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Existence of fixed point for generalized weak contraction satisfying rational type expression in partially ordered metric spaces

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Abstract

In this paper, we obtain some fixed point theorems of mappings satisfying a generalized rational type weak contractive condition in partially ordered metric spaces. The presented results generalize and extend various fixed point theorems of the literature. We also provide an example which supports our new results, but it contradicts the previously established results. Furthermore, we discuss the application of these results to the existence and uniqueness of solutions for first-order periodic boundary value problems arising in ordinary differential equations.

Keywords: fixed point, generalized weak contraction, rational type expression, ordered metric spaces
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1 Introduction

The Banach contraction principle is one of the most pivotal results in the study of nonlinear analysis. Due to its applicability, this principle has been generalized and extended in different directions by various researchers in the literature. In 1997, the concept of weak contraction was introduced by Alber and Guerre-Delabriere [1] in Hilbert spaces and proved the corresponding fixed point result. Later, this result is also valid in complete metric spaces, as proved by Rhoades [16]. Furthermore, in this direction, Dutta and Choudhary [6] proved a generalized result by defining (ψ, ϕ)-weak contraction in complete metric spaces. However, in 2009, Zhang and Song [17] introduced a proper extension of ϕ-weak contraction, namely, generalized ϕ-weak contraction. Thereafter, Doric [5] extended the result of Zhang and Song [17] by defining a more generalized (ψ, ϕ)-weak contraction and proved some fixed point theorems. Also, the existence of fixed points in partially ordered metric spaces was first investigated by Ran and Reurings [15] in 2004, and subsequently Nieto et al. [11] extended their results for non-decreasing mappings and found the solution of first order ordinary differential equation with periodic boundary conditions. Moreover, results on weakly contractive mappings in partially ordered metric spaces were obtained by Harjani and Sadarangani [8] and also extended the result of Dutta and Choudhary [6] with application to differential equations (see [9]). There are many other researchers who have contributed some fixed point theorems for weakly contractive mappings in partially ordered metric spaces (see [2], [7], [10], [12], [13], [14], and references therein).

In 1975, Dass and Gupta [4] proved the following fixed point theorem.

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Theorem 1.1. Let (U,ϱ) be a comlplete metric space and Q : U $\longrightarrow$ U be a self mapping such that their exist $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 < 1$ satisfying

$$\varrho(Q\mu, Q\nu) \leq \beta_1 \frac{\varrho(\nu, Q\nu)[1 + \varrho(\mu, Q\mu)]}{1 + \varrho(\mu, \nu)} + \beta_2 \varrho(\mu, \nu)$$

for all $\mu, \nu \in$ U. Then Q has a fixed point.

Also, Cabrera et al. [3] extended the result of Dass and Gupta [4] and proved the following fixed point theorem in partially ordered metric spaces.

Theorem 1.2. Let $(U,\leq)$ be a partially ordered set and suppose that there exist a metric ϱ in U such that (U,ϱ) is a complete metric space. Let Q : $U \to U$ be a continuous and non-decreasing mapping such that there exist $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 < 1$ satisfying

$$\varrho(Q\mu, Q\nu) \leq \beta_1 \frac{\varrho(\nu, Q\nu)[1 + \varrho(\mu, Q\mu)]}{1 + \varrho(\mu, \nu)} + \beta_2 \varrho(\mu, \nu)$$

for all $\mu, \nu \in$ U with $\mu \leq \nu$. If there exist $\mu_0 \in U$ such that $\mu_0 \leq Q\mu_0$ then Q has a fixed point.

Now, the motive of this paper is to establish the fixed point results for generalized weak contraction mapping satisfying a rational type expression in partially ordered metric spaces, which generalizes the result due to Cabrera et al. [3]. To illustrate the applicability of our results, there is a proper example which supports our results, but not that of Cabrera et al. [3]. Additionally, we demonstrate how our results can be applied to prove the existence and uniqueness of a solution for first order periodic boundary value problem that arises in ordinary differential equations.

2 Preliminaries

Throughout the discussion of the paper, the following definitions will be necessary.

Definition 2.1. The triple $(U, \varrho, \leq)$ is called a partially ordered metric space, if $(U, \leq)$ is a partially ordered set together with (U, ϱ) is a metric space

Definition 2.2. If (U, ϱ) is a complete metric space, then the triple $(U, \varrho, \leq)$ is called a partially ordered complete metric space.

Definition 2.3. Let $(U, \leq)$ be a partially ordered set. A self-mapping $Q : U \to U$ is said to be strictly increasing, if $Q(\mu) < Q(\nu)$, for all $\mu, \nu \in U$ with $\mu < \nu$ and is also said to be strictly decreasing, if $Q(\mu) > Q(\nu)$, for all $\mu, \nu \in U$ with $\mu < \nu$.

