visualization of Temporal Evolution



Alternative Visualization: Timeline Representation



The temporal graph shows how the network structure evolves over time:

1. At $t = 1$: Only A and C are connected.
2. At $t = 2$: Only A and B are connected.
3. At $t = 3$: A is connected to C, and B is connected to C (forming a path A-C-B).
4. At $t = 4$: A is connected to B, and B is connected to C (forming a path A-B-C).
5. At $t = 5$: Returns to the configuration at $t = 1$ (only A-C connected).

This example illustrates how temporal graphs capture the *dynamics* of network connections, which is essential for modeling time-dependent phenomena in communication networks, transportation systems, social networks, and biological systems.

Example 4.3. Consider a mobile phone network in a small office building during a typical workday (9 AM to 5 PM). We'll track phone calls between four colleagues: Alex (A), Barbara (B), Chris (C), and Diana (D). The phone call network is represented as $\mathcal{TG}_{\text{phone}} = (V, E, \mathcal{T}, \rho, w)$ where:

1. $V = \{A, B, C, D\}$ (the four colleagues)
2. $E = \{(A, B), (A, C), (A, D), (B, C), (B, D), (C, D)\}$ (all possible call connections)
3. $\mathcal{T} = [9 : 00, 17 : 00]$ (continuous 8-hour workday, in hours)
4. $\rho : E \times \mathcal{T} \to \{0, 1\}$ where $\rho((X, Y), t) = 1$ iff X and Y are engaged in a phone call at time t
5. $w : E \times \mathcal{T} \to \mathbb{R}$ where $w((X, Y), t) = 1$ for all active calls (unweighted model)

The following calls occurred during the workday:

Call #	**Participants**	**Start Time**	**Duration**
1	Alex ↔ Barbara	9:15	10 minutes
2	Barbara ↔ Chris	10:30	15 minutes
3	Alex ↔ Diana	11:00	5 minutes
4	Chris ↔ Diana	11:45	20 minutes
5	Alex ↔ Barbara ↔ Chris (conference call)	13:30	25 minutes
6	Barbara ↔ Diana	14:45	10 minutes
7	Alex ↔ Chris	15:30	15 minutes

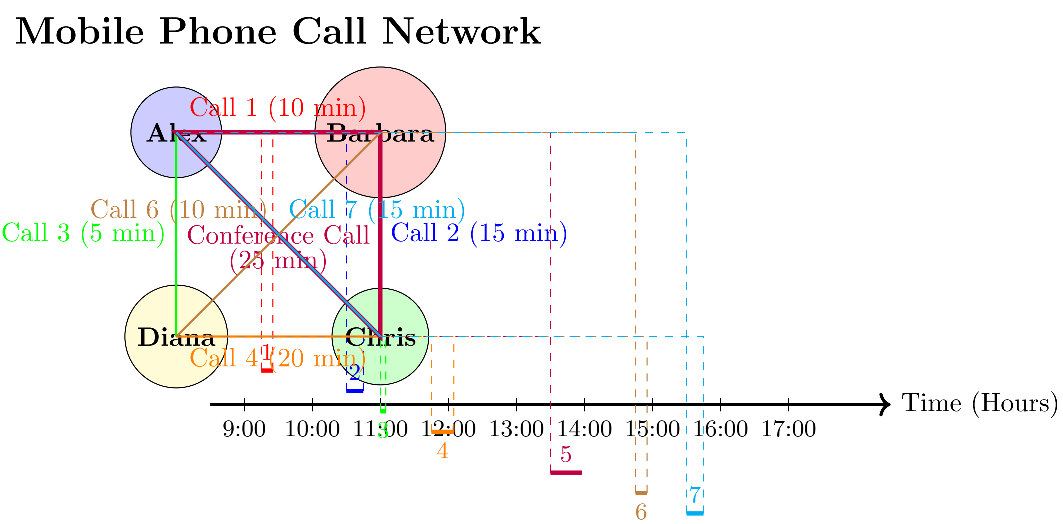
The presence function ρ can be derived from the schedule:

$$\rho((X, Y), t) = \begin{cases} 1 & \text{if } t \in [\text{call start, call end}] \text{ for participants } X \text{ and } Y \\ 0 & \text{otherwise} \end{cases}$$

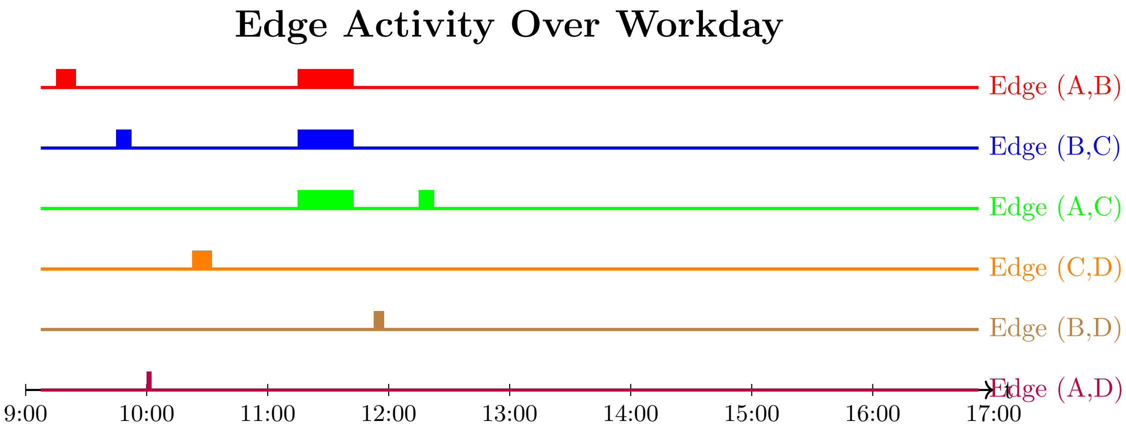
For the conference call at 13:30-13:55, we have:

