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On tetranacci functions with tetranacci numbers

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Abstract

In this research work, we define a tetranacci function $\zeta : \mathbb{R} \to \mathbb{R}$ satisfying

$$\zeta(4+\omega) = \zeta(3+\omega) + \zeta(2+\omega) + \zeta(1+\omega) + \zeta(\omega),$$

for all $\omega \in \mathbb{R}$. We apply induction technique to obtain useful results for tetranacci functions with tetranacci numbers and also prove that $\lim\limits_{\omega\to\infty} \frac{\zeta(\omega+1)}{\zeta(\omega)} = \delta > 1$, where δ is one of the zeros of the equation $\omega^4 - \omega^3 - \omega^2 - \omega - 1 = 0$.

Keywords: Tetranacci number, Tetranacci function, Quotient of tetranacci functions, Induction method
2020 MSC: 39A10, 11B39

1 Introduction

In 13th century, Itlaian Leonardo Pisano created Fibonacci numbers [1, 3], which were gotten by sum of two prior values starting with zero and one. Fibonacci numbers and Fibonacci sequences and related results have been published in [2, 5, 6, 8, 9, 10, 11]. Particularly, Han et al. [7] proved the useful results for Fibonacci function by applying hypothesis of $f-$ odd and $f-$ even functions. In 19th century, Feinberg [4] innovated tribonacci numbers, which were gotten by sum of three prior values starting with zero, zero and one. Parizi and Eshaghi Gordji [12] developed the notation of tribonacci functions for the tribonacci numbers.

Now, we study tetranacci numbers and define tetranacci functions on real numbers. We apply induction technique to obtain useful results for tetranacci functions with tetranacci numbers and also prove that $\lim\limits_{\omega\to\infty} \frac{\zeta(\omega+1)}{\zeta(\omega)} = \delta > 1$, where δ is one of the zeros of the equation $\omega^4 - \omega^3 - \omega^2 - \omega - 1 = 0$.

2 Tetranacci numbers and tetranacci functions

Definition 2.1. Tetranacci numbers are obtained by adding four preceding terms starting with g_1, g_2, g_3, g_4, where g_1, g_2, g_3, g_4 are arbitrary non-negative integers.

Example 2.2. Let us take the values of $g_1 = g_2 = 0$ and $g_3 = g_4 = 1$. The first few terms of tetranacci numbers are $0, 0, 1, 1, 2, 4, 8, 15, 29, 56, 108, \cdots$.

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Definition 2.3. If $\zeta : \mathbb{R} \to \mathbb{R}$ is a function

$$\zeta(4+\omega) = \zeta(3+\omega) + \zeta(2+\omega) + \zeta(1+\omega) + \zeta(\omega)$$

for all $\omega \in \mathbb{R}$, then ζ is called a tetranacci function.

Example 2.4. Let $\zeta(\omega) = b^{\omega}$ be a function on real numbers and b be a positive real number. Then

$$b^{4+\omega} = \zeta(4+\omega) = \zeta(3+\omega) + \zeta(2+\omega) + \zeta(1+\omega) + \zeta(\omega) = b^{\omega}(b^3 + b^2 + b + 1).$$

Since b is positive, we obtain $b^4 - b^3 - b^2 - b - 1 = 0$ and this equation has a solution, say, δ. That is, $b = \delta > 1$. Thus $\zeta(\omega) = \delta^{\omega}$ is a tetranacci function.

Example 2.5. Let $\{p_n\}_{-\infty}^{\infty}, \{q_n\}_{-\infty}^{\infty}, \{r_n\}_{-\infty}^{\infty}$ and $\{s_n\}_{-\infty}^{\infty}$ be tetranacci sequences, and define $\zeta(\omega) = s_{[\omega]}l^3 + r_{[\omega]}l^2 + q_{[\omega]}l + p_{[\omega]}$, where $l = \omega - [\omega], 0 < l < 1$. Then

$$\begin{aligned}
\zeta(\omega+4) =& s_{[\omega]+4}l^3 + q_{[\omega]+4}l + r_{[\omega]+4}l^2 + p_{[\omega]+4} \\
=& (p_{[\omega]+3} + p_{[\omega]+2} + p_{[\omega]+1} + p_{[\omega]}) + (q_{[\omega]+3} + q_{[\omega]+2} + q_{[\omega]+1} + q_{[\omega]})l \\
& + (r_{[\omega]+3} + r_{[\omega]+2} + r_{[\omega]+1} + r_{[\omega]})l^2 + (s_{[\omega]+3} + s_{[\omega]+2} + s_{[\omega]+1} + s_{[\omega]})l^3 \\
=& (p_{[\omega]+3} + q_{[\omega]+3}l + r_{[\omega]+3}l^2 + s_{[\omega]+3}l^3) + (p_{[\omega]+2} + q_{[\omega]+2}l + r_{[\omega]+2}l^2 + s_{[\omega]+2}l^3) \\
& + (p_{[\omega]+1} + q_{[\omega]+1}l + r_{[\omega]+1}l^2 + s_{[\omega]+1}l^3) + (p_{[\omega]} + q_{[\omega]}l + r_{[\omega]}l^2 + s_{[\omega]}l^3) \\
=& \zeta(\omega+3) + \zeta(\omega+2) + \zeta(\omega+1) + \zeta(\omega)
\end{aligned}$$

for all $\omega \in \mathbb{R}$. Hence ζ is a tetranacci function.

Example 2.6. If a tetranacci function ζ is differentiable on $\mathbb{R}$, then

$$\begin{aligned}
\zeta(4+\omega) &= \zeta(3+\omega) + \zeta(2+\omega) + \zeta(1+\omega) + \zeta(\omega), \\
\zeta'(4+\omega) &= \zeta'(3+\omega) + \zeta'(2+\omega) + \zeta'(1+\omega) + \zeta'(\omega)
\end{aligned}$$

for all $\omega \in \mathbb{R}$. Thus ζ' is a tetranacci function.

Example 2.7. If ζ is a tetranacci function and we define $\rho_1(\omega) = \zeta(l^3 + l^2 + l + \omega)$, where $l, \omega \in \mathbb{R}$, then

$$\begin{aligned}
\rho_1(\omega+4) = \zeta(l^3 + l^2 + l + \omega + 4) &= \zeta(l^2 + 3 + l + l^3 + \omega) + \zeta(l^2 + l + l^3 + 2 + \omega) \\
&\quad + \zeta(\omega + l^2 + l + l^3 + 1)\zeta(\omega + l^2 + l + l^3) \\
&= \rho_1(3+\omega) + \rho_1(2+\omega) + \rho_1(1+\omega) + \rho_1(\omega)
\end{aligned}$$

for all $\omega \in \mathbb{R}$. Thus ρ_1 is a tetranacci function.

Example 2.8. If $\zeta(\omega) = \delta^{\omega}$ is a tetranacci function and we define $\rho_2(\omega) = (\delta)^{l+l^2+l^3+\omega}$, where $l, \omega \in \mathbb{R}$, then

$$\rho_2(\omega) = (\delta)^{l+l^2+l^3+\omega} = (\delta)^{\omega}(\delta)^{l+l^2+l^3} = (\delta)^{l+l^2+l^3}\zeta(\omega)$$

for all $\omega \in \mathbb{R}$. Thus ρ_2 is a tetranacci function.

3 Tetranacci functions with tetranacci numbers

Theorem 3.1. Let $\zeta(\omega)$ be a tetranacci function and $\{\zeta_k\}, \{\zeta_k'\}, \{\zeta_k''\}$ and $\{\zeta_k'''\}$ be sequences of tetranacci numbers with $\zeta_0 = 0$, $\zeta_1 = 1$, $\zeta_2 = 2$, $\zeta_3 = 4$, $\zeta_4 = 7$ and $\zeta_0' = \zeta_1' = \zeta_2' = 0, \zeta_3' = \zeta_4' = 1$ and $\zeta_0'' = 0, \zeta_1'' = 1, \zeta_2'' = \zeta_3'' = 0, \zeta_4'' = 1$ and $\zeta_0''' = 1, \zeta_1''' = \zeta_2''' = \zeta_3''' = 0, \zeta_4''' = 1$. Then

$$\zeta(\omega+k) = \zeta_k'\zeta(\omega+3) + \zeta_{k-3}\zeta(\omega+2) + \zeta_k''\zeta(\omega+1) + \zeta_k'''\zeta(\omega)$$

for all $\omega \in \mathbb{R}$, where $k \geq 4$ is an integer such that $\zeta_k = \zeta_{k+2}' + \zeta_{k+1}' + \zeta_k'$, $\zeta_k'' = \zeta_{k-2}' + \zeta_{k-1}'$ and $\zeta_k''' = \zeta_{k-1}'$.