Definition 2.4. Suppose $(U, \leq)$ be a partially ordered set and let $Q : U \to U$ be a self mapping. Q is said to be monotone non-decreasing mapping if for all $\mu, \nu \in U$,

$$\mu \leq \nu \text{ implies } Q\mu \leq Q\nu.$$

and Q is said to be monotone non-increasing mapping if for all $\mu, \nu \in U$,

$$\mu \leq \nu \text{ implies } Q\mu \geq Q\nu.$$

Definition 2.5. Let (U,$\leq$) be a partially ordered set and let Q : U $\longrightarrow$ U be a self mapping. Then

(1) Element $\mu, \nu \in$ U are comparable, if $\mu \leq \nu$ or $\nu \leq \mu$ holds.
(2) A non-empty set U is called well ordered set. If two elements of it are comparable.

Definition 2.6. A partially ordered metric space $(U, \varrho, \leq)$ is called an ordered complete, if for each convergent sequence $\{\mu_n\}_{n=0}^{\infty} \subseteq U$, one of the following condition holds

(i) if $\{\mu_n\}$ is a non-decreasing sequence in U such that $\mu_n \to \mu$ implies $\mu_n \leq \mu$, for all $n \in N$ that is, $\mu = \sup\{\mu_n\}$ or

(ii) if $\{\mu_n\}$ is a non-increasing sequence in U such that $\mu_n \to \mu$ implies $\mu \leq \mu_n$, for all $n \in N$ that is, $\mu = \inf\{\mu_n\}$.

Definition 2.7. Let (U, ϱ) be a metric space and $Q : U \to U$ is said to be a ϕ - weak contraction if it satisfies the condition

$$\varrho(Q\mu, Q\nu) \leq \varrho(\mu, \nu) - \phi(\varrho(\mu, \nu))$$

for all $\mu, \nu \in U$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\phi(t) = 0$ if and only if $t = 0$.

3 Main Results

Theorem 3.1. Let $(U,\leq)$ be a partially ordered set and suppose that there exist a metric ϱ in U such that (U,ϱ) is a complete metric space. Let Q is a continuous self mapping on U, Q is monotone non-decreasing mapping satisfying the following inequality

$$\varrho(Q\mu, Q\nu) \leq M(\mu, \nu) - \phi(M(\mu, \nu)), for\ all\ \mu, \nu \in U with\ \mu \leq \nu, \tag{3.1}$$

where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a lower semi-continuous function with ϕ(t) = 0 if and only if t = 0, and

$$M(\mu, \nu) = \max\left\{\frac{\varrho(\nu, Q\nu)[1 + \varrho(\mu, Q\mu)]}{1 + \varrho(\mu, \nu)}, \varrho(\mu, \nu)\right\}.$$

If there exist $\mu_0 \in U$ such that $\mu_0 \leq Q\mu_0$, then Q has a fixed point.

Proof . If $Q\mu_0 = \mu_0$, then the theorem is proved. So, suppose that $\mu_0 < Q\mu_0$. Since, Q is monotone non-decreasing mapping. Therefore, by using mathematical induction, we get

$$\mu_0 < Q\mu_0 \leq Q^2\mu_0 \leq \ldots \leq Q^n\mu_0 \leq Q^{n+1}\mu_0 \leq \ldots.$$

This gives a sequence $\{\mu_n\}$ in U such that $\mu_{n+1} = Q\mu_n$ for every $n \geq 0$. Since, Q is monotone non-decreasing mapping, we have

$$\mu_0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \mu_{n+1} \leq \ldots.$$

If there exist $n \geq 1$ such that $\mu_{n+1} = Q\mu_n = \mu_n$, μ_n is a fixed point, and the proof is finished. So, we suppose that $\mu_{n+1} \neq \mu_n$ for all $n \geq 0$. Since, $\mu_n \leq \mu_{n+1}$ for any $n \in N$, for $n \geq 1$ and using contractive condition 3.1, we get

$$\begin{aligned}
\varrho(\mu_n, \mu_{n+1}) &= \varrho(Q\mu_{n-1}, Q\mu_n) \\
&\leq \max\left\{\frac{\varrho(\mu_n, Q\mu_n)[1 + \varrho(\mu_{n-1}, Q\mu_{n-1})]}{1 + \varrho(\mu_{n-1}, \mu_n)}, \varrho(\mu_{n-1}, \mu_n)\right\} \\
&\quad -\phi\left(\max\left\{\frac{\varrho(\mu_n, Q\mu_n)[1 + \varrho(\mu_{n-1}, Q\mu_{n-1})]}{1 + \varrho(\mu_{n-1}, \mu_n)}, \varrho(\mu_{n-1}, \mu_n\right\}\right) \\
&= \max\left\{\varrho(\mu_n, \mu_{n+1}), \varrho(\mu_{n-1}, \mu_n)\right\} - \phi\left(\max\left\{\varrho(\mu_n, \mu_{n+1}), \varrho(\mu_{n-1}, \mu_n)\right\}\right).
\end{aligned} \tag{3.2}$$