- $\rho((A, B), t) = 1$ for $t \in [13 : 30, 13 : 55]$
- $\rho((A, C), t) = 1$ for $t \in [13 : 30, 13 : 55]$
- $\rho((B, C), t) = 1$ for $t \in [13 : 30, 13 : 55]$

Visualization: Network Evolution



Alternative View: Edge Activity Timeline



Each bar shows when an edge (phone call) is active

Real-World Analysis and Applications

This temporal graph representation enables several practical analyses:

1. **Communication Patterns**: Identify who communicates with whom and when
2. **Network Connectivity**: Determine if information can flow through the network over time
3. **Busy Periods**: Identify peak communication times (e.g., conference call at 13:30)
4. **Information Flow**: Track how information might propagate (e.g., if Alex hears news at 9:15 from Barbara, who could he tell and when?)

The temporal graph formalism allows us to model not just *who* communicates, but *when* they communicate, which is crucial for understanding dynamic networks in telecommunications, epidemiology (contact tracing), social networks, and transportation systems.

Definition 4.4 (Infinite Temporal Graph). A temporal graph is a tuple $\mathcal{TG} = (V, E, \mathcal{T}, \rho, w)$, where:

1. V is a (potentially infinite) set of vertices.
2. $E \subseteq V \times V$ is the underlying set of edges.
3. $\mathcal{T} \subseteq \mathbb{R}^+$ is the time domain.
4. $\rho : E \times \mathcal{T} \to \{0, 1\}$ is the **presence function**. $\rho(e, t) = 1$ iff edge e is active at time t.
5. $w : E \times \mathcal{T} \to \mathbb{R}$ is the **weight function**, defined for all (e, t) where $\rho(e, t) = 1$.

When V is infinite, we typically require some additional structure (like metric space structure) for meaningful analysis.

Example 4.5. Consider an idealized global network of ocean buoys measuring temperature, salinity, and pollution levels. The buoys are distributed across the Earth's oceans.

1. **Vertices:** $V = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$ where R is Earth's radius.
   - This represents all possible positions on Earth's surface (using a 2D projection).
   - In practice, we'd consider $\mathbb{S}^2$ (the sphere), but for visualization we use a disk.
   - The vertex set is uncountably infinite (continuum).
2. **Underlying Edges:** $E = \{(v_1, v_2) \in V \times V : d(v_1, v_2) \leq D\}$
   - Two buoys can potentially communicate if they're within distance D (say 100km).
   - $d(v_1, v_2)$ is the great-circle distance on Earth's surface.
3. **Time Domain:** $\mathcal{T} = [0, \infty)$ (continuous time over a long period)
4. **Presence Function:** $\rho((v_1, v_2), t)$ depends on several factors:
   - Weather conditions (storms disrupt communication)

- Buoy power cycles (solar-powered buoys sleep at night)
- Ocean currents (buoys drift, changing distances)
- Satellite visibility (for relay through satellites)

5. **Weight Function:** For simplicity, we define $w((v_1, v_2), t) = 1$ for all active edges. In practice, weights could represent signal strength, data rate, or reliability.

Let's define ρ more formally. For vertices at positions $\mathbf{x}, \mathbf{y} \in V$:

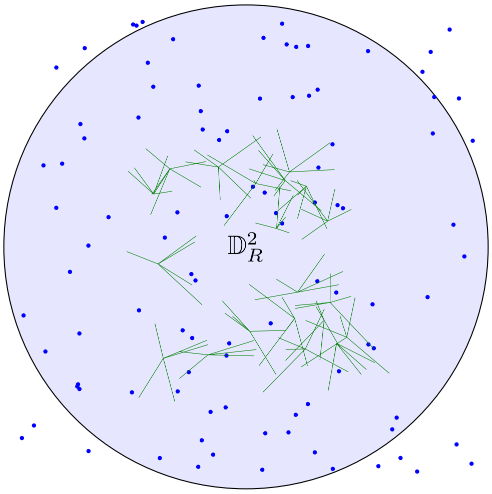
$$\rho((\mathbf{x}, \mathbf{y}), t) = \begin{cases} 1 & \text{if all conditions hold:} \\ & 1.\, d(\mathbf{x}(t), \mathbf{y}(t)) \leq D \\ & 2.\, W(\mathbf{x}, \mathbf{y}, t) = 0 \text{ (no storm between them)} \\ & 3.\, P(\mathbf{x}, t) = 1 \text{ and } P(\mathbf{y}, t) = 1 \text{ (both powered)} \\ & 4.\, \text{Either line-of-sight or satellite relay available} \\ 0 & \text{otherwise} \end{cases}$$

where:

1. $\mathbf{x}(t), \mathbf{y}(t)$: Positions at time t (buoys drift with currents)
2. $d(\cdot, \cdot)$: Distance function
3. $W(\mathbf{x}, \mathbf{y}, t)$: Weather disruption function
4. $P(\mathbf{x}, t)$: Power availability function (depends on time of day and solar panel efficiency)

