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Extrapolation method on a long interval for a class of nonlinear system of Volterra integral equations of the second kind

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Abstract

In this paper, we develop and apply the extrapolation method to a large class of nonlinear systems of Volterra integral equations. These systems arise in many physical and medical phenomena models, such as plasma and diseases, which can be investigated through mathematical models. The solutions of such systems require to be obtained on a long independent variable interval. We show that for such kernels, this method converges on a long interval. We apply the method on the unit interval $[0, 1]$ and obtain the solution through $[0, 1]$. Then, we apply the method on $[1, 2]$, and use the solution on $[0, 1]$ as the lag term; hence, we obtain the solution on $[0, 2]$. We continue this procedure on $[0, n]$, for some positive integer n. We show that in the presented kernel, the error on $[0, n]$ is not propagated to $[n, n + 1]$. Hence, we can use the proposed method on a long interval and also for stiff problems. We give a total algorithm for the proposed method and then establish a convergence analysis for it. As we shall see in the last example, the method is a powerful technique for stiff problems. Finally, we show the applicability and accuracy of the method in some sample problems.

Keywords: Extrapolation method, Nonlinear Volterra integral system, Stiff problems, Lag terms, Long interval
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1 Introduction

Many physical, engineering, and medical phenomena are mathematically modeled as Volterra integral and integro-differential equations. Numerical solution of such problems is very important due to its engineering and medical applications. Some methods are based on base functions [19, 12, 11, 7] and some methods are based on numerical integrations. This study is concerned with application of methods based on numerical integrations. Hence, we consider the following system of nonlinear Volterra integral equations of the second kind

$$X(t) = F(t) + \int_0^t K(\tau, X(\tau))d\tau, \quad 0 < t, \tag{1.1}$$

where

$$X(t) = (x_1(t), ..., x_d(t))^T, \ F(t) = (f_1(t), ..., f_d(t))^T, \tag{1.2}$$

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$$K(\tau, X(\tau)) := (k_1(\tau, X(\tau)), ..., k_d(\tau, X(\tau)))^T, \tag{1.3}$$

$$\int_0^t K(\tau, X(\tau))d\tau := \left(\int_0^t k_1(\tau, X(\tau))d\tau, ..., \int_0^t k_d(\tau, X(\tau))d\tau\right)^T. \tag{1.4}$$

Here $d \in \mathbb{N}$ is the dimension of the system, $X(t)$ is the unknown vector valued function, and $F(t), K(\tau, X(\tau)$ are known vector valued functions.
This system operates directly in several medical models such as those of the endemic infectious diseases [9], which lead to autonomous systems. Autonomous systems are special cases of (1.1), where the argument τ in the kernel function K dose not appear; in other words, $K(\tau, X) = K(X)$. In several other models, system (1.1) operates in the differential form

$$\begin{cases} \dot{X}(\tau) = K(\tau, X(\tau)), & \tau \geq 0 \\ X(0) = X_0, \end{cases}, \tag{1.5}$$

which is convertible to (1.1), when integrating on $\tau \in [0, t]$. Hethcote et al. (2004) [10], consider a predator-prey model with infected prey, Kuznetsov et al. (1994) [13], investigate nonlinear dynamics of immunogenic tumors, Nyabadza et al. (2011) [15], modelled the HIV/AIDS epidemic trends in South Africa. All of the above models comply with the differential form (1.5) and must be solved on a long time interval. In [1] the authors converted a wave equation model to the form (1.1) and used the extrapolation method to solve it. The results were very accurate and corresponded with practical cases. Suppose N_i, represents the number of partial intervals and we want to solve Eq. (1.1) on $[0, N_i] = \bigcup_{k=1}^{N_i}[k-1, k]$. Since the method dose not employ higher derivatives of functions and the round off error dose not propagate between the partial intervals of $[0, N_i]$; therefore, it is an appropriate technique for stiff problems. Stiff differential equations are characterized as those whose exact solutions have a term in the form of e^{-ct}, where c is a large positive constant. This is usually only a part of the solution, called the transient solution, the more important portion of the solution is called the steady-state solution. A transient portion of a stiff equation will rapidly decay to zero as t increases, but since the nth derivative of this term has magnitude $c^n e^{-ct}$, the derivative does not decay quickly. In fact, since the derivative in the error term is evaluated not at t, but at a number between zero and t, the derivative terms may increase as t increases and very rapidly indeed [5]. These terms do not appear in our method and hence it is a very good choice for stiff problems. As we see in Example 5.4, the proposed method is a good approach to stiff problems on a long variable interval. There are at least two novelties in this paper. First, the proposed method can be applied to long intervals that are favorable to various scientific problems, especially mathematical models of incurable diseases [3]. Secondly, several mathematical problems are stiff, and methods that use higher derivatives are inefficient in such problems. In mathematical models of medical problems, it is necessary to solve the problem in a long period of time and for this purpose, the simplest quadrature rules should be used. Provided that previously mentioned factors do not cause stiffness. This issue is fully observed in the proposed method. A combination of this method with Newton's method for solving a coronavirus model is given in [4].

2 Composite trapezoidal iteration algorithm

We give some theorems about the method and its convergence.

Theorem 2.1. Suppose $g(t) = (g_1(t), ..., g_d(t))^T$ is a vector valued function, where every g_i, $i = 1, ..., d$ is smooth on $[a, b]$. For $l = 1, ..., d$ assume

$$I(g_l) = \int_a^b g_l(t)dt. \tag{2.1}$$

For sufficiently large N, put $h = (b-a)/N$, $t_i = a + ih, i = 0, 1..., N$. If $g_l \in C^2[a, b]$ then the composite trapezoidal rule

$$Q_N(g_l) = \frac{h}{2}(g_l(t_0) + g_l(b)) + h\sum_{i=1}^{N-1} g_l(t_i), \tag{2.2}$$

has the following asymptotic expansion

$$E_N(g_l) := Q_N(g_l) - I(g_l) = \frac{b-a}{12} g_l^{''}(\theta)h^2, \tag{2.3}$$

where $\theta \in (a, b)$.

Proof . See [5] Theorem 4.5. □

For $k \in \{1, .., N_i\}$, we divide each partial interval $[k-1, k]$ to N_p subintervals, with common length $h = \frac{1}{N_p}$. Suppose we have solved Eq. (1.1) on $[0, n]$, for some $n \in \{0, 1, 2, ...\}$, and we want to solve the problem on $[n, n+1]$. To achieve this aim, we put $t = t_{n,i} := n + ih, i = 1, ..., N_p$ in equation (1.1) and obtain

$$X(t_{n,i}) - \int_0^{t_{n,i}} K(\tau, X(\tau))d\tau = F(t_{n,i}), \tag{2.4}$$

or its equivalence which is

$$X(t_{n,i}) - \int_n^{t_{n,i}} K(\tau, X(\tau))d\tau = F(t_{n,i}) + G_n, \tag{2.5}$$

where

$$G_n = \int_0^n K(\tau, X(\tau))d\tau = X(n) - F(n). \tag{2.6}$$

Hence, we have

$$X(t_{n,i}) - \int_n^{t_{n,i}} K(\tau, X(\tau))d\tau = X(n) + F(t_{n,i}) - F(n), \tag{2.7}$$

where $X(n)$ obtained from the lag interval $[0, n]$, is known, especially for $n = 0$, $X(0) = F(0)$. Employing the quadrature formula (2.2) we have

$$X_{n,i} = X_n + F(t_{n,i}) - F(n) + \frac{h}{2}\left(K(n, X_n) + K(t_{n,i}, X_{n,i})\right) + h\sum_{j=1}^{i-1} K(t_{n,j}, X_{n,j}),\ i = 1, ..., N_p, \tag{2.8}$$

where $X_{n,i}$ is the approximation of $X(t_{n,i})$ and X_n is the approximation of $X(n)$. For $i = 1$, the term containing Σ is zero. The equations (2.8) are a nonlinear diagonal system of equations with unknown vectors $X_{n,i}, i = 1, ..., N_p$. The following algorithm approximate $X_{n,i}$s by the sequence $\left\{\widetilde{X}_{n,i}\right\}_{i=1}^{N_p}$ with at most $\epsilon > 0$ distinction. In this paper, we suppose $\|.\|$, is an equivalence norm of $\mathbb{R}^d$, such as Euclidean or infinity norms (see chapter 1 of [2]).

Algorithm 1 (Composite Trapezoidal iteration method)
Step 1. Set $\widetilde{X}_{n,0} = X_n$ (note that $X_n = X_{n-1,N_p}$ is obtained from the lag interval $[0, n]$, and $X_0 = F(0)$), and set $i = 1$;

Step 2. Set $X^0_{n,i} = \widetilde{X}_{n,i-1}, \quad m = 0$;

Step 2.1. Set

$$X^{m+1}_{n,i} = X_n + F(t_{n,i}) - F(n) + \frac{h}{2}\left(K(n, X_n) + K(t_{n,i}, X^m_{n,i})\right) + h\sum_{j=1}^{i-1} K(t_{n,j}, X_{n,j}); \tag{2.9}$$

Step 2.2. If $\left\|X^{m+1}_{n,i} - X^m_{n,i}\right\| \le \epsilon$, set $\widetilde{X}_{n,i} = X^{m+1}_{n,i}$ and if $i < N_p$ set $i = i + 1$ then go to Step2, otherwise, go to Step 3;

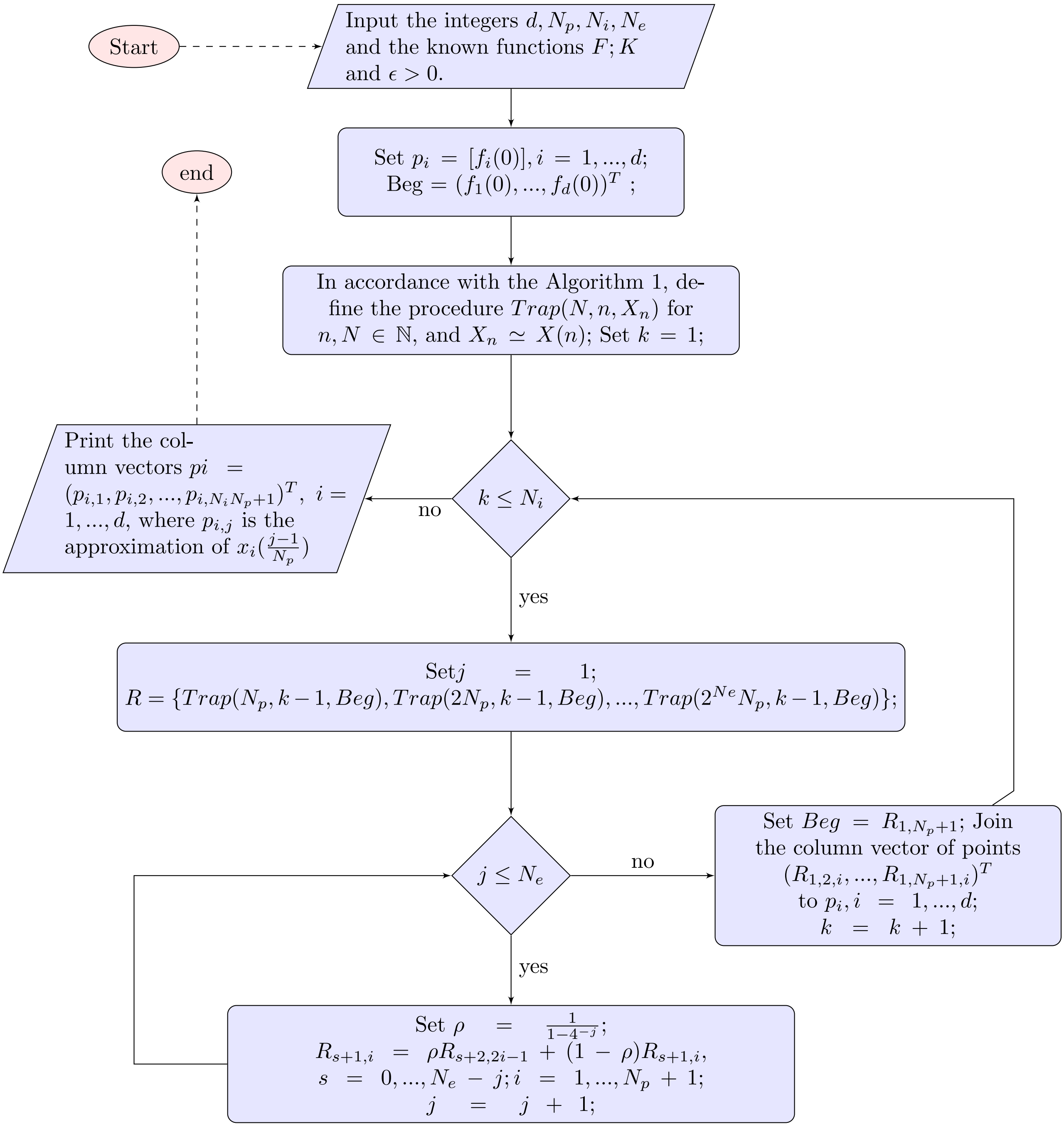
If $\left\|X^{m+1}_{n,i} - X^m_{n,i}\right\| > \epsilon$ (Otherwise of the first If), set $m = m + 1$ and go to Step 2.1

Step3. Print $\left\{\widetilde{X}_{n,i}\right\}_{i=0}^{N_p}$ as the approximation of $\{X_{n,i}\}_{i=0}^{N_p}$, and the end the algorithm.

Remark 2.2. In the above algorithm, we denote the ordered list $\left\{\widetilde{X}_{n,i}\right\}_{i=0}^{N_p}$ by $Trap(N_p; n; X_n)$. Note that every element of $Trap(N_p; n; X_n)$ is a d-tuple column vector. The first element of $Trap(N_p; n; X_n)$ is $\widetilde{X}_{n,0}$, and in general for $i = 1, ..., N_p + 1$, the ith element of $Trap(N_p; n; X_n)$ is $\widetilde{X}_{n,i-1}$. For $i = 1, ..., N_p + 1$ and $j = 1, ..., d$ the (i, j)th element of $Trap(N_p; n; X_n)$ is the jth component of the d-tuple column vector $\widetilde{X}_{n,i-1}$. All of these ordered list can be defined by any software. Particularly, in Mathematica programming these ordered lists are defined by the command "Table".

3 Extrapolation and total algorithm

In this section, we extend the Romberg extrapolation technique from [17] for numerical solution of (2.4). For a simple case of Romberg extrapolation technique, see [16], algorithm 7.1. We demonstrate this algorithm by the following flow chart



In the first rectangle, every p_i is a column vector with one component and during process of the algorithm, the other components will be added to them. We sketch the algorithm so as to make the jth component of the column vector p_i be the approximation of $x_i(\frac{j-1}{N_p})$ for $i = 1, ..., d$ d and $j = 1, ..., N_iN_p + 1$. In the above flow chart, R is an ordered list, with $N_e + 1$ element, where $N_e \in \mathbb{N}$ is the number of extrapolations. For $m = 1, ..., N_e + 1$, the mth element of R is $Trap(2^{m-1}N_p; k-1, Beg)$, we denote this element by R_m. For $m = 1, ..., N_e + 1$, $i = 1, ..., 2^{m-1}N_p + 1$ and $j = 1, ..., d$ and we denote the (i, j)th element of $Trap(2^{m-1}N_p; k-1, Beg)$ by $R_{m,i,j}$ which is the (m, i, j)th element of R. In the last rectangle, the Romberg extrapolation is applied on the interval $[k-1, k]$, and we obtain $(R_{1,2,i}, ..., R_{1,N_p+1,i})^T$ as the approximation of $(x_i(k-1+\frac{1}{N_p}), ..., x_i(k-1+\frac{N_p}{N_p}))^T$. In the fourth rectangle, we join this vector to p_i. Approximation of $X(k)$ is R_{1,N_p+1} and hence we can set the beginning of the next interval by $Beg = R_{1,N_p+1}$, then, we would be able to add one value to k, and go to the next interval (the first rhombus).

4 Convergence Analysis

For the convergence analysis of the method, the operator K must be satisfied by Lipschitz condition, this means that there must be a positive constant $L > 0$ such that

$$\|K(X) - K(Y)\| \leq L\|X - Y\|, \quad X, Y \in E, \tag{4.1}$$

where $E \subseteq \mathbb{R}^d$ is the domain of K. We offer sufficient condition for this purpose. The following theorem cites the main criterion.

Theorem 4.1. (Generalized Taylor's Theorem) Let $K : V \longrightarrow W$ be an operator between two Banach spaces, such that K is n times continuously differentiable in a neighborhood $b(X, r)$, $r > 0$, of X. Then, for all $\widetilde{X}$ in the interior of $b(X, r)$

$$\left\| K(X) - K(\widetilde{X}) - \sum_{i=1}^{n-1} \frac{1}{i!} K^{(i)}(X)(X - \widetilde{X})^i \right\| \leq \sup_{Y \in l(X,\widetilde{X})} \|K^{(n)}(Y)\| \frac{\|X - \widetilde{X}\|^n}{n!}.$$

where $l(X, \widetilde{X})$ is the line segment between X and $\widetilde{X}$.

Proof . See [14], Theorem 5.8. □

Theorem 4.2. (Frechet or Jacobian matrix) Suppose K maps an open set $E \subseteq \mathbb{R}^d$ in to $\mathbb{R}^d$, and K is differentiable at a point $X \in E$. Then, the partial derivatives $(D_j K_i)(X) = \frac{\partial K_i}{\partial x_j}(X)$ exist, and

$$K^{'}(X) = \begin{bmatrix} (D_1K_1)(X) & \dots & (D_dK_1)(X) \\ \vdots & \vdots & \vdots \\ (D_1K_d)(X) & \dots & (D_dK_d)(X) \end{bmatrix}. \tag{4.2}$$

Proof . See [18], Theorem 9.17. □

The following Lemma gives a criterion for Lipschitz continuity of K.

Lemma 4.3. Let K be a differentiable operator on an open set $E \subseteq \mathbb{R}^d$. Moreover, suppose

$$L = \sup_{X \in E} \max_{1 \leq i \leq d} \sum_{j=1}^{d} |(D_j K_i)(X)| < \infty,$$

then condition (4.1) is satisfied with infinity norm.

Proof . Suppose $X, Y \in E$, then using theorem 4.1 with $n = 1$ implies that

$$\begin{aligned} \|K(X) - K(Y)\| &\leq \sup_{Z \in l(X,Y)} \|K^{'}(Z)\| \|X - Y\| \\ &= \sup_{Z \in l(X,Y)} \max_{1 \leq i \leq d} \sum_{j=1}^{d} |(D_j K_i)(Z)| \, \|X - Y\| \\ &\leq L\|X - Y\|, \end{aligned} \tag{4.3}$$

where in accordance with [5] Theorem7.11, $\|K^{'}(Z)\| = \|K^{'}(Z)\|_\infty = \max_{1 \leq i \leq d} \sum_{j=1}^{d} |(D_j K_i)(Z)|$. □ We shall use the following version of Gronwall inequality in this paper [20].

Lemma 4.4. [20] Suppose that $\{y_n\}_{n=0}^{N}$ is a nonnegative sequence of real numbers, and satisfies

$$y_0 = 0, \quad y_n \leq A + Bh \sum_{j=0}^{n-1} y_j, 1 \leq n \leq N, \quad h = \frac{1}{N}, \tag{4.4}$$

where A, and B are positive constants independent of h, then

$$\max_{0\leq i\leq N} y_i \leq Ae^B. \tag{4.5}$$

Theorem 4.5. Assume that h is sufficiently small, then the solutions of the nonlinear discrete equations (2.8) exist and are unique, and the simple iteration (2.9) is geometrically convergent.

Proof . If the Eq. (2.8) has another solution such as

$$Y_{n,i} = X_n + F(t_{n,i}) - F(n) + \frac{h}{2}\left(K(n, X_n) + K(t_{n,i}, Y_{n,i})\right) + h\sum_{j=1}^{i-1} K(t_{n,j}, Y_{n,j}),\ j = 1, ..., N_p, \tag{4.6}$$

then the differences $Z_{n,i} = Y_{n,i} - X_{n,i}$ satisfy

$$Z_{n,0} = Y_{n,0} - X_{n,0} = X_n - X_n = 0,$$

and for $i = 1, ..., N_p$,

$$Z_{n,i} = \frac{h}{2}\left(K(t_{n,i}, Y_{n,i}) - K(t_{n,i}, X_{n,i})\right) + h\sum_{j=1}^{i-1}\left(K(t_{n,j}, Y_{n,j}) - K(t_{n,j}, X_{n,j})\right). \tag{4.7}$$

For $y_i = \|Z_{n,i}\|$ in Lemma 4.4, and according to the Lipschitz continuity of the operator K, we obtain

$$\|Z_{n,i}\| \leq \frac{hL}{2}\|Z_{n,i}\| + hL\sum_{j=1}^{i-1}\|Z_{n,j}\|. \tag{4.8}$$

This inequality is equivalent with

$$\|Z_{n,i}\| \leq Bh\sum_{j=1}^{i-1}\|Z_{n,j}\|, \quad B = \frac{1}{1 - Lh/2}. \tag{4.9}$$

Inequality (4.9) is in the form (4.4) with $A = 0$. So Lemma 4.4 forces $\|Z_{n,i}\| \leq Ae^B = 0$, and hence $\|Z_{n,i}\| = 0$. This means $Z_{n,i} = 0$.

Secondly, the simple iteration (2.9) yields that

$$\begin{aligned}
\|X_{n,i}^{m+1} - X_{n,i}^{m}\| &= \Bigg\| X_n + F(t_{n,i}) - F(n) + \frac{h}{2}\left(K(n, X_n) + K(t_{n,i}, X_{n,i}^{m})\right) + h\sum_{j=1}^{i-1} K(t_{n,j}, X_{n,j}) \\
&\quad - \left(X_n + F(t_{n,i}) - F(n) + \frac{h}{2}\left(K(n, X_n) + K(t_{n,i}, X_{n,i}^{m-1})\right) + h\sum_{j=1}^{i-1} K(t_{n,j}, X_{n,j})\right)\Bigg\| \\
&= \frac{h}{2}\|K(X_{n,i}^{m}) - K(X_{n,i}^{m-1})\| \leq \frac{hL}{2}\|X_{n,i}^{m} - X_{n,i}^{m-1}\| \leq \frac{1}{2}\|X_{n,i}^{m} - X_{n,i}^{m-1}\|,
\end{aligned} \tag{4.10}$$

where we can choose $0 < h$ so small that $Lh \leq 1$. Hence the iterative Eq. (2.9) is geometrically convergent, and its solution is the only solution of (2.8). □

Theorem 4.6. There is a positive constant C independent of h such that

$$\max_{0\leq i\leq N_p}\|X_{n,i} - \widetilde{X}_{n,i}\| \leq C\epsilon h, \tag{4.11}$$

where $\widetilde{X}_{n,i} = \begin{cases} X_{n,i}^{m+1}, & i = 1, ..., N_p, \\ X_n, & i = 0. \end{cases}$, is defined by Algorithm 1

Proof . By defining $V_{n,i} = X_{n,i} - \widetilde{X}_{n,i},\ V_0 = 0$. For $i = 1, ..., N_p$, subtracting (2.9) from (2.8) yields

$$V_{n,i} = X_{n,i} - X_{n,i}^{m+1} = \frac{h}{2}\left(K(t_{n,i}, X_{n,i}) - K(t_{n,i}, X_{n,i}^{m})\right) + h\sum_{j=1}^{i-1}\left(K(t_{n,j}, X_{n,j}) - K(t_{n,j}, \widetilde{X}_{n,j})\right). \qquad (4.12)$$

The Lipschitz condition (4.1) forces

$$\begin{aligned} \|V_{n,i}\| &\leq \frac{Lh}{2}\|X_{n,i} - X_{n,i}^{m}\| + Lh\sum_{j=1}^{i-1}\|V_{n,j}\| \\ &\leq \frac{Lh}{2}\left(\|X_{n,i} - X_{n,i}^{m+1}\| + \|X_{n,i}^{m+1} - X_{n,i}^{m}\|\right) + Lh\sum_{j=1}^{i-1}\|V_{n,j}\|, \end{aligned} \qquad (4.13)$$

so

$$\left(1 - \frac{Lh}{2}\right)\|V_{n,i}\| \leq \frac{\epsilon Lh}{2} + Lh\sum_{j=1}^{i-1}\|V_{n,j}\|. \qquad (4.14)$$

Hence

$$\|V_{n,i}\| \leq A + Bh\sum_{j=1}^{i-1}\|V_{n,j}\|, \quad A = \frac{\epsilon h}{2}, B = \frac{L}{1 - Lh/2}. \qquad (4.15)$$

Using Lemma 4.4 forces that

$$\max_{0\leq i\leq N_p}\|V_{n,i}\| \leq C\epsilon h, \quad C = \frac{1}{2}e^{B}. \qquad (4.16)$$

□

Lemma 4.7. Suppose $T_{m,k}$ is the approximation of $\int_a^b f(t)dt$, by using $N_p = 2^k$ subintervals and m times use extrapolation method. Then, $\left|T_{m,k} - \int_a^b f(t)dt\right| = O\left(h^{2(m+1)}\right)$, where $h = (b-a)/2^k$.

Proof . See [17] Chapter 4 Section 4.10.□

Lemma 4.8. By the hypothesis of Lemma 4.7 and $n \in \mathbb{N}$, $\|X(n) - X_n\|$ has the accuracy $nO(h^{2(m+1)})$. In particular, for $n = 2^k$, $\|X(n) - X_n\|$ has the $O(h^{2m+1})$ accuracy, and for $n = N_p^2 = 4^k$, this accuracy is $O(h^{2m})$.

Proof . For $n = 0$, $\|X(0) - X_0\| = \|0\| = 0$. Now suppose $n \in \mathbb{N}$, $j \in \{1, ..., n\}$, and $T_{m,k,j}$ is the approximation of $\int_{j-1}^{j} K(X(\tau))d\tau$ by using $N_p = 2^k$ subintervals and m times use of extrapolation method. From the Lemma 4.7, we obtain

$$\begin{aligned} \|X(n) - X_n\| &= \int_0^n K(\tau, X(\tau))d\tau + F(n) - \left(F(n) + \sum_{j=1}^{n} T_{m,k,j}\right) \\ &= \left\|\sum_{j=1}^{n}\left(\int_{j-1}^{j} K(\tau, X(\tau))d\tau - T_{m,k,j}\right)\right\| \\ &\leq \sum_{j=1}^{n}\left\|\int_{j-1}^{j} K(\tau, X(\tau))d\tau - T_{m,k,j}\right\| = nO(h^{2(m+1)}) \end{aligned} \qquad (4.17)$$

Since $h = 1/2^k$, for $n = 2^k$, we obtain $nh = 1$, and in this case $\|X(n) - X_n\|$ has the $O(h^{2m+1})$ accuracy. For $n = 4^k$, $nh^2 = 1$, and hence the accuracy of $\|X(n) - X_n\|$ is $O(h^{2m})$. □ Another version of the Gronwall inequality that we use in this paper is as follows [6].

Lemma 4.9. [6] Suppose n_0 and N are two integers with $n_0 < N$. Let $\{x_i\}_{i=n_0}^N$, $\{a_i\}_{i=n_0}^N$ and $\{b_i\}_{i=n_0}^N$ be sequences of real numbers, with $b_i \geq 0$, which satisfies

$$x_i \leq a_i + \sum_{j=n_0}^{i-1} b_j x_j, \quad i = n_0, ..., N, \tag{4.18}$$

then

$$x_i \leq a^* \prod_{j=n_0}^{i-1} (1 + b_j), \quad a^* = \max_{j=n_0,...,N} a_j. \tag{4.19}$$

Theorem 4.10. There is a positive constant C independent of h such that

$$\max_{0 \leq i \leq N_p} \|e_{n,i}\| \leq Ch^2, \tag{4.20}$$

where $e_{n,i} := X(t_{n,i}) - X_{n,i}$, and $X_{n,i}$ is the solution of Eq. (2.8).

Proof . Without losing generality, we suppose that $n = 4^k$ is sufficient for practical problems. From Lemma 4.8, the accuracy of $\|e_{n,0}\| := \|X(n) - X_n\|$ is $O(h^{2m})$. Using Theorem 2.1 for the Eq. (2.7) yields that

$$\begin{aligned} X(t_{n,i}) &= \frac{h}{2}\left(K(n, X(n)) + K(t_{n,i}, X(t_{n,i}))\right) \\ &+ h\sum_{j=1}^{i-1} K(t_{n,j}, X(t_{n,j})) + h^2(c_{i1}, ..., c_{id})^T + X(n) + F(t_{n,i}) - F(n) \end{aligned} \tag{4.21}$$

where $c_{i1}, ..., c_{id}$ are constants. Subtracting Eq. (2.8) from (4.21) forces

$$\begin{aligned} e_{n,i} &= X(t_{n,i}) - X_{n,i} = \frac{h}{2}\left(K(n, X(n)) - K(n, X_n) + K(t_{n,i}, X(t_{n,i})) - K(t_{n,i}, X_{n,i})\right) \\ &+h\sum_{j=1}^{i-1} \left(K(t_{n,j}, X(t_{n,j})) - K(t_{n,j}, X_{n,j})\right) + X(n) - X_n + h^2(c_{i1}, ..., c_{id})^T. \end{aligned} \tag{4.22}$$

Lipschitz condition (4.1) results in

$$\begin{aligned} \|e_{n,i}\| &= \frac{Lh}{2}\left(\|X(n) - X_n\| + \|e_{n,i}\|\right) \\ &+ hL\sum_{j=1}^{i-1} \|e_{n,j}\| + \|X(n) - X_n\| + h^2\|(c_{i1}, ..., c_{id})^T\|. \end{aligned} \tag{4.23}$$

Since the accuracy of $\|X(n) - X_n\|$ is $O(h^{2m})$, then there exists a positive constant c^*, such that $\left(1 + \frac{Lh}{2}\right)\|X(n) - X_n\| + h^2\|(c_{i1}, ..., c_{id})^T\| \leq c^* h^2$, for all $i = 1, ..., N_p$.

Thus, we have

$$\left(1 - \frac{Lh}{2}\right)\|e_{n,i}\| \leq c^* h^2 + hL\sum_{j=1}^{i-1} \|e_{n,j}\|. \tag{4.24}$$

Using Lemma 4.9 with $n_0 = 0$, $N = N_p$, $x_i = \|e_{n,i}\|$, $a_i = \frac{c^* h^2}{1-hL/2} = \frac{c^* N_p^{-2}}{1-N_p^{-1}L/2} =: a_{N_p}$, $b_j = \frac{Lh}{1-hL/2} = \frac{L}{N_p - L/2} =: b_{N_p}$, $j = 1, ..., N_p$, $a^* = a_{N_p}$, we obtain

$$x_i \leq h^2 \frac{c^*}{1 - LN_p^{-1}/2}(1 + b_{N_p})^{N_p} =: h^2 c_{N_p}. \tag{4.25}$$

Since $\lim_{N\to\infty} b_N = 0$, then $\lim_{N\to\infty} c_N = c^* \exp\{\lim_{N\to\infty} N b_N\} = c^* e^L$. So for sufficiently large N_p, the inequality (4.20), is obtained with $C = c^* e^L$. □

Theorem 4.11. For $N_i \leq 4^k$, all of the solutions have at least $O(h^{2m})$ accuracy on $[0, N_i]$, where $h = 1/N_p, N_p = 2^k$.

Proof .

We prove this procedure by a mathematical induction. For $n = 1$, according to Theorem 4.10 the application of Algorithm 1 causes $O(h^2)$ accuracy in the interval $[0, 1]$. Then, from Lemma 4.7, using m times extrapolation yields at least $O(h^{2m})$ accuracy on $[0, 1]$. Now suppose $n \in \{1, ..., N_i - 1\}$, and theorem is true on $[0, n]$, which concludes that $\|X(n) - X_n\|$ has at least $O(h^{2m})$ accuracy. Again, according to Theorem 4.10, the application of Algorithm 1 causes $O(h^2)$ accuracy in the interval $[0, n + 1]$. Then, from Lemma 4.8, using m times extrapolation on $[0, n + 1]$ implies at least $O(h^{2m})$ accuracy on $[0, n + 1]$.□

5 Numerical examples

Example 5.1. In problem (1.1)-(1.4), for

$$F(t) = \begin{bmatrix} te^{-t} \\ t^2e^{-t} \end{bmatrix}, \qquad K(t, X) = \begin{bmatrix} x_1^2 - e^{-t}x_2 \\ x_2^2 - tx_1x_2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

the integral system associated with this problem is (1.1), which has the exact solution

$$X(t) = \begin{bmatrix} te^{-t} \\ t^2e^{-t} \end{bmatrix}.$$

Since $\lim_{t\to\infty} x_j(t) = 0,\ j = 1, 2$, then the appropriate criterion for error analysis is the absolute error. In Table 1, columns 2 and 3 show absolute errors of $\widetilde{x}_j,\ j = 1, 2$ at $t_i = 2i, i = 1, ..., 10$. $x_j,\ j = 1, 2$ are exact solutions and $\widetilde{x}_j$, $j = 1, 2$ are evaluated by the extrapolation method with $N_p = 128, N_e = 3, N_i = 20$. Figure 1 shows variation of these solutions as functions of t for Example 5.1. In all of the following figures, the dashed lines show approximations and the solid lines show the exact solutions. In this example

$$\begin{aligned} L =& sup_{X\in E} \max_{1\leq i\leq d} \sum_{j=1}^{d} |(D_jK_i)(X)| \\ =& \sup_{t\geq 0} \max \left\{2\,|x_1(t)| + e^{-t}, |tx_2(t)| + |2x_2(t) - tx_1(t)|\right\} \\ =& \sup_{t\geq 0} \max \left\{(1 + 2t)e^{-t}, (t^2 + t^3)e^{-t}\right\} \\ =& \max \left\{2e^{-\frac{1}{2}}, (1 + \sqrt{3})^2(2 + \sqrt{3})e^{-1-\sqrt{3}}\right\} = 1.813, \end{aligned}$$

which provides the sufficient conditions for the convergence of the method.

Table 1: Absolute errors of $\widetilde{x}_j$, $j = 1, 2$ at $t_i = 2i, i = 1, ..., 10$ in Example 5.1.

i	Errors of $\widetilde{x}_1$	Errors of $\widetilde{x}_2$
1	*Negligible*	*Negligible*
2	2.776×10^{-17}	5.551×10^{-17}
3	2.602×10^{-17}	1.249×10^{-16}
4	2.472×10^{-17}	1.353×10^{-16}
5	2.439×10^{-17}	1.396×10^{-16}
6	2.434×10^{-17}	1.405×10^{-16}
7	2.434×10^{-17}	1.408×10^{-16}
8	2.434×10^{-17}	1.409×10^{-16}
9	2.434×10^{-17}	1.409×10^{-16}
10	2.434×10^{-17}	1.409×10^{-16}

Example 5.2. In problem (1.1)-(1.4), for

$$F(t) = \begin{bmatrix} t^2 + \frac{1}{1+t} + ln(1 + t) \\ 1 + t - \frac{1}{2}\left(\frac{t(1+t)}{1+t^2} + \arctan t\right) \\ \frac{5}{3} + \frac{4\pi}{9\sqrt{3}} + t^2 - \frac{2+t}{3(1+t+t2)} - \frac{8\arctan\left(\frac{1+2t}{\sqrt{3}}\right)}{3\sqrt{3}} \end{bmatrix},$$

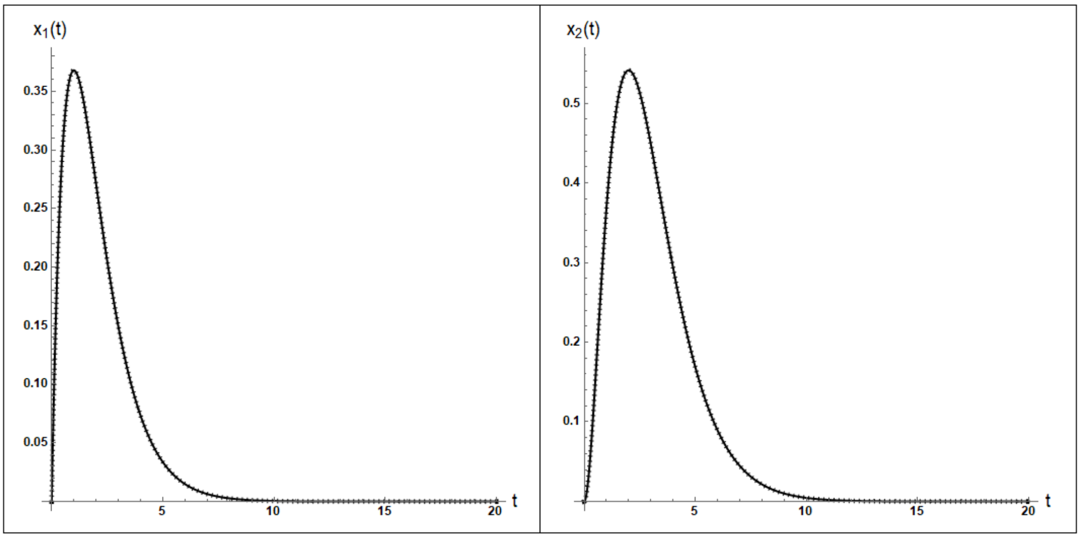


Figure 1: Variations of $x_j(t)$ (solid lines) and $\widetilde{x}_j(t)$ (dashed lines), $j = 1, 2$ as functions of t in Example 5.1.

$$K(X) = \begin{bmatrix} \frac{x_1}{x_2^2} \\ \frac{x_2}{x_3^2} \\ \frac{x_3}{x_1^2} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

the integral system associated with this problem is (1.1), which has the exact solution

$$X(t) = \begin{bmatrix} t^2 + t + 1 \\ t + 1 \\ t^2 + 1 \end{bmatrix}.$$

In Table 2, columns $2, 3$ and 4 show the absolute errors of $\widetilde{x}_j$, $j = 1, 2, 3$ at $t_i = 25i, i = 1, ..., 10$. Columns $2, 3$ and 4 of Table 3 show the relative errors of $\widetilde{x}_j$, $j = 1, 2, 3$ at $t_i = 25i, i = 1, ..., 10$. x_j, $j = 1, 2, 3$ are the exact solutions and $\widetilde{x}_j$, $j = 1, 2, 3$ are evaluated by the extrapolation method with $N_p = 128, N_e = 3, N_i = 256$. Figure 2 shows variation of these solutions as functions of t in Example 5.2. In this example,

$$L = \sup_{X \in E} \max_{1 \le i \le d} \sum_{j=1}^{d} |(D_j K_i)(X)| = \sup_{t \ge 0} \max \left\{ 2 \left| \frac{x_1}{x_2^3} \right| + \frac{1}{x_2^2}, 2 \left| \frac{x_2}{x_3^3} \right| + \frac{1}{x_3^2}, 2 \left| \frac{x_3}{x_1^3} \right| + \frac{1}{x_1^2} \right\} \le 3,$$

provided the sufficient conditions for the convergence of the method.

Table 2: Absolute errors of $\widetilde{x}_j$, $j = 1, 2, 3$ at $t_i = 25i, i = 1, ..., 10$ in Example 5.2.

i	Errors of $\widetilde{x}_1$	Errors of $\widetilde{x}_2$	Errors of $\widetilde{x}_3$
1	3.958×10^{-5}	5.950×10^{-6}	1.040×10^{-6}
2	4.820×10^{-5}	5.950×10^{-6}	1.039×10^{-6}
3	5.320×10^{-5}	5.950×10^{-6}	1.039×10^{-6}
4	5.672×10^{-5}	5.950×10^{-6}	1.039×10^{-6}
5	5.944×10^{-5}	5.950×10^{-6}	1.039×10^{-6}
6	6.166×10^{-5}	5.950×10^{-6}	1.039×10^{-6}
7	6.353×10^{-5}	5.950×10^{-6}	1.039×10^{-6}
8	6.515×10^{-5}	5.950×10^{-6}	1.039×10^{-6}
9	6.657×10^{-5}	5.950×10^{-6}	1.039×10^{-6}
10	6.785×10^{-5}	5.950×10^{-6}	1.039×10^{-6}

Example 5.3. The condition $L < \infty$ is sufficient for the convergence of the method, but it is not necessary. In problem (1.5), for $X_0 = (0, 1)^T, K(X) = (x_2 - x_1^2, 2(x_1 x_2 - x_1^3))^T, X = (x_1, x_2)^T$, the integral equation associated

Table 3: Relative errors of $\widetilde{x}_j$, $j = 1, 2, 3$ at $t_i = 25i, i = 1, ..., 10$ in Example 5.2.

i	Errors of $\widetilde{x}_1$	Errors of $\widetilde{x}_2$	Errors of $\widetilde{x}_3$
1	6.080×10^{-8}	2.289×10^{-7}	1.662×10^{-9}
2	1.890×10^{-8}	1.167×10^{-7}	4.154×10^{-0}
3	9.331×10^{-9}	7.830×10^{-8}	1.846×10^{-10}
4	5.615×10^{-9}	5.891×10^{-8}	1.038×10^{-10}
5	3.774×10^{-9}	4.722×10^{-8}	6.646×10^{-11}
6	2.722×10^{-9}	3.940×10^{-8}	4.615×10^{-11}
7	2.063×10^{-9}	3.381×10^{-8}	3.391×10^{-11}
8	1.621×10^{-9}	2.960×10^{-8}	2.596×10^{-11}
9	1.309×10^{-9}	2.633×10^{-8}	2.051×10^{-11}
10	1.081×10^{-9}	2.371×10^{-8}	1.661×10^{-11}

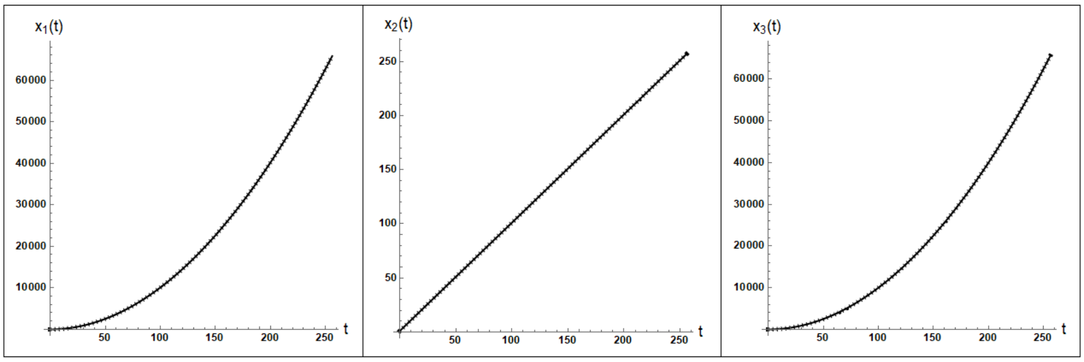


Figure 2: Variations of $x_j(t)$ (solid lines) and $\widetilde{x}_j(t)$ (dashed lines), $j = 1, 2, 3$ as functions of t in Example 5.2.

with this problem is, $X(t) - \int_0^t K(X(\tau))d\tau = X_0$, which has the exact solution $X(t) = (t, t^2 + 1)^T$. In this example $L = \sup_{X \in E} \max_{1 \le i \le d} \sum_{j=1}^{d} |(D_j K_i)(X)| = \sup_{t \ge 0} \max \left\{1 + 2\,|x_1|\,, 2\,|x_1| + 2\left|-3x_1^2 + x_2\right|\right\} = \infty$, but as we see the method is convergent.

In Table 4, columns 2, 3, 4 and 5 show the absolute and the relative errors of $\widetilde{x}_1$ and $\widetilde{x}_2$ at $t_i = 25i, i = 1, ..., 10$. x_1, x_2 are the exact solutions, and $\widetilde{x}_1, \widetilde{x}_2$ are evaluated by the extrapolation method with $N_p = 128, N_e = 3, N_i = 256$. Figure 3 shows variation of these solutions as functions of t in Example 5.3.

Table 4: Absolute and Relative errors of $\widetilde{x}_1$ and $\widetilde{x}_2$ at $t_i = 25i, i = 1, ..., 10$ in Example 5.3.

i	Absolute errors of $\widetilde{x}_1$	Relative errors of $\widetilde{x}_1$	Absolute errors of $\widetilde{x}_2$	Relative errors of $\widetilde{x}_2$
1	5.18×10^{-12}	2.07×10^{-13}	2.60×10^{-10}	4.15×10^{-13}
2	3.02×10^{-11}	6.04×10^{-13}	3.02×10^{-9}	1.21×10^{-12}
3	4.32×10^{-11}	5.76×10^{-13}	6.47×10^{-9}	1.15×10^{-12}
4	2.24×10^{-11}	2.24×10^{-13}	4.47×10^{-9}	4.47×10^{-13}
5	2.99×10^{-10}	2.39×10^{-12}	7.47×10^{-8}	4.78×10^{-12}
6	1.81×10^{-9}	1.21×10^{-11}	5.43×10^{-7}	2.41×10^{-11}
7	2.43×10^{-9}	1.39×10^{-11}	8.49×10^{-7}	2.77×10^{-11}
8	4.02×10^{-9}	2.01×10^{-11}	1.61×10^{-6}	4.02×10^{-11}
9	1.25×10^{-8}	5.54×10^{-11}	5.61×10^{-6}	1.11×10^{-10}
10	2.80×10^{-8}	1.12×10^{-10}	1.40×10^{-5}	2.24×10^{-10}

Example 5.4. The proposed method is a powerful technique for the stiff system of differential equations. In accordance with [5], Example 1 of section 5.11, the following is a stiff system

$$\begin{bmatrix} \dot{x_1}(\tau) \\ \dot{x_2}(\tau) \end{bmatrix} = \begin{bmatrix} 9 & 24 \\ -24 & -51 \end{bmatrix} \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} + \begin{bmatrix} 5\cos\tau - \frac{1}{3}\sin\tau \\ -9\cos\tau + \frac{1}{3}\sin\tau \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \end{bmatrix}.$$

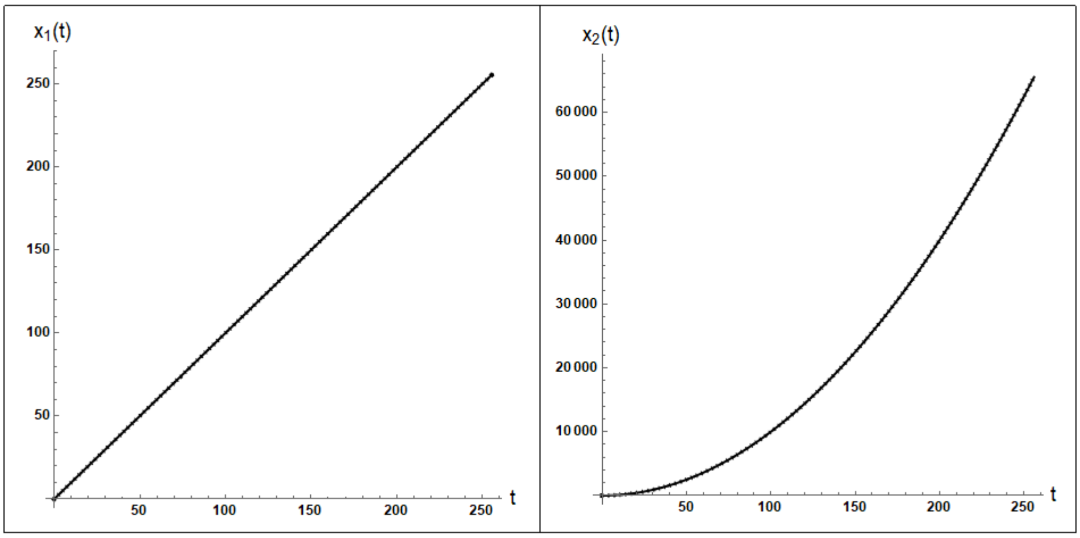


Figure 3: Variations of $x_1(t), x_2(t)$ (solid lines) and $\widetilde{x}_1, \widetilde{x}_2$ (dashed lines) as functions of t in Example 5.3.

By integrating on $\tau \in [0, t]$, for $t > 0$, we obtain Eq. (1.1) with

$$F(t) = \begin{bmatrix} 1 + \frac{1}{3}\cos t + 5\sin t \\ 1 - \frac{1}{3}\cos t - 9\sin t \end{bmatrix}, \quad K(t, X) = K(X) = AX, \quad A = \begin{bmatrix} 9 & 24 \\ -24 & -51 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

With the above data, the integral system (1.1) has the exact solution

$$X(t) = \begin{bmatrix} 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t \\ -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t \end{bmatrix}.$$

The transient term e^{-39t} in the above solution causes this system to be stiff. Since $x_j(t),\ j = 1, 2$ are bounded, then the appropriate criterion for error analysis is the absolute error. In Table 5, columns 2 and 3 show the absolute errors of $\widetilde{x}_j,\ j = 1, 2$ at $t_i = 2i, i = 1, ..., 10$. $x_j,\ j = 1, 2$ are the exact solutions and $\widetilde{x}_j,\ j = 1, 2$ are evaluated by the extrapolation method with $N_p = 128, N_e = 3, N_i = 20$. Figure 4 shows variation of these solutions as functions of t in Example 5.4. In this example $L = \sup_{X \in E} \max_{1 \le i \le d} \sum_{j=1}^{d} |(D_j K_i)(X)| = \sup_{t \ge 0} \max\{9 + 24, 24 + 51\} = 75$, which provides the sufficient conditions for the convergence of the method. However as Example 1 of section 5.11 of [5] shows, the Runge-Kutta fourth order method for systems leads to the disastrous results on $[0, 1]$ with the step size $h = 0.1$. Hence, applicability and accuracy of the method for stiff problems is illustrated numerically by this Example.

Table 5: Absolute errors of $\widetilde{x}_j, j = 1, 2$ at $t_i = 2i, i = 1, ..., 10$ in Example 5.4.

i	Errors of $\widetilde{x}_1$	Errors of $\widetilde{x}_2$
1	4.248×10^{-7}	2.955×10^{-7}
2	3.267×10^{-7}	3.057×10^{-7}
3	3.546×10^{-7}	3.789×10^{-7}
4	2.988×10^{-8}	1.047×10^{-8}
5	3.795×10^{-7}	3.702×10^{-7}
6	2.860×10^{-7}	3.186×10^{-7}
7	1.415×10^{-7}	1.050×10^{-7}
8	4.038×10^{-7}	4.060×10^{-7}
9	1.946×10^{-7}	2.329×10^{-7}
10	2.418×10^{-7}	2.122×10^{-7}

6 Conclusions

In this paper, we present a powerful method for solving the nonlinear Volterra Integral Equations of the second kind. Given that such equations appear in mathematical models of many infectious diseases [3, 9], and that these

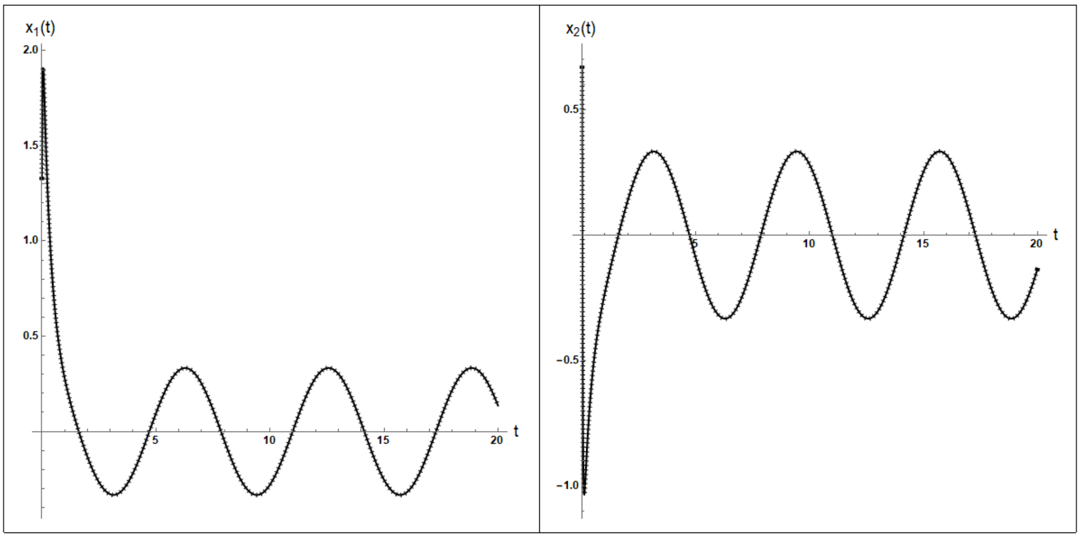


Figure 4: Variations of $x_1(t), x_2(t)$ (solid lines) and $\widetilde{x}_1, \widetilde{x}_2$ (dashed lines) as functions of t in Example 5.4.

models must be solved on a long period of time, we have proposed a method that can predict the system on a long period of time. In addition, because the proposed method uses the simplest type of quadrature rules, so it is a suitable method for stiff problems. Moreover, we have used extrapolation to the limit to increase the accuracy of the approximation. The error analysis used shows clearly that the proposed method has excellent efficiency and accuracy for several stiff problems and models related to infectious disease problems, such as mathematical models concerning influenza and coronavirus; However, the proposed method is also applicable to physical problems [8, 1]. The present paper has offered a convergent analysis for equations (1.1) system. The study reveals the applicability and accuracy of its employed method by using several sample problems. To clarify the method farther, we have presented its algorithm in a flow-chart.

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