Proof . For $k=4$, we obtain

$$\begin{aligned}\zeta(4+\omega) &= \zeta(3+\omega)+\zeta(2+\omega)+\zeta(1+\omega)+\zeta(\omega)\\ &= \zeta_4'\zeta(3+\omega)+\zeta_1\zeta(2+\omega)+\zeta_4''\zeta(1+\omega)+\zeta_4'''\zeta(\omega).\end{aligned}$$

For $k=5$, we obtain

$$\begin{aligned}\zeta(5+\omega) =&\zeta(4+\omega)+\zeta(3+\omega)+\zeta(2+\omega)+\zeta(1+\omega)\\ =&\zeta_4'\zeta(3+\omega)+\zeta_1\zeta(2+\omega)+\zeta_4''\zeta(1+\omega)+\zeta_4'''\zeta(\omega)+1\zeta(3+\omega)\\ &+0\zeta(2+\omega)+0\zeta(1+\omega)+0\zeta(\omega)+0\zeta(3+\omega)+1\zeta(2+\omega)+0\zeta(1+\omega)\\ &+0\zeta(\omega)+0\zeta(3+\omega)+0\zeta(2+\omega)+1\zeta(1+\omega)+0\zeta(\omega)\\ =&(\zeta_4'+\zeta_3'+\zeta_2'+\zeta_1')\zeta(\omega+3)+(\zeta_1+\zeta_0+\zeta_{-1}+\zeta_{-2})\zeta(\omega+2)\\ &+(\zeta_4''+\zeta_3''+\zeta_2''+\zeta_1'')\zeta(\omega+1)+(\zeta_4'''+\zeta_3'''+\zeta_2'''+\zeta_1''')\zeta(\omega)\\ =&\zeta_5'\zeta(\omega+3)+\zeta_2\zeta(\omega+2)+\zeta_5''\zeta(\omega+1)+\zeta_5'''\zeta(\omega).\end{aligned}$$

For $k=6$, we have

$$\begin{aligned}\zeta(6+\omega) =&\zeta(5+\omega)+\zeta(4+\omega)+\zeta(3+\omega)+\zeta(2+\omega)\\ =&\zeta_5'\zeta(\omega+3)+\zeta_2\zeta(\omega+2)+\zeta_5''\zeta(\omega+1)+\zeta_5'''\zeta(\omega)+\zeta_4'\zeta(\omega+3)+\zeta_1\zeta(\omega+2)\\ &+\zeta_4''\zeta(\omega+1)+\zeta_4'''\zeta(\omega)+\zeta_3'\zeta(\omega+3)+\zeta_0\zeta(\omega+2)+\zeta_3''\zeta(\omega+1)+\zeta_3'''\zeta(\omega)\\ &+\zeta_2'\zeta(\omega+3)+\zeta_{-1}\zeta(\omega+2)+\zeta_2''\zeta(\omega+1)+\zeta_2'''\zeta(\omega)\\ =&(\zeta_5'+\zeta_4'+\zeta_3'+\zeta_2')\zeta(\omega+3)+(\zeta_2+\zeta_1+\zeta_0+\zeta_{-1})\zeta(\omega+2)\\ &+(\zeta_5''+\zeta_4''+\zeta_3''+\zeta_2'')\zeta(\omega+1)+(\zeta_5'''+\zeta_4'''+\zeta_3'''+\zeta_2''')\zeta(\omega)\\ =&\zeta_6'\zeta(\omega+3)+\zeta_3\zeta(\omega+2)+\zeta_6''\zeta(\omega+1)+\zeta_6'''\zeta(\omega).\end{aligned}$$

For $k=7$, we have

$$\begin{aligned}\zeta(7+\omega) =&\zeta(6+\omega)+\zeta(5+\omega)+\zeta(4+\omega)+\zeta(3+\omega)\\ =&\zeta_6'\zeta(\omega+3)+\zeta_3\zeta(\omega+2)+\zeta_6''\zeta(\omega+1)+\zeta_6'''\zeta(\omega)+\zeta_5'\zeta(\omega+3)+\zeta_2\zeta(\omega+2)\\ &+\zeta_5''\zeta(\omega+1)+\zeta_5'''\zeta(\omega)+\zeta_4'\zeta(\omega+3)+\zeta_1\zeta(\omega+2)+\zeta_4''\zeta(\omega+1)+\zeta_4'''\zeta(\omega)\\ &+\zeta_3'\zeta(\omega+3)+\zeta_0\zeta(\omega+2)+\zeta_3''\zeta(\omega+1)+\zeta_3'''\zeta(\omega)\\ =&(\zeta_6'+\zeta_5'+\zeta_4'+\zeta_3')\zeta(\omega+3)+(\zeta_3+\zeta_2+\zeta_1+\zeta_0)\zeta(\omega+2)\\ &+(\zeta_6'+\zeta_5''+\zeta_4''+\zeta_3''+)\zeta(\omega+1)+(\zeta_6''+\zeta_5'''+\zeta_4'''+\zeta_3''')\zeta(\omega)\\ =&\zeta_7'\zeta(\omega+3)+\zeta_4\zeta(\omega+2)+\zeta_7''\zeta(\omega+1)+\zeta_7'''\zeta(\omega).\end{aligned}$$

By induction hypothesis, assume that the outcome is true for $k+3$. Then we will show that the outcome is true for $k+4$. We have

$$\begin{aligned}\zeta(\omega+k+4) =&\zeta(\omega+k+3)+\zeta(\omega+k+2)+\zeta(\omega+k+1)+\zeta(k+\omega)\\ =&\zeta_{k+3}'\zeta(\omega+3)+\zeta_k\zeta(\omega+2)+\zeta_{k+3}''\zeta(\omega+1)+\zeta_{k+3}'''\zeta(\omega)+\zeta_{k+2}'\zeta(\omega+3)\\ &+\zeta_{k-1}\zeta(\omega+2)+\zeta_{k+2}''\zeta(\omega+1)+\zeta_{k+2}'''\zeta(\omega)\\ &+\zeta_{k+1}'\zeta(\omega+3)+\zeta_{k-2}\zeta(\omega+2)+\zeta_{k+1}''\zeta(\omega+1)+\zeta_{k+1}'''\zeta(\omega)\\ &+\zeta_k'\zeta(\omega+3)+\zeta_{k-3}\zeta(\omega+2)+\zeta_k''\zeta(\omega+1)+\zeta_k'''\zeta(\omega)\\ =&(\zeta_{k+3}'+\zeta_{k+2}'+\zeta_{k+1}'+\zeta_k')\zeta(\omega+3)+(\zeta_k+\zeta_{k-1}+\zeta_{k-2}+\zeta_{k-3})\zeta(\omega+2)\\ &+(\zeta_{k+3}''+\zeta_{k+2}''+\zeta_{k+1}''+\zeta_k'')\zeta(\omega+1)+(\zeta_{k+3}'''+\zeta_{k+2}'''+\zeta_{k+1}'''+\zeta_k''')\zeta(\omega)\\ =&\zeta_{k+4}'\zeta(\omega+3)+\zeta_{k+1}\zeta(\omega+2)+\zeta_{k+4}''\zeta(\omega+1)+\zeta_{k+4}'''\zeta(\omega).\end{aligned}$$

This completes the proof. □

Corollary 3.2. If $\{\zeta_k'\}$ is a sequence of tetranacci numbers with $\zeta_1'=\zeta_2'=0$ and $\zeta_3'=\zeta_4'=1$, then

$$\delta^k=\zeta_k'\delta^3+(\zeta_{k-3}'+\zeta_{k-2}'+\zeta_{k-1}')\delta^2+(\zeta_{k-2}'+\zeta_{k-1}')\delta+\zeta_{k-1}', \tag{3.1}$$

where $\delta>1$ is a solution of the equation $\omega^4-\omega^3-\omega^2-\omega-1=0$.