Visualization of Infinite Network

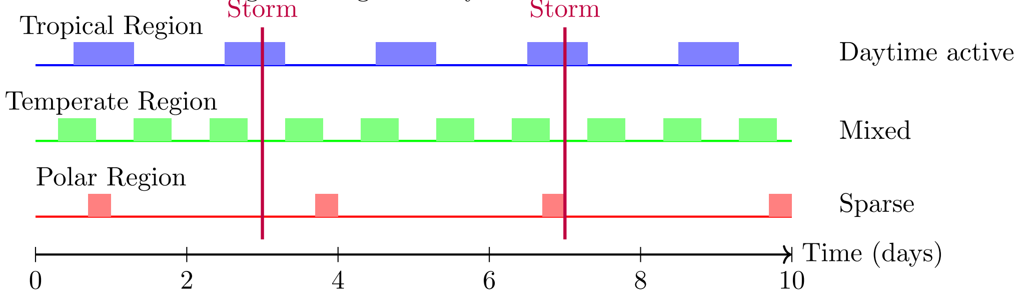
Infinite Vertex Set $V \subset \mathbb{R}^2$ (Ocean Buoy Positions)



Edges: communication within range D

Temporal Dynamics Visualization

Regional Edge Density Over Time



4.1 Mathematical Properties of the Infinite Temporal Graph

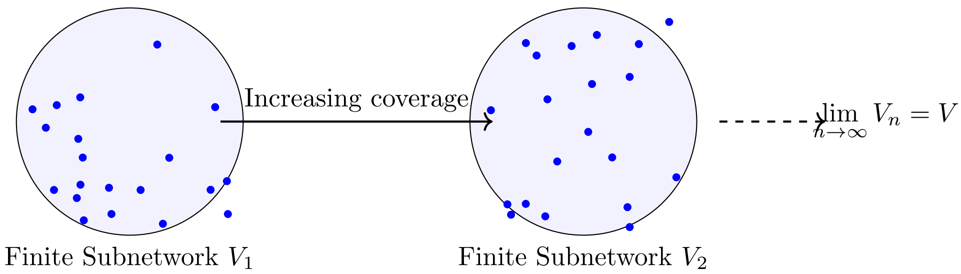
1. **Uncountable Vertex Set:** $|V| = \mathfrak{c}$ (continuum).
2. **Edge Density:** Locally finite-each vertex has finite neighborhood.

$$\deg(v) = |\{u \in V : d(v, u) \leq D\}| < \infty \quad \text{for all } v \in V.$$

3. **Temporal Patterns:** The graph exhibits:
   (a) **Diurnal cycles:** $\rho(e, t)$ depends on $t \mod 24\text{hr}$.
   (b) **Seasonal cycles:** Polar regions more connected in summer.
   (c) **Stochastic events:** Storms randomly disrupt connections.
4. **Spatial Homogeneity:** For regions far from land, the graph is approximately homogeneous and isotropic.
5. **Scale Invariance:** At different spatial scales, similar connectivity patterns emerge.

4.2 Analysis Through Finite Approximations

While the full graph has infinite vertices, we can study it through finite subnetworks:



For analysis, we consider finite subsets $V_n \subset V$ that increase to cover V:

$$V_1 \subset V_2 \subset \cdots \subset V, \quad \bigcup_{n=1}^{\infty} V_n = V$$

The corresponding temporal graphs $\mathcal{TG}_n = (V_n, E_n, \mathcal{T}, \rho_n, w_n)$ approximate the infinite graph.

4.3 Applications of Infinite Temporal Graphs

1. **Continuum Limits:** Studying infinite networks helps understand asymptotic behavior of large finite networks
2. **Percolation Theory:** When do infinite connected components emerge in temporal graphs?

$$\mathbb{P}[\text{infinite temporal path exists}] = ?$$

3. **Information Diffusion:** How quickly can information spread in an infinite temporal network?
4. **Sensor Network Design:** What density of sensors is needed for global connectivity?
5. **Climate Modeling:** Understanding how ocean temperature anomalies propagate

4.4 Theoretical Challenges

Working with infinite temporal graphs presents unique challenges:

1. **Measure Theory Needed:** Instead of counting vertices, we measure regions

$$\mu(A) = \text{“number” of vertices in region } A \subset V.$$

2. **Continuum Graph Theory:** Extending discrete concepts to continuous spaces.
3. **Temporal Continuity:** Studying properties like $\lim_{\Delta t \to 0} \rho(e, t + \Delta t)$.
4. **Stochastic Geometry:** Modeling random vertex distributions (Poisson point processes).

This example demonstrates that temporal graph theory naturally extends to infinite vertex sets, with important applications in:

1. Large-scale sensor networks
2. Wireless ad-hoc networks
3. Epidemic modeling on continuous populations
4. Cosmic ray detection networks
5. Satellite constellations

The infinite case forces us to use analytical methods rather than combinatorial ones, bridging graph theory with analysis and probability theory.

4.5 Birth, Death, and Lifespan in Temporal Graphs

In temporal graphs, both vertices and edges have explicit temporal existence. This extends the basic temporal graph definition by tracking the precise moments when vertices and edges come into and go out of existence.

Definition 4.6 (Vertex Lifespan). For a vertex $v \in V$ in a temporal graph $\mathcal{TG} = (V, E, \mathcal{T}, \rho, w)$, we define:

1. A finite collection of **birth moments** $\{t_b^{(v,1)}, t_b^{(v,2)}, \dots\}$ and corresponding **death moments** $\{t_d^{(v,1)}, t_d^{(v,2)}, \dots\}$.
2. The **lifespan** of v as the union of disjoint intervals:

$$\mathcal{L}(v) = \bigcup_{i=1}^{k_v} [t_b^{(v,i)}, t_d^{(v,i)}] \subseteq \mathcal{T}$$

   where $k_v \in \mathbb{N}$ is the number of distinct life intervals for v.
3. The **duration** of the i-th life interval as $\Delta^{(v,i)} = t_d^{(v,i)} - t_b^{(v,i)}$.

We say v is **alive** at time t if $t \in \mathcal{L}(v)$, and **dead** otherwise.

Definition 4.7 (Edge Lifespan). For an edge $e = (u, v) \in E$ in a temporal graph $\mathcal{TG} = (V, E, \mathcal{T}, \rho, w)$, we define:

1. A finite collection of **birth moments** $\{t_b^{(e,1)}, t_b^{(e,2)}, \dots\}$ and corresponding **death moments** $\{t_d^{(e,1)}, t_d^{(e,2)}, \dots\}$.
2. The **lifespan** of e as the union of disjoint intervals:

$$\mathcal{L}(e) = \bigcup_{j=1}^{m_e} [t_b^{(e,j)}, t_d^{(e,j)}] \subseteq \mathcal{T}$$

   where $m_e \in \mathbb{N}$ is the number of distinct active intervals for e.
3. The **duration** of the j-th active interval as $\delta^{(e,j)} = t_d^{(e,j)} - t_b^{(e,j)}$.

The presence function satisfies $\rho(e, t) = 1$ if and only if $t \in \mathcal{L}(e)$.

Definition 4.8 (Sleeping Vertex). A vertex v is said to be **sleeping** on an interval $I \subseteq \mathcal{T}$ if:

1. $I \subseteq \mathcal{L}(v)$ (the vertex is alive throughout I), and
2. For every $t \in I$, for every edge e incident to v, we have $t \notin \mathcal{L}(e)$.

A sleeping vertex exists and has the *potential capability* to interact (receive or send edges) but is currently isolated.

Key Distinction: A sleeping vertex is within its lifespan but has no active edges. A dead vertex is outside its lifespan and cannot participate in any interactions.

Example 4.9 (Classroom Dynamics with Multiple Life Intervals). Consider a classroom scenario from 8:00 to 10:00. Let $V = \{T, S_1, S_2, \dots, S_n\}$ represent a teacher and students.

1. **Vertex Lifespans:** Each person enters and leaves the classroom multiple times:
   (a) Teacher T: $\mathcal{L}(T) = [8:00, 8:45] \cup [9:15, 10:00]$ (leaves for a break).

   (b) Student S_1: $\mathcal{L}(S_1) = [8:10, 8:50] \cup [9:05, 9:55]$ (goes to bathroom).
   (c) Total duration for T: $\Delta^{(T,1)} + \Delta^{(T,2)} = 45 + 45 = 90$ minutes.
2. **Edge Lifespans:** Each spoken sentence creates an edge:
   (a) T asks question to S_1: $\mathcal{L}((T, S_1)) = [8:15:00, 8:15:10]$ (10-second question).
   (b) S_1 responds: $\mathcal{L}((S_1, T)) = [8:15:20, 8:15:35]$ (15-second answer).
   (c) Duration of first interaction: $\delta^{((T,S_1),1)} = 10$ seconds.
3. **Sleeping States:**
   (a) During [8:20, 8:30], all students work silently: all vertices are sleeping.
   (b) During [8:45, 9:15], teacher T is dead (outside classroom), cannot receive questions.

This models the dynamic nature of classroom interactions where vertices have multiple life intervals.

Example 4.10 (US Airports and Flights Network). Consider US airports during the first week of 2025 ($\mathcal{T} = [\text{Jan }1, \text{Jan }7]$).

- **Vertices (Airports):** $V = \{\text{JFK}, \text{LAX}, \text{ORD}, \dots\}$
  - $\mathcal{L}(\text{JFK}) = [\text{continuously open}]$ (never closes)
  - $\mathcal{L}(\text{small regional}) = \bigcup_{\text{days}}[6:00, 22:00]$ (closes nightly)
  - During closure: airport is dead, no flights possible
- **Edges (Flights):** Directed edges represent flights
  - Flight AA100: JFK $\to$ LAX
  - $\mathcal{L}(\text{AA100}) = [\text{Jan 2, 10:00}, \text{Jan 2, 13:00}]$
  - Birth: takeoff at JFK (t_b), Death: landing at LAX (t_d)
  - Duration: $\delta = 3$ hours
- **Sleeping Airports:**
  - JFK at 3:00 AM: alive but no active flights $\Rightarrow$ sleeping
  - Small airport at 2:00 AM: dead (outside lifespan), different from sleeping
- **Multiple Flights:** Same route can have multiple edges:
  - AA100: $\mathcal{L} = [\text{Jan 2, 10:00}, \text{Jan 2, 13:00}]$
  - AA102: $\mathcal{L} = [\text{Jan 2, 14:00}, \text{Jan 2, 17:00}]$
  - Two distinct edges with separate lifespans

This example shows how flight networks naturally model as temporal graphs with explicit birth/death moments for edges (flights) and vertices (airports).

Example 4.11 (Mobile Phone Network with Device States). Extending the earlier phone call example, consider employees in a company with mobile phones:

- **Vertices (Phones):** $V = \{\text{Phone}_A, \text{Phone}_B, \dots\}$
  - $\mathcal{L}(\text{Phone}_A) = [9:00, 17:00]$ (work hours)
  - Phone turned off overnight: dead state
  - Phone on but no calls: sleeping state
- **Edges (Calls):** Directed edges represent calls
  - Call from A to B: $\mathcal{L} = [10:15, 10:25]$

  - Birth: call answered (t_b), Death: call ended (t_d)
  - Duration: $\delta = 10$ minutes

- **Device Failure:** If Phone$_C$ breaks at 14:30:
  - Modified lifespan: $\mathcal{L}(\text{Phone}_C) = [9:00, 14:30]$
  - After 14:30: dead, cannot make/receive calls
  - Different from sleeping (which occurs within lifespan)

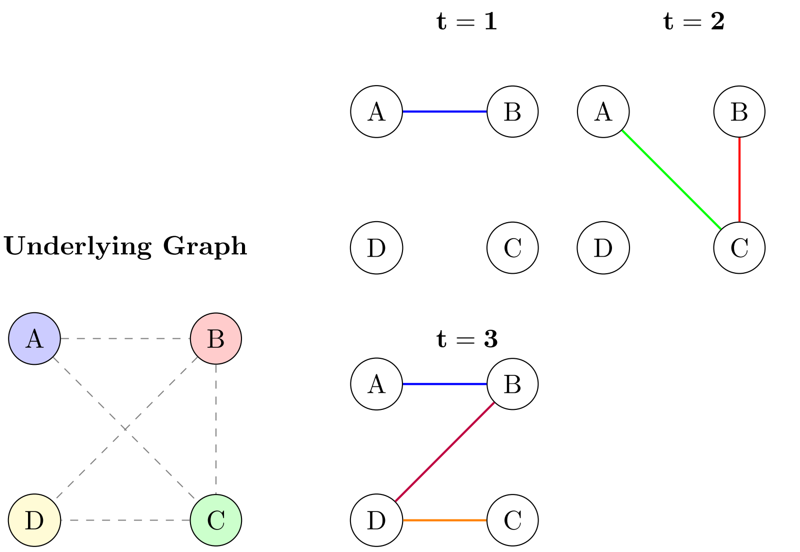
Definition 4.12 (Temporal Walk). A sequence $W = \{(v_0, v_1, t_1), (v_1, v_2, t_2), \ldots, (v_{k-1}, v_k, t_k)\}$ is a temporal walk if:

1. $(v_{i-1}, v_i) \in E$ for all $1 \leq i \leq k$.
2. $\rho((v_{i-1}, v_i), t_i) = 1$.
3. $t_1 \leq t_2 \leq \cdots \leq t_k$ (Time-respecting condition).

Example 4.13. Consider a finite temporal graph $\mathcal{TG}_{\text{fin}} = (V, E, \mathcal{T}, \rho, w)$ with:

1. **Vertices:** $V = \{A, B, C, D\}$
2. **Edges:** $E = \{(A, B), (B, C), (C, D), (A, C), (B, D)\}$
3. **Time Domain:** $\mathcal{T} = \{1, 2, 3, 4, 5\}$ (discrete time steps)
4. **Presence Function:** ρ defined as:
5. **Weight Function:** $w(e, t) = 1$ for all active edges (unweighted)

t	$\rho((A,B),t)$	$\rho((B,C),t)$	$\rho((C,D),t)$	$\rho((A,C),t)$	$\rho((B,D),t)$
1	1	0	0	0	0
2	0	1	0	1	0
3	1	0	1	0	1
4	0	1	0	0	0
5	0	0	1	1	0



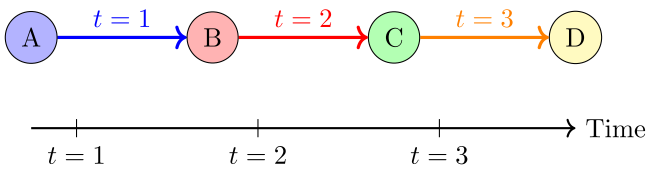
Consider the walk from vertex A to vertex D:

$$W_{\text{fin}} = \{(A, B, 1), (B, C, 2), (C, D, 3)\}.$$

1. **Edge existence:** $(A, B), (B, C), (C, D) \in E$
2. **Activity at specified times:**
   (a) $\rho((A, B), 1) = 1$

   (b) $\rho((B, C), 2) = 1$
   (c) $\rho((C, D), 3) = 1$
3. **Time-respecting:** $1 \leq 2 \leq 3$

This is a valid temporal walk of length 3 from A to D.



Temporal Walk: $A \xrightarrow{t=1} B \xrightarrow{t=2} C \xrightarrow{t=3} D$

Alternative Walks in the Same Graph

1. $\{(A, C, 2), (C, D, 3)\}$ (shorter walk, same destination)
2. $\{(A, B, 1), (B, D, 3)\}$ (alternative route)
3. $\{(A, B, 1), (B, C, 4), (C, D, 5)\}$ (longer waiting times)

Example 4.14. Consider an infinite temporal graph $\mathcal{TG}_\infty = (V, E, \mathcal{T}, \rho, w)$ with:

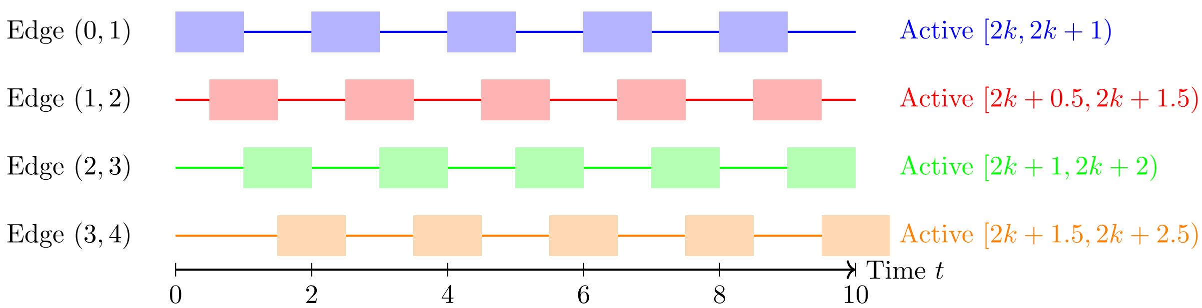
- **Vertices:** $V = \mathbb{Z}$ (all integers)
- **Edges:** $E = \{(n, n+1) : n \in \mathbb{Z}\}$ (infinite line graph)
- **Time Domain:** $\mathcal{T} = [0, \infty)$ (continuous time)
- **Presence Function:** For edge $e = (n, n+1)$,

$$\rho(e, t) = \begin{cases} 1 & \text{if } t \in \bigcup_{k=0}^{\infty} \left[2k + \frac{|n|}{2},\ 2k + 1 + \frac{|n|}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

  Each edge is active periodically, but with a phase shift depending on $|n|$.

- **Weight Function:** $w(e, t) = 1$ for all active edges (unweighted)

Activation Intervals for Edges



Consider a walk starting at vertex 0 and moving to vertex 3:

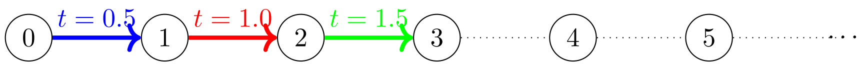
$$W_\infty = \{(0, 1, 0.5), (1, 2, 1.0), (2, 3, 1.5)\}$$

Verification:

1. **Edge existence:** $(0, 1), (1, 2), (2, 3) \in E$

2. **Activity at specified times:**
   - $\rho((0,1),0.5) = 1$ since $0.5 \in [0,1)$
   - $\rho((1,2),1.0) = 1$ since $1.0 \in [0.5,1.5)$
   - $\rho((2,3),1.5) = 1$ since $1.5 \in [1,2)$
3. **Time-respecting:** $0.5 \leq 1.0 \leq 1.5$

This is a valid temporal walk in the infinite graph.



Temporal Walk on Infinite Line: $0 \xrightarrow{t=0.5} 1 \xrightarrow{t=1.0} 2 \xrightarrow{t=1.5} 3$

We can extend this indefinitely. Consider the infinite walk moving rightward forever:

$$W_\infty^{\text{inf}} = \{(n, n+1, t_n) : n = 0, 1, 2, \dots\}$$

where $t_0 = 0.5$ and for $n \geq 1$, we choose t_n such that:

- $t_n \geq t_{n-1}$ (time-respecting)
- t_n falls within an active interval of edge $(n, n+1)$
- The increments are small enough that we don't "miss" the next activation window

For example, we could choose:

$$t_n = 0.5 + \frac{n}{2} \quad \text{for } n \geq 0$$

Then:

- Edge $(0,1)$: $t_0 = 0.5 \in [0,1)$
- Edge $(1,2)$: $t_1 = 1.0 \in [0.5,1.5)$
- Edge $(2,3)$: $t_2 = 1.5 \in [1,2)$
- Edge $(3,4)$: $t_3 = 2.0 \in [1.5,2.5)$ (corrected: $2.0 \in [1.5,2.5)$)
- And so on ...

This gives the infinite temporal walk:

$$W_\infty^{\text{inf}} = \{(0,1,0.5), (1,2,1.0), (2,3,1.5), (3,4,2.0), (4,5,2.5), \dots\}$$

- **Finite vs Infinite:**
  - Finite walks have bounded length and exist in finite temporal graphs.
  - Infinite walks can exist in infinite temporal graphs (either infinite vertices or infinite time domain).
- **Waiting Allowed:** The time-respecting condition $t_i \leq t_{i+1}$ permits waiting at vertices. In the infinite example, we didn't need to wait long because edge activations overlapped suitably.
- **Constructability:** For infinite graphs with periodic or recurrent edge activations, infinite walks often exist if the activation patterns permit progressive movement without unbounded waiting.
- **Mathematical Significance:** The existence of infinite temporal walks relates to percolation and connectivity properties in infinite temporal graphs, which is an active research area in temporal graph theory.

Properties of Temporal Walks

- **Not necessarily simple:** A temporal walk can revisit vertices if the timing allows.
- **Not closed under reversal:** If W is a temporal walk from u to v, the reverse sequence may not be a valid temporal walk (due to time ordering).
- **Composition:** If W_1 is a temporal walk from u to v ending at time t, and W_2 is a temporal walk from v to w starting at time $\geq t$, their concatenation is a temporal walk from u to w.
- **Subwalks:** Any contiguous subsequence of a temporal walk is itself a temporal walk.

5 Fundamental Vertex States

Let $\mathcal{T} \subseteq \mathbb{R}$ be the continuous time domain. Let V be the set of all potential vertices in the universe of the system.

5.1 Lifespan and Capability

For every vertex $v \in V$, we define a **Lifespan** function $\mathcal{L} : V \to \mathcal{P}(\mathcal{T})$, which returns the set of time intervals during which v physically exists and possesses the *intrinsic capability* to interact.

Definition 5.1 (The Tri-State Vertex Model). At any specific time instance $t \in \mathcal{T}$, the state of a vertex v is defined as follows:

1. **Active:**
$$t \in \mathcal{L}(v) \quad \text{AND} \quad \deg(v, t) > 0$$
The vertex exists and is currently participating in at least one interaction (edge).
2. **Sleeping:**
$$t \in \mathcal{L}(v) \quad \text{AND} \quad \deg(v, t) = 0$$
The vertex exists and is capable of interaction (potentiality), but is currently isolated. It contributes to the denominator of density calculations but not the numerator.
3. **Dead:**
$$t \notin \mathcal{L}(v)$$
The vertex does not exist in the system or has permanently lost the capability to interact. It is topologically null and must be excluded from the graph's order ($|V|$).

Example 5.2 (The Classroom Dynamics). Consider a graph where nodes are students in a classroom ($08:00$ to $10:00$).

- **Active:** At 08:30, Student A talks to Student B. Edge (A, B) exists.
- **Sleeping:** At 08:45, silence reigns during an exam. Student A is present (in $\mathcal{L}(v)$) but has degree 0.
- **Dead:** At 09:00, Student C falls ill and leaves the school. For $t > 09:00$, Student C is removed from the system. An edge cannot be formed even if someone tries to call them (topology change).

Example 5.3 (Neuroscience).
- **Active:** Neuron firing (Action Potential).
- **Sleeping:** Neuron at Resting Potential (Polarized, ready to fire).
- **Dead:** Neuron undergoing Apoptosis (Cell death).

6 Edge Dynamics and Flow Continuity

Let $E \subseteq V \times V \times \mathcal{T}$ be the set of temporal edges. The weight function is $w : E \to \mathbb{R}$.

Definition 6.1 (Flow Types).
- **Continuous Flow:** The map $t \mapsto w_{uv}(t)$ is continuous on intervals. (e.g., Fluid in a pipe, Voltage on a wire).
- **Discrete Flow (Impulsive):** The weight exists only as Dirac deltas or discrete packets. (e.g., Email, SMS).

7 The (f, g)-Homomorphism Framework

We introduce a generalized morphism that maps both structure and time, crucial for recognizing patterns that occur at different speeds or scales.

Definition 7.1 ((f,g)-Homomorphism). Let $\mathcal{G} = (V_G, E_G, \mathcal{T}_G, \rho_G, w_G)$ and $\mathcal{H} = (V_H, E_H, \mathcal{T}_H, \rho_H, w_H)$ be temporal graphs. An (f, g)-homomorphism is a pair $\Phi = (f, g)$ where:

1. $f : V_G \to V_H$ is a vertex mapping.
2. $g : \mathcal{T}_G \to \mathcal{T}_H$ is a temporal mapping.

Satisfying the preservation conditions:

1. **Edge Preservation:** $\forall u, v \in V_G, \forall t \in \mathcal{T}_G : \quad \rho_G((u, v), t) = 1 \implies \rho_H((f(u), f(v)), g(t)) = 1$
2. **Weight Preservation:** $\forall u, v \in V_G, \forall t \in \mathcal{T}_G : \quad \rho_G((u, v), t) = 1 \implies w_H((f(u), f(v)), g(t)) = w_G((u, v), t)$

Example 7.2 (Affine Scaling: Time Zone Translation). The social network example demonstrates Class I homomorphism with $g(t) = t - 8$, preserving the temporal structure while translating between time zones.

Example 7.3 (Time Inversion: Reversible Processes). The chemical reaction network illustrates Class II with $g(t) = T/t$, showing how temporal graphs can model processes reversible in time.

Example 7.4 (Nonlinear Warping: Accelerating Dynamics). The epidemic model exemplifies Class III with $g(t) = 10(1 - e^{-t/5})$, capturing systems where time scales nonlinearly between different contexts.

7.1 Classification of Time Mapping $g(t)$

The power of this framework lies in the choice of g:

1. **Class I (Affine Scaling):** $g(t) = \alpha t + \beta$.
2. **Class II (Inversion):** $g(t) = t^{-1}$. Used to study reversible processes.
3. **Class III (Nonlinear Warping):** $g(t)$ is monotonic nonlinear (e.g., $g(t) = e^t$). Used for modeling systems with accelerating dynamics.

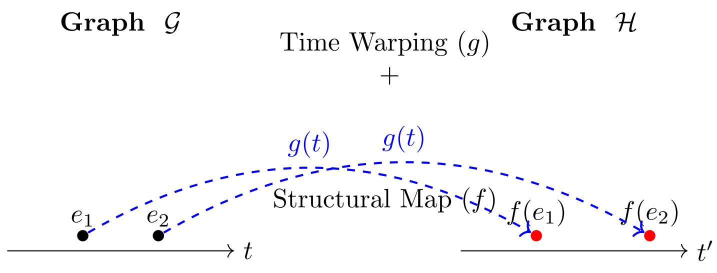


Figure 2: Schematic of an (f, g)-homomorphism where g expands the time interval between events (Time Dilation).

Theorem 7.5 (Composition). The class of temporal graphs with (f, g)-homomorphisms forms a Category.

Proof . Standard categorical proof via associativity of function composition f and g. □

7.2 Case Study: Human Motion and Video Speed

Scenario: We have a set of vertices representing 10 key points on a human body (Head, Neck, Shoulders, Elbows, Hands, Knees, Feet).

- **Vertices (V):** The 10 body joints.
- **Edges (E):** Kinematic links (e.g., Knee to Ankle) or spatial proximity. The weights $w_{uv}(t)$ represent the Euclidean distance or angle between joints at time t.
- **Graph G:** Represents a video of a person walking at **Normal Speed**.
- **Graph H:** Represents a video of the same person walking, but **Fast-Forwarded** ($2\times$).

7.2.1 The Morphism Components

1. The Spatial Map (f): Identity Since it is the same person (or a digital twin), the mapping f sends the "Right Knee" in the normal video to the "Right Knee" in the fast video.

$$f(v^G_{knee}) = v^H_{knee}$$

2. The Temporal Map (γ): Time Warping This is the critical component. Graph H runs twice as fast as G.

- An event happening at $t = 10$ seconds in the normal video (G) occurs at $t' = 5$ seconds in the fast video (H).
- The mapping is defined as:

$$\gamma(t) = \frac{1}{2}t$$

7.2.2 Verification of Structure

Suppose in Graph G (Normal), at $t = 2.0s$, the right foot hits the ground (a specific edge configuration or weight change peaks). Using the morphism:

- Vertex map f: Right Foot $\to$ Right Foot.
- Time map γ: $2.0s \to 1.0s$.

Observation in Graph H (Fast): At $t' = 1.0s$, the right foot hits the ground. Thus, the structure is preserved:

$$w^G_{u,v}(t) = w^H_{f(u),f(v)}(\gamma(t))$$

Remark 7.6 (Slow Motion). If Graph K represents a **Slow Motion** video ($0.5\times$ speed), the temporal map would be an expansion:

$$\gamma(t) = 2t$$

Here, an event at $t = 1$ in real life takes place at $t = 2$ in the video.

8 Advanced Structural Features

8.1 Temporal Bridge

Definition 8.1. An edge $e = (u, v)$ existing in interval I is a Temporal Bridge if its removal permanently disconnects a subset of vertices for all $t > \sup(I)$.

Example 8.2. The last ferry leaving an island. If you miss the edge at $t_{departure}$, the path to the mainland is destroyed for the rest of the night (Dead path).

9 Open Problems and Future Research

The framework proposed in this paper serves as a foundation. We outline significant open problems divided into pure mathematics and applied sciences.

9.1 Part I: Pure Mathematics and Analysis

Problem 9.1 (Fixed Point Theory on Temporal Graphs). Let Ω be the space of all temporal graphs on a fixed vertex set V and edge set E, over a bounded time domain $\mathcal{T} \subset \mathbb{R}^+$. We define a metric $d_{\mathcal{T}}(\mathcal{G}_1, \mathcal{G}_2)$ based on the L^p difference of their presence functions:

$$d_{\mathcal{T}}(\mathcal{G}_1, \mathcal{G}_2) = \left(\int_{\mathcal{T}} \sum_{e \in E} |\rho_1(e,t) - \rho_2(e,t)|^p dt \right)^{1/p}$$

Question: If $\Psi : \Omega \to \Omega$ is a contractive operator describing the evolution of the network (e.g., an update rule), under what conditions does a unique "Limit Graph" or *Temporal Fixed Point* exist? This extends the Banach Contraction Principle to dynamic graph spaces.

Problem 9.2 (Temporal Persistent Homology). In algebraic topology, Betti numbers count holes. In a temporal graph, a "hole" might appear and disappear. **Question:** Can we construct a *Temporal Persistence Module* that captures "spatio-temporal voids"? For instance, a cycle $A \to B \to C \to A$ that is completed only over a specific time interval $[t_1, t_2]$. What are the stability properties of such barcodes under (f, g)-perturbations?

Problem 9.3 (Spectral Theory of Non-Autonomous Operators). The adjacency matrix $A(t)$ of a temporal graph is a time-dependent operator. **Question:** How do the eigenvalues $\lambda_i(t)$ evolve? Can we define a "Temporal Spectrum" as the set of Lyapunov exponents of the dynamical system $\dot{x} = -A(t)x$? This would link graph theory with the theory of non-autonomous differential equations. (Note: This requires regularity assumptions on $A(t)$, e.g., continuity or measurability.)

Problem 9.4 (Ramsey Theory for Dynamic Structures). Question: What is the *Temporal Ramsey Number* $R_T(k, l, \delta)$? Defined as the minimum size of a temporal graph such that there exists either a clique of size k persisting for duration δ, or an independent set of size l persisting for duration δ.

9.2 Part II: Advanced Applications and Frontiers

Research Question 9.5 (Quantum Causal Graphs). In quantum networks, entanglement swapping allows for connections to be established between nodes that never interacted directly. **Challenge:** Does the (f, g)-framework hold when $g(t)$ acts on complex time $t \in \mathbb{C}$? This is crucial for modeling "indefinite causal order" in quantum gravity and quantum internet protocols.

Research Question 9.6 (Blockchain and DAG Dynamics). Cryptocurrencies like IOTA use a Directed Acyclic Graph (DAG) called the "Tangle" instead of a blockchain. The security depends on the temporal rate of incoming transactions. **Challenge:** Can we use Class III (Nonlinear) homomorphisms to model "Double-Spending Attacks" as topological anomalies where two conflicting paths exist in the same temporal window?

Research Question 9.7 (Smart City Traffic Control). Traffic flow is a continuous temporal graph. **Challenge:** Can we design an optimization problem $\min \int \text{Congestion}(t)dt$ where the control variables are the local time-delays (traffic lights) at each node? This maps the traffic problem to finding an optimal (Id, g)-homomorphism that "smooths" the flow.

10 Conclusion

We have presented a unified algebraic framework for Temporal Graphs. By introducing the (f, g)-homomorphism, we bridged the gap between static graph theory and nonlinear dynamics. The proposed open problems, particularly regarding Fixed Point Theory and Temporal Homology, suggest that this field is ripe for exploration by functional analysts and topologists alike.

References

[1] F. Harary and G. Gupta, *Dynamic graph models*, Math. Comput. Modell. **25** (1997), no. 7, 79–87.

[2] D. Kempe, J. Kleinberg, and A. Kumar, *Connectivity and inference problems for temporal networks*, J. Comput. Syst. Sci. **64** (2002), no. 4, 820–842.

[3] P. Holme and J. Saramäki, *Temporal networks*, Phys. Rep. **519** (2012), no. 3, 97–125.

[4] N. Masuda and R. Lambiotte, *A Guide to Temporal Networks*, World Scientific, 2016.

[5] O. Michail, *Introduction to temporal graphs: An algorithmic perspective*, Internet Math. **12** (2016), no. 4, 239–280.

[6] A. Casteigts, P. Flocchini, W. Quattrociocchi, and N. Santoro, *Time-varying graphs and dynamic networks*, Int. J. Parallel Emergent Distrib. Syst. **27** (2012), no. 5, 387–408.