Suppose that their exists m_0 such that $\varrho(\mu_{m_0}, \mu_{m_0+1}) > \varrho(\mu_{m_0-1}, \mu_{m_0})$, then from 3.2 we have

$$\begin{aligned}
\varrho(\mu_{m_0}, \mu_{m_0+1}) &\leq \max\left\{\varrho(\mu_{m_0}, \mu_{m_0+1}), \varrho(\mu_{m_0-1}, \mu_{m_0})\right\} - \phi\left(\max\left\{\varrho(\mu_{m_0}, \mu_{m_0+1}), \varrho(\mu_{m_0-1}, \mu_{m_0})\right\}\right) \\
&= \varrho(\mu_{m_0}, \mu_{m_0+1}) - \phi(\varrho(\mu_{m_0}, \mu_{m_0+1})) \\
&< \varrho(\mu_{m_0}, \mu_{m_0+1})
\end{aligned}$$

which is a contradiction. Hence, $\varrho(\mu_n, \mu_{n+1}) \leq \varrho(\mu_{n-1}, \mu_n)$ for all $n \geq 1$.Since, $\varrho(\mu_n, \mu_{n+1})$ is a non-increasing sequence of postive real numbers, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} \varrho(\mu_n, \mu_{n+1}) = \alpha.$$

Now, we shall prove that $\alpha = 0$. Suppose to the contray, that $\alpha > 0$. Applying limit as $n \to \infty$ in 3.2 and using the properties of the function ϕ, we obtain

$$\alpha \leq \alpha - \liminf_{n\to\infty} \phi(\max\{\varrho(\mu_n, \mu_{n+1}), \varrho(\mu_{n-1}, \mu_n)\}) = \alpha - \phi(\alpha) < \alpha$$

which is a contradiction. Therefore, $\alpha = 0$ that is

$$\lim_{n\to\infty} \varrho(\mu_n, \mu_{n+1}) = 0. \tag{3.3}$$

Also, we show that $\{\mu_n\}$ is a cauchy sequence. Assume to the contrary that $\{\mu_n\}$ is not a cauchy sequence. Then, there exist $\epsilon > 0$ such that we can find subsequences $\{\mu_{m(l)}\}$, $\{\mu_{n(l)}\}$ of $\{\mu_n\}$ with $l \leq m(l) < n(l)$ satisfying

$$\varrho(\mu_{m(l)}, \mu_{n(l)}) \geq \epsilon \tag{3.4}$$

Further, corresponding to $m(l)$, we can choose $n(l)$ in such a way that it is the smallest integer with $l \leq m(l) < n(l)$ satisfying 3.4. Hence,

$$\varrho(\mu_{m(l)}, \mu_{n(l)-1}) < \epsilon.$$

Thus, $\epsilon \leq \varrho(\mu_{m(l)}, \mu_{n(l)}) \leq \varrho(\mu_{m(l)}, \mu_{n(l)-1}) + \varrho(\mu_{n(l)-1}, \mu_{n(l)}) < \epsilon + \varrho(\mu_{n(l)-1}, \mu_{n(l)})$. Applying $l \to \infty$ and using 3.3, we get

$$\lim_{l\to\infty} \varrho(\mu_{m(l)}, \mu_{n(l)}) = \epsilon. \tag{3.5}$$

Now, using triangular inequality, we have

$$\varrho(\mu_{m(l)}, \mu_{n(l)}) \leq \varrho(\mu_{m(l)}, \mu_{m(l)-1}) + \varrho(\mu_{m(l)-1}, \mu_{n(l)-1}) + \varrho(\mu_{n(l)-1}, \mu_{n(l)})$$

and

$$\varrho(\mu_{m(l)-1}, \mu_{n(l)-1}) \leq \varrho(\mu_{m(l)-1}, \mu_{m(l)}) + \varrho(\mu_{m(l)}, \mu_{n(l)}) + \varrho(\mu_{n(l)}, \mu_{n(l)-1}).$$