Proof . Since $\zeta(\omega)=\delta^{\omega}$ is a tetranacci function, by Theorem 3.1,

$$\begin{aligned}\zeta(\omega+k) &= \zeta'_k\zeta(\omega+3)+\zeta_{k-3}\zeta(\omega+2)+\zeta''_k\zeta(\omega+1)+\zeta'''_k\zeta(\omega)\\ \delta^{\omega+k} &= \zeta'_k\delta^{\omega+3}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})\delta^{\omega+2}+(\zeta'_{k-1}+\zeta'_{k-2})\delta^{\omega+1}+\zeta'_{k-1}\delta^{\omega}.\end{aligned} \tag{3.2}$$

Since δ is greater than one, (3.2) implies (3.1). □

Theorem 3.3. Let $\{p_k\}$ be a tetranacci sequence. Then

$$P_{[\omega+k]}=\zeta'_kP_{[\omega+3]}+(\zeta'_{k-1}+\zeta'_{k-2}+\zeta'_{k-3})P_{[\omega+2]}+(\zeta'_{k-1}+\zeta'_{k-2})P_{[\omega+1]}+\zeta'_{k-1}P_{[\omega]},$$

$$\begin{aligned}P_{[\omega+k]-1}&=\zeta'_kP_{[\omega+2]}+(\zeta'_{k-1}+\zeta'_{k-2}+\zeta'_{k-3})P_{[\omega+1]}+(\zeta'_{k-1}+\zeta'_{k-2})P_{[\omega]}+\zeta'_{k-1}P_{[\omega]-1},\\ P_{[\omega+k]-2}&=\zeta'_kP_{[\omega+1]}+(\zeta'_{k-1}+\zeta'_{k-2}+\zeta'_{k-3}))P_{[\omega]}+(\zeta'_{k-1}+\zeta'_{k-2})P_{[\omega-1]}+\zeta'_{k-1}P_{[\omega]-2},\\ P_{[\omega+k]-3}&=\zeta'_kP_{[\omega]}+(\zeta'_{k-1}+\zeta'_{k-2}+\zeta'_{k-3})P_{[\omega-1]}+(\zeta'_{k-1}+\zeta'_{k-2})P_{[\omega-2]}+\zeta'_{k-1}P_{[\omega]-3}.\end{aligned}$$

Proof . We know that $\zeta(\omega)=P_{[\omega]-3}l^3+P_{[\omega]-2}l^2+P_{[\omega]-1}l+P_{[\omega]}$ is a tetranacci function. By Theorem 3.1, we have

$$\begin{aligned}&P_{-1+[k+\omega]}l+P_{-2+[k+\omega]}l^2+P_{-3+[k+\omega]}l^3+P_{[k+\omega]}=\zeta(\omega+k)\\ =&\zeta'_k\zeta(\omega+3)+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})\zeta(\omega+2)+(\zeta'_{k-1}+\zeta'_{k-2})\zeta(\omega+1)+\zeta'_{k-1}\zeta(\omega)\\ =&(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})(P_{[2+\omega]}+P_{-1+[2+\omega]}l+P_{-2+[2+\omega]}l^2+P_{-3+[2+\omega]}l^3)\\ &+(\zeta'_{k-1}+\zeta'_{k-2})(P_{[1+\omega]}+P_{-1+[1+\omega]}l+P_{-2+[1+\omega]}l^2+P_{-3+[1+\omega]}l^3)\\ &+\zeta'_k(P_{[\omega+3]}+P_{[\omega+3]-1}l+P_{[\omega+3]-2}l^2+P_{[\omega+3]-3}l^3)+\zeta'_{k-1}(P_{[\omega]}+P_{-1+[\omega]}l+P_{-2+[\omega]}l^2+P_{-3+[\omega]}l^3)\\ =&\zeta'_{k-1}P_{[\omega]}+\zeta'_kP_{[3+\omega]}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})P_{[2+\omega]}+(\zeta'_{k-1}+\zeta'_{k-2})P_{[1+\omega]}\\ &+(\zeta'_kP_{[\omega+2]}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})P_{[\omega+1]}+(\zeta'_{k-1}+\zeta'_{k-2})P_{[\omega]}+\zeta'_{k-1}P_{[\omega]-1})l\\ &+(\zeta'_kP_{[\omega+1]}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})P_{[\omega]}+(\zeta'_{k-1}+\zeta'_{k-2})P_{[\omega-1]}+\zeta'_{k-1}P_{[\omega]-2})l^2\\ &+(\zeta'_kP_{[\omega]}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})P_{[\omega-1]}+(\zeta'_{k-1}+\zeta'_{k-2})P_{[\omega-2]}+\zeta'_{k-1}P_{[\omega]-3})l^3.\end{aligned}$$

This completes the proof. □

Corollary 3.4. If $k\geq 4$, then

$$\zeta_{[\omega+k]}=\zeta'_k\zeta_{[\omega]+3}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})\zeta_{[\omega]+2}+(\zeta'_{k-1}+\zeta'_{k-2})\zeta_{[\omega]+1}+\zeta'_{k-1}\zeta_{[\omega]}, \tag{3.3}$$

$$\zeta_{[\omega+k]-1}=\zeta'_k\zeta_{[\omega]+2}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})\zeta_{[\omega]+1}+(\zeta'_{k-1}+\zeta'_{k-2})\zeta_{[\omega]}+\zeta'_{k-1}\zeta_{[\omega]-1}, \tag{3.4}$$

$$\zeta_{[\omega+k]-2}=\zeta'_k\zeta_{[\omega]+1}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})\zeta_{[\omega]}+(\zeta'_{k-1}+\zeta'_{k-2})\zeta_{[\omega]-1}+\zeta'_{k-1}\zeta_{[\omega]-2}, \tag{3.5}$$

$$\zeta_{[\omega+k]-3}=\zeta'_k\zeta_{[\omega]}+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})\zeta_{[\omega]-1}+(\zeta'_{k-1}+\zeta'_{k-2})\zeta_{[\omega]-2}+\zeta'_{k-1}\zeta_{[\omega]-3}. \tag{3.6}$$

Corollary 3.5.

$$\zeta_{[k+4]}=\zeta'_k\zeta_7+(\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})\zeta_6+(\zeta'_{k-1}+\zeta'_{k-2})\zeta_5+\zeta'_{k-1}\zeta_4 \tag{3.7}$$

Proof . Putting $\omega=4$ in (3.3) or $\omega=5$ in (3.4) or $\omega=6$ in (3.5) or $\omega=7$ in (3.6), we get the desired results. □

4 Quotients of tetranacci functions

Theorem 4.1. If $\zeta(\omega)$ is a tetranacci function and $\frac{\zeta(\omega+1)}{\zeta(\omega)}$ is a quotient of $\zeta(\omega)$, then $\lim\limits_{\omega\to\infty}\frac{\zeta(\omega+1)}{\zeta(\omega)}$ exists.

Proof . Let us take $\alpha_1=\zeta(\omega),\alpha_2=\zeta(\omega+1),\alpha_3=\zeta(\omega+2)$ and $\alpha_4=\zeta(\omega+3)$, which are all positive.

Case (i) Since ω' is any real number, it may be rewritten as $\omega' = \omega + k$, where k is an integer.

$$\begin{aligned}
\lim_{\omega\to\infty} \frac{\zeta(\omega'+1)}{\zeta(\omega')} &= \lim_{k\to\infty} \frac{\zeta(\omega+k+1)}{\zeta(\omega+k)} \\
&= \lim_{k\to\infty} \frac{\zeta'_{k+1}\alpha_4 + (\zeta'_{k-2}+\zeta'_{k-1}+\zeta'_k)\alpha_3 + (\zeta'_{k-1}+\zeta'_k)\alpha_2 + \zeta'_k\alpha_1}{\zeta'_k\alpha_4 + (\zeta'_{k-3}+\zeta'_{k-2}+\zeta'_{k-1})\alpha_3 + (\zeta'_{k-2}+\zeta'_{k-1})\alpha_2 + \zeta'_{k-1}\alpha_1} \\
&= \lim_{k\to\infty} \frac{\zeta'_k}{\zeta'_{k-1}} \frac{\frac{\zeta'_{k+1}}{\zeta'_k}\alpha_4 + \left(\frac{\zeta'_{k-2}+\zeta'_{k-1}}{\zeta'_k}+1\right)\alpha_3 + \left(\frac{\zeta'_{k-1}}{\zeta'_k}+1\right)\alpha_2 + \alpha_1}{\frac{\zeta'_k}{\zeta'_{k-1}}\alpha_4 + \left(\frac{\zeta'_{k-3}+\zeta'_{k-2}}{\zeta'_{k-1}}+1\right)\alpha_3 + \left(\frac{\zeta'_{k-2}}{\zeta'_{k-1}}+1\right)\alpha_2 + \alpha_1} \\
&=\delta \left[\frac{\delta\alpha_4 + \left(\frac{1}{\delta}+\frac{1}{\delta}+1\right)\alpha_3 + \left(\frac{1}{\delta}+1\right)\alpha_2 + \alpha_1}{\delta\alpha_4 + \left(\frac{1}{\delta}+\frac{1}{\delta}+1\right)\alpha_3 + \left(\frac{1}{\delta}+1\right)\alpha_2 + \alpha_1}\right] = \delta,
\end{aligned}$$

where $\delta = \lim\limits_{k\to\infty} \frac{\zeta'_{k+1}}{\zeta'_k}$ is a solution of the equation $\omega^4 - \omega^3 - \omega^2 - \omega - 1 = 0$ for which $\delta \in (1, 2)$. Hence $\lim\limits_{\omega\to\infty} \frac{\zeta(\omega+1)}{\zeta(\omega)} = \delta$.

Case (ii) Since ω is any positive real number, it may be rewritten as $\omega = \sigma_1 + 2\sigma_2$, where σ_1 is a real number and σ_2 is a natural number.

$$\begin{aligned}
\frac{\zeta(\omega+1)}{\zeta(\omega)} &= \frac{\zeta(\sigma_1+2\sigma_2+1)}{\zeta(\sigma_1+2\sigma_2)} \\
&= \frac{\zeta(\sigma_1+2\sigma_2)+\zeta(\sigma_1+2\sigma_2-1)+\zeta(\sigma_1+2\sigma_2-2)+\zeta(\sigma_1+2\sigma_2-3)}{\zeta(\sigma_1+2\sigma_2)} \\
&= \left[1 + \frac{\zeta(\sigma_1+2\sigma_2-1)+\zeta(\sigma_1+2\sigma_2-2)+\zeta(\sigma_1+2\sigma_2-3)}{\zeta(\sigma_1+2\sigma_2)}\right] < 2,
\end{aligned}$$

since $\dfrac{\zeta(\sigma_1+2\sigma_2-1)+\zeta(\sigma_1+2\sigma_2-2)+\zeta(\sigma_1+2\sigma_2-3)}{\zeta(\sigma_1+2\sigma_2)} < 1$. Consider the sequence $\left\{\frac{\zeta(\sigma_1+2\sigma_2+1)}{\zeta(\sigma_1+2\sigma_2)}\right\}_{\sigma_2=1}^{\infty}$. Then we have to prove that this sequence is monotonically increasing. Let us take two consecutive sequences

$$\begin{aligned}
\frac{\zeta(\sigma_1+2(\sigma_2+1)+1)}{\zeta(\sigma_1+2(\sigma_2+1))} - \frac{\zeta(\sigma_1+2\sigma_2+1)}{\zeta(\sigma_1+2\sigma_2)} &= \frac{\zeta(\sigma_1+2\sigma_2+3)}{\zeta(\sigma_1+2\sigma_2+2})) - \frac{\zeta(\sigma_1+2\sigma_2+1)}{\zeta(\sigma_1+2\sigma_2)} \\
&= \frac{\zeta(\sigma_1+2\sigma_2+3)\zeta(\sigma_1+2\sigma_2) - \zeta(\sigma_1+2\sigma_2+1)\zeta(\sigma_1+2\sigma_2+2)}{\zeta(\sigma_1+2\sigma_2+2)\zeta(\sigma_1+2\sigma_2)} \\
&> \frac{3\zeta(\sigma_1+2\sigma_2)^2}{\zeta(\sigma_1+2\sigma_2+2)\zeta(\sigma_1+2\sigma_2)} \geq 0,
\end{aligned}$$

since the numerator part is non-negative. Hence it is increasing monotonically, and so there exists the limit. That is., $\lim\limits_{\sigma_2\to\infty} \frac{\zeta(\sigma_1+2\sigma_2+1)}{\zeta(\sigma_1+2\sigma_2)} = \lim\limits_{\omega\to\infty} \frac{\zeta(\omega+1)}{\zeta(\omega)}$ exists. □

5 Conclusion

In this research work, we studied tetranacci numbers and defined tetranacci functions on real numbers. We applied induction technique to obtain useful results for tetranacci functions with tetranacci numbers and also proved that the limit of quotient of tetranacci function exists. This analysis can also be compassed for higher levels (like pentanacci, hexanacci, heptanacci, $\cdots$) in future.

Declarations

Availablity of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Competing interest
The authors declare that they have no competing interests.

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