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A novel investigation of multiple solution forms for the extended fractional Korteweg–de Vries equation

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Abstract

The extended fractional Korteweg–de Vries (K-dV) equation is examined in this paper through the Jacobi elliptic method. This approach is a powerful technique capable of generating multiple forms of solutions, making it highly useful for representing various types of wave behaviors. The graphical interpretations clearly illustrate the dynamics of the obtained solutions. To the best of our knowledge, this work is the first to provide exact analytical solutions for the fractional K-dV equation, and the findings hold significant theoretical and practical value in areas such as optical signal transmission, plasma wave dynamics, and ocean wave propagation.

Keywords: The Jacobi elliptic method, the extended Korteweg-de Vries(K-dV) equation
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1 Introduction

As a generalization of the integer derivative, the fractional derivative was first mentioned in a communication between Leibniz and L'Hospital in 1695. Fractional derivatives were found to depict phenomena with memory, which enriched their physical importance and attracted a lot of attention from mathematical physicists. There are many types of fractional derivatives, such as the Caputo type, the Riemann–Liouville type, the Grünwald type, and the Marchaud type. Many important and significant problems in engineering, experimental sciences, and social sciences are expressed in the form of mathematical models. These models usually lead to the determination of a function and should apply to equations that include derivatives of this unknown function. In the 18th century, the partial differential equation began with the work of mathematicians such as Euler, Lagrange, De Albert and Laplace with an analytical study of physical models. The theory of equations, which includes ordinary differential equations and differential equations with partial derivatives, is a branch of mathematical analysis that has a close relationship with physics. Physical concepts such as the movement of a vibrating string, gravity, fluid flow, heat conduction, electricity and magnetism have all led to the emergence of different differential equations. The answers of these equations correctly answer the effects of these physical phenomena. Many physical phenomena cannot be expressed with ordinary derivative models, and the observations and results obtained in these models are closer to modeling with fractional derivatives. In addition to this concept of ordinary derivative which is interpreted with speed in mechanical physics, it also takes on another aspect in fractional derivatives. This aspect can be expressed by increasing or decreasing the moving acceleration. In

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recent years, many new methods have been presented to find exact solutions of differential equations with nonlinear partial derivatives, and one of the best methods is the modified Sardar sub-equation method [4, 9]. The modified Sardar sub-equation method is an efficient and effective solution method for calculating exact wave solutions and one of the most direct and effective algebraic methods for finding exact solutions of differential equations with nonlinear partial derivatives [7, 12, 13, 14, 15]. The difference between this method and other old methods is to avoid complex and tedious calculations and more accurate and explicit conclusions to determine the isolated wave solutions with high accuracy. In recent years, this method is useful and useful for determining the exact solutions (soliton, alternating and exponential solutions) of the Schrödinger equation, Eckhaus equation and various differential equations with other non-linear partial derivatives that are widely used in various fields of science and engineering. , has been successfully used [1, 2, 3, 5, 6, 8, 10, 16]. The present paper applied the modified Sardar sub-equation method to derive traveling wave solutions for the extended fractional Korteweg-de Vries(K-dV) equation

$$D_\tau^\alpha \varphi_1 + \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - \left(C_2(1-f) + D_2 f\sigma^2\right)\right] D_\xi^{2\alpha}\varphi_1 + \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right)\left(D_\xi^\alpha \varphi_1\right)^3 + \frac{\lambda^3}{2} D_\xi^{3\alpha}\varphi_1 = 0. \tag{1.1}$$

This importance has caused many researchers to conduct many researches and studies in order to find different and effective methods, which has led to the discovery of widely used methods. From [11], we have

$$\left((1-f)C_1 + fD_1\sigma - \frac{1}{\lambda^2}\right)\varphi_2 = \left[\frac{3}{2\lambda^4} - \left((1-f)C_2 + fD_2\sigma^2\right)\right](\varphi_1)^2,$$

$$C_1 = \frac{\kappa_c - \frac{1}{2}}{\kappa_c - \frac{3}{2}}, \qquad D_1 = \frac{\kappa_h - \frac{1}{2}}{\kappa_h - \frac{3}{2}}$$

$$C_2 = \frac{(\kappa_c - \frac{1}{2})(\kappa_c + \frac{1}{2})}{2(\kappa_c - \frac{3}{2})^2}, \qquad D_2 = \frac{(\kappa_h - \frac{1}{2})(\kappa_h + \frac{1}{2})}{2(\kappa_h - \frac{3}{2})^2}$$

and

$$\frac{\partial n_1}{\partial \tau} - \lambda\frac{\partial n_3}{\partial \xi} + \frac{\partial (n_2 u_1)}{\partial \xi} + \frac{\partial (n_1 u_2)}{\partial \xi} + \frac{\partial u_3}{\partial \xi} = 0,$$

$$\frac{\partial u_1}{\partial \tau} + u_2\frac{\partial u_1}{\partial \xi} + u_1\frac{\partial u_2}{\partial \xi} - \lambda\frac{\partial u_3}{\partial \xi} + \frac{\partial \varphi_3}{\partial \xi} = 0,$$

$$\frac{\partial^2 \varphi_1}{\partial \xi^2} + \left[\frac{3}{2\lambda^4} - \left((1-f)C_2 + f\sigma^2 D_2\right)\right](\varphi_1)^2 = \frac{1}{\lambda^2}\varphi_3 + 2\left((1-f)C_2 + f\sigma^2 D_2\right)(\varphi_1\varphi_2) + \left(C_3(1-f) + f\sigma^3 D_3\right)\varphi_1^3 - n_3.$$

Now with solving (1-3) – (1-5) we have

$$\frac{\partial \varphi_1}{\partial \tau} + \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - \left(C_2(1-f) + D_2 f\sigma^2\right)\right]\frac{\partial \varphi_1^2}{\partial \xi} + \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right)\frac{\partial \varphi_1^3}{\partial \xi} + \frac{\lambda^3}{2}\frac{\partial^3 \varphi_1}{\partial \xi^3} = 0,$$

and

$$C_3 = \frac{(\kappa_c - \frac{1}{2})(\kappa_c + \frac{1}{2})(\kappa_c + \frac{3}{2})}{6(\kappa_c - \frac{3}{2})^3}, \quad D_3 = \frac{(\kappa_h - \frac{1}{2})(\kappa_h + \frac{1}{2})(\kappa_h + \frac{3}{2})}{6(\kappa_h - \frac{3}{2})^3}.$$

In the following, we present the method in question in the second part, and in the third part, we describe the application of the method along with the graphical analysis of the behavior of the answers. And in the final part, we present the results of the discussion.

2 Modified Jacobi elliptic function method

The main idea of the Jacobi function approach is expressed as the following steps 13.

At first we consider the nonlinear PDE

$$\mathbb{L}(\psi, \psi_x, \psi_t, \psi_{xx}, \ldots) = 0. \tag{2.1}$$

Using the transformations $\psi = \varphi(\xi)$ and $\xi = \sigma x - lt$, we reduce the nonlinear partial differential equation to the following ordinary differential equation:

$$\mathcal{L}(\varphi, \sigma\varphi', -l\varphi', \ldots) = 0, \tag{2.2}$$

where the values of σ and l will be found later.

Step (2-1): assume that Eq. (2.2) has the following solution:

$$\varphi(\xi) = g_0 + \sum_{j=1}^{M}\left[g_j\left(\frac{z(\xi)}{1+z^2(\xi)}\right) + f_j\left(\frac{1-z^2(\xi)}{1+z^2(\xi)}\right)\right]\left(\frac{z(\xi)}{1+z^2(\xi)}\right)^{j-1}, \tag{2.3}$$

$z(\xi)$ satisfy the following status:

$$z(\xi) = \sqrt{a + bz^2(\xi) + cz^4(\xi)}, \tag{2.4}$$

here $g_0, g_j, f_j, a, b, c\,(j = 1, 2, ..., N)$ are constants that need to be evaluated.

Step (2-2): the positive integer Mcan be considered by balancing between the non-linear terms and highest order derivatives in Eq. (2.2). Consequently by Substituting with the proposed solution (2.4) with Eq. (2.3) into Eq. (2.2), one can get a polynomial of $z(\xi)$.

Step (2-3): Equating all coefficients of $z^j$of the resulted polynomial in step (2-2) to be zero gives a system of algebraic equations that can be handled by the aid of software program like mathematica or maple to identifying the unknown parameters such as $g_0, g_j, f_j\,(j = 1, 2, ..., N)$.

Step (2-4): General solutions for Eq. (2.4) can be determined as follows:

If $a = 1, b = -1 - m^2, c = m^2$the general solution of (2.4) is$sn(\xi, m)$.

If $a = 1 - m^2, b = 2m^2 - 1, c = -m^2$the general solution of (2-4) is$cn(\xi, m)$.

If $a = 1 - m^2, b = 2m^2 - 1, c = -m^2$the general solution of (2-4) is$cn(\xi, m)$.

If $a = -m^2, b = 2m^2 - 1, c = 1 - m^2$the general solution of (2-4) is$nc(\xi, m)$.

If $a = \frac{1}{4}, b = \frac{1-2m^2}{2}, c = \frac{1}{4}$the general solution of (2-4) is$ns(\xi, m) \pm cs(\xi, m)$.

If $a = \frac{1-m^2}{4}, b = \frac{1+m^2}{2}, c = \frac{1-m^2}{4}$,the general solution of (2-4) is$nc(\xi, m) \pm sc(\xi, m)$.

By substituting with the determined parameters $g_0, g_j, f_j\,(j = 1, 2, ..., N)\,, c$with the general solutions of Eq. (2.4) into the solution (2.3), one can get various exact travelling wave solutions for the differential equations.

3 Mathematical Analysis

The main goal is to introduce equation (1.1) using the agreement derivatives as follows[16]

$$D_\tau^\alpha \varphi_1 + \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - \left(C_2(1-f) + D_2 f\sigma^2\right)\right] D_\xi^{2\alpha}\varphi_1 + \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right)\left(D_\xi^\alpha \varphi_1\right)^3 + \frac{\lambda^3}{2} D_\xi^{3\alpha}\varphi_1 = 0, \tag{3.1}$$

We consider

$$\varphi_1(\xi, \tau) = U(\chi), \qquad \chi = \left(\frac{1}{\alpha}\right)\xi^\alpha - V\left(\frac{1}{\alpha}\right)\tau^\alpha \tag{3.2}$$

so

$$-VU + \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - \left(C_2(1-f) + D_2 f\sigma^2\right)\right] U^2 + \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right) U^3 + \frac{\lambda^3}{2} U'' = 0. \tag{3.3}$$

We consider

$$-VU + s_1 U^2 + s_2 U^3 + \frac{\lambda^3}{2} U'' = 0$$

where

$$s_1 = \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - \left(C_2(1-f) + D_2 f\sigma^2\right)\right]$$
$$s_2 = \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right).$$

In (3-2) by HBM between $\varphi_1^3$and $d^2\varphi_1/d\chi^2$ we have $M=1$. So solution takes the following form

$$U(\chi)=g_0+g_1\left(\frac{z(\chi)}{1+z^2(\chi)}\right)+f_1\left(\frac{1-z^2(\chi)}{1+z^2(\chi)}\right), \tag{3.4}$$

Inserting equation (3.7) into (3.6) and equating all coefficients of z^j to be zero gives a system of algebraic equations that can be handled by the aid of software program like maple to identifying the unknown parameters as following cases:

Family 3-1: If we consider $a=1, b=-1-m^2, c=m^2$and substituting in algebraic equation we obtain following cases of solutions

Case 3-1-1:

$$\begin{array}{l} g_0=0, g_1=\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}, f_1=\frac{(-v_2{s_2}^2)^{(1/3)}}{2s_2}, \\ V=\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}, m=0. \end{array} \tag{3.5}$$

where

$$s_1=\frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4}-\left(C_2(1-f)+D_2 f\sigma^2\right)\right], s_2=\frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6}-C_3(1-f)-f\sigma^3 D_3\right)$$

In this case $sn(\chi,0)$ is $\sin(\chi)$. So Eq. (3.1) has the following solution (see Image 1):

$$\varphi_1(\xi,\tau)=\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{\sin\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\sin^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right)+$$
$$\frac{(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{1-\sin^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\sin^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right),$$

Case 3-1-2:

$$\begin{array}{l} g_0=\frac{i\sqrt{3}}{2s_2}, g_1=-\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}, f_1=\frac{(v_2{s_2}^2)^{(1/3)}}{2s_2}, \\ V=\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}, m=1. \end{array} \tag{3.6}$$

where

$$s_1=\frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4}-\left(C_2(1-f)+D_2 f\sigma^2\right)\right], s_2=\frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6}-C_3(1-f)-f\sigma^3 D_3\right).$$

In this case $sn(\chi,1)$ is $\tanh(\chi)$. So Eq. (3.1) has the following solution:

$$\varphi_1(\xi,\tau)=\frac{i\sqrt{3}}{2s_2}-\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{\tanh\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\tanh^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right)+$$
$$\frac{(v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{1-\tanh^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\tanh^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right),$$

Family 3-2: If we consider $a=1-m^2, b=2m^2-1, c=-m^2$and substituting in algebraic equation we obtain following cases of solutions

Case3-2-1:

$$\begin{array}{l} g_0=\frac{(-v_2{s_2}^2)}{2s_2}, g_1=\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}, f_1=0, \\ V=\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}}, m=0. \end{array} \tag{3.7}$$

where

$$s_1=\frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4}-\left(C_2(1-f)+D_2 f\sigma^2\right)\right], s_2=\frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6}-C_3(1-f)-f\sigma^3 D_3\right)$$

In this case $cn(\chi,0)$ is $\cos(\chi)$. So Eq. (3.1) has the following solution:

$$\varphi_1(\xi,\tau)=\frac{(-v_2{s_2}^2)}{2s_2}+\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{\cos\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\cos^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right),$$

Case 3-2-2:

$$g_0 = \frac{i\sqrt{3}}{2s_2}, g_1 = 0, f_1 = \frac{(v_2 {s_2}^2)^{(1/3)}}{2s_2}, V = \frac{\lambda^3 v_1 s_2}{(i\sqrt{3}+1)}, m = 1.$$

In this case $sn\,(\chi, 1)$ is $\sec h\,(\chi)$. So Eq. (3.1) has the following solution:

$$\varphi_1\,(\xi,\tau) = \frac{i\sqrt{3}}{2s_2} + \frac{(v_2 {s_2}^2)^{(1/3)}}{2s_2}\left(\frac{1-\sec h^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\sec h^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right),$$

Family 3-3: If we consider $a = -m^2, b = 2m^2 - 1, c = 1 - m^2$ and substituting in algebraic equation we obtain following cases of solutions

Case 3-3-1:

$$g_0 = -\frac{i\sqrt{3}}{2s_2}, g_1 = \frac{(-v_2 {s_2}^2)^{(1/3)}}{2s_2}, f_1 = \frac{(v_2 {s_2}^2)^{(1/3)}}{2s_2}, \quad V = \frac{\lambda^3 v_1 s_2}{(-v_2 {s_2}^2)^{(1/3)}}, m = 0. \tag{3.8}$$

In this case, $nc\,(\chi, 0)$ is deduced to $\sec h\,(\chi)$. So Eq. (3.1) has the following solution:

$$\varphi_1\,(\xi,\tau) = -\frac{i\sqrt{3}}{2s_2} + \frac{(-v_2 {s_2}^2)^{(1/3)}}{2s_2}\left(\frac{\sec h\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(-v_2 {s_2}^2)^{(1/3)}}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\sec h^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(-v_2 {s_2}^2)^{(1/3)}}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right) + \frac{(v_2 {s_2}^2)^{(1/3)}}{2s_2}\left(\frac{1-\sec h^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(-v_2 {s_2}^2)^{(1/3)}}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\sec h^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(-v_2 {s_2}^2)^{(1/3)}}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right),$$

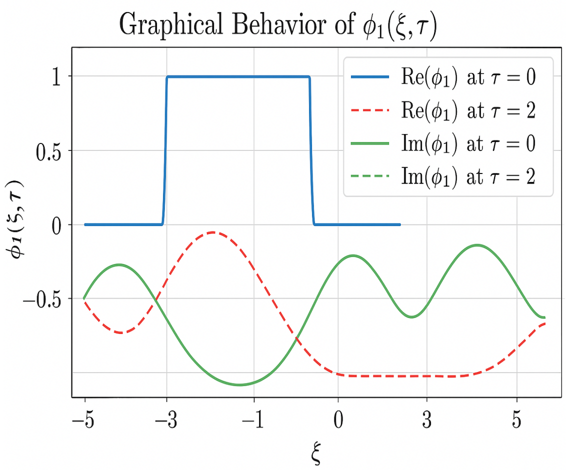
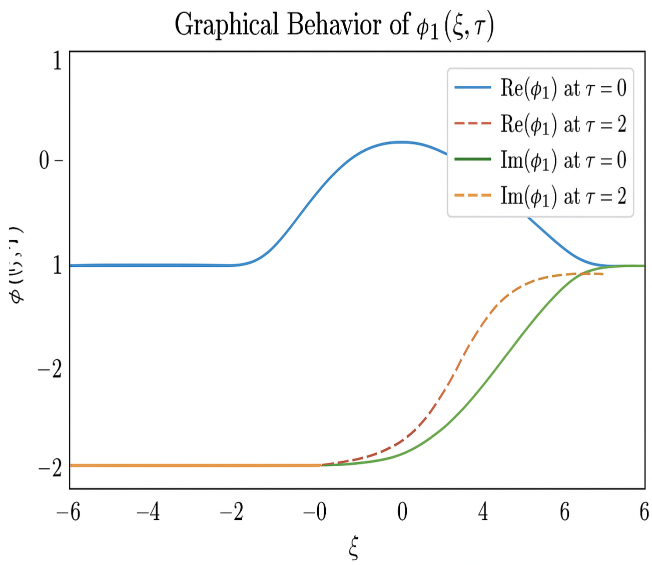


Fig.1 Graphical RevolutionPlot3D representations related to solution φ_1 - case 3-1-1.



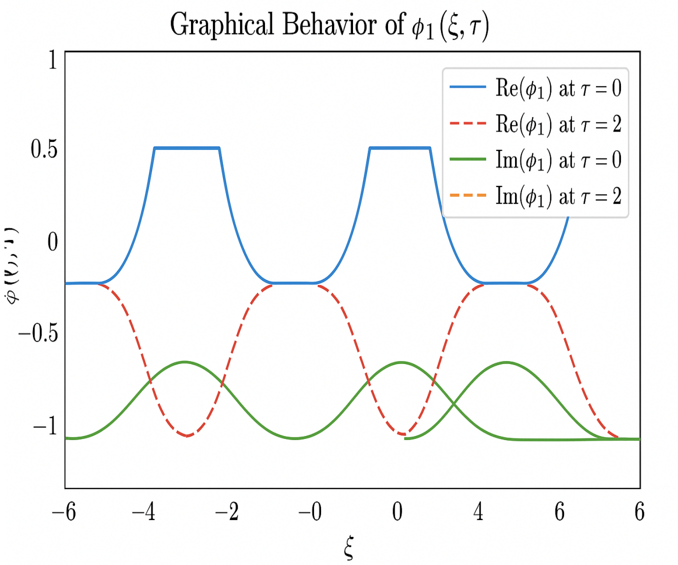


Fig.2 Graphical RevolutionPlot3D representations related to solution φ_1 - case 3-2-1 and 3-3-1 respectively.

Case 3-3-2:

$$\begin{aligned} &g_0 = 0, g_1 = -\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}, f_1 = \frac{1}{2}s_2, \\ &V = \frac{\lambda^3 v_1 s_2}{(i\sqrt{3}+1)}, m = 1. \end{aligned} \tag{3.9}$$

In this case, $nc\,(\chi, 1)$ is deduced to $\cosh(\chi)$. So Eq. (3.1) has the following solution:

$$\varphi_1(\xi,\tau) = -\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{\cosh\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\cosh^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right) + \frac{1}{2}s_2\left(\frac{1-\cosh^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}{1+\cosh^2\left(\left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha\right)}\right),$$

Family 3-4: If we consider $a = \frac{1}{4}, b = \frac{1-2m^2}{2}, c = \frac{1}{4}$ and substituting in algebraic equation we obtain following cases of solutions

$$\begin{aligned} &g_0 = \frac{i\sqrt{3}}{2s_2}, g_1 = 0, f_1 = \frac{(-v_2{s_2}^2)^{(1/3)}}{2s_2}, \\ &V = \frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}, m = 1. \end{aligned} \tag{3.10}$$

In this case, $ns\,(\chi, 1) + cs\,(\chi, 1)$ is deduced to $\coth(\chi) + \operatorname{csc}h\,(\chi)$. So Eq. (3.1) has the following solution:

$$\varphi_1(\xi,\tau) = \frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{\coth(\chi)+\operatorname{csc}h(\chi)}{1+(\coth(\chi)+\operatorname{csc}h(\chi))^2}\right) + \frac{(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{1-(\coth(\chi)+\operatorname{csc}h(\chi))^2}{1+(\coth(\chi)+\operatorname{csc}h(\chi))^2}\right),$$

and $\chi = \left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1 s_2}{(-v_2{s_2}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha$.

Family 3-5: If we consider $a = \frac{1-m^2}{4}, b = \frac{1+m^2}{2}, c = \frac{1-m^2}{4}$, and substituting in algebraic equation we obtain following solution

$$\begin{aligned} &g_0 = \frac{i\sqrt{3}}{2s_2}, g_1 = -\frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}, f_1 = \frac{(v_2{s_2}^2)^{(1/3)}}{2s_2}, \\ &V = \frac{\lambda^3 v_1}{-v_2 s_2}, m = 1. \end{aligned} \tag{3.11}$$

In this case $nc\,(\chi, 1) + sc\,(\chi, 1)$ is$\cosh(\chi) + \sinh(\chi)$. So Eq. (3.1) has the following solution:

$$\varphi_1(\xi,\tau) = \frac{i\sqrt{3}(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{\cosh(\chi)+\sinh(\chi)}{1+(\cosh(\chi)+\sinh(\chi))^2}\right) + \frac{(-v_2{s_2}^2)^{(1/3)}}{2s_2}\left(\frac{1-(\cosh(\chi)+\sinh(\chi))^2}{1+(\cosh(\chi)+\sinh(\chi))^2}\right),$$

and $\chi = \left(\frac{1}{\alpha}\right)\xi^\alpha - \frac{\lambda^3 v_1}{-v_2 s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha$.

4 Conclusion

In this work, employing the definitions of fractional derivatives, we aimed to derive new analytical solutions for the KdV equation through the extended Sardar–Sub equation method. This approach enabled us to obtain a new class of solutions supported by graphical analysis. One of the notable strengths of this method is its broad applicability, as it can be effectively utilized for a wide variety of equations and can generate multiple types of solutions.

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