Taking $l \to \infty$ and using 3.5 in the above inequalities, we obtain

$$\lim_{l\to\infty} \varrho(\mu_{m(l)-1}, \mu_{n(l)-1}) = \epsilon. \tag{3.6}$$

Since $n(l) > m(l)$, $\mu_{n(l)-1} > \mu_{m(l)-1}$, from 3.1 we have

$$\begin{aligned}
\varrho(\mu_{n(l)}, \mu_{m(l)}) &= \varrho(Q\mu_{n(l)-1}, Q\mu_{m(l)-1}) \\
&\leq \max\left\{\frac{\varrho(\mu_{m(l)-1}, Q\mu_{m(l)-1})[1 + \varrho(\mu_{n(l)-1}, Q\mu_{n(l)-1})]}{1 + \varrho(\mu_{n(l)-1}, \mu_{m(l)-1})}, \varrho(\mu_{n(l)-1}, \mu_{m(l)-1})\right\} \\
&\quad -\phi\left(\max\left\{\frac{\varrho(\mu_{m(l)-1}, Q\mu_{m(l)-1})[1 + \varrho(\mu_{n(l)-1}, Q\mu_{n(l)-1})]}{1 + \varrho(\mu_{n(l)-1}, \mu_{m(l)-1})}, \varrho(\mu_{n(l)-1}, \mu_{m(l)-1})\right\}\right) \\
&= \max\left\{\frac{\varrho(\mu_{m(l)-1}, \mu_{m(l)})[1 + \varrho(\mu_{n(l)-1}, \mu_{n(l)})]}{1 + \varrho(\mu_{n(l)-1}, \mu_{m(l)-1})}, \varrho(\mu_{n(l)-1}, \mu_{m(l)-1})\right\} \\
&\quad -\phi\left(\max\left\{\frac{\varrho(\mu_{m(l)-1}, \mu_{m(l)})[1 + \varrho(\mu_{n(l)-1}, \mu_{n(l)})]}{1 + \varrho(\mu_{n(l)-1}, \mu_{m(l)-1})}, \varrho(\mu_{n(l)-1}, \mu_{m(l)-1})\right\}\right).
\end{aligned}$$

Taking $l \to \infty$ and using 3.5, 3.6 and properties of the function ϕ in the above inequality, we obtain

$$\epsilon \leq \max\{0, \epsilon\} - \phi(\max\{0, \epsilon\}) = \epsilon - \phi(\epsilon) < \epsilon,$$

which is a contradiction. Therefore, $\{\mu_n\}$ ia a cauchy sequence. Since, U is a complete metric space, their exist $\mu \in U$ such that $\lim_{n\to\infty} \mu_n = \mu$. Also, Q is continuous, so we get

$$Q\mu = Q(\lim_{n\to\infty} \mu_n) = \lim_{n\to\infty} Q\mu_n = \lim_{n\to\infty} \mu_{n+1} = \mu.$$

Hence, μ is a fixed point. □

Now, we shall prove that Theorem 3.1 is still valid for Q not necessarily continuous, assuming the following hypothesis in U:

$$\text{If } \{\mu_n\} \text{ is a non-decreasing sequence in } U \text{ such that } \mu_n \to \mu, \text{ then } \mu = \sup\{\mu_n\}. \tag{3.7}$$

Theorem 3.2. Let $(U, \leq)$ be a partially ordered set and suppose that there exist a metric ϱ in U such that (U, ϱ) is a complete metric space. Suppose that Q be a self mapping on U, Q is monotone non-decreasing mapping and

$$\varrho(Q\mu, Q\nu) \leq M(\mu,\nu) - \phi(M(\mu,\nu)), \textit{for all } \mu,\nu \in U \textit{ with } \mu \leq \nu, \tag{3.8}$$

where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a lower semi-continuous function with $\phi(\text{t}) = 0$ if and only if t = 0, and

$$M(\mu,\nu) = \max\left\{\frac{\varrho(\nu,Q\nu)[1+\varrho(\mu,Q\mu)]}{1+\varrho(\mu,\nu)}, \varrho(\mu,\nu)\right\}.$$

Assume that $\{\mu_n\}$ is a non-decreasing sequence in U such that $\mu_n \to \mu$, then $\mu = \sup\{\mu_n\}$, and if there exist $\mu_0 \in U$ such that $\mu_0 \leq Q\mu_0$, then Q has a fixed point.

Proof . Following the proof of the Theorem 3.1 we have $\{\mu_n\}$ is a cauchy sequence. Since, $\{\mu_n\}$ is a non-decreasing sequence in U such that $\mu_n \to \mu$, then $\mu = \sup\{\mu_n\}$. Particularly, $\mu_n \leq \mu$, for all $n \in$ N.

Since, Q is a monotone non-decreasing mapping $Q\mu_n \leq Q\mu$, for all $n \in$ N. Moreover, as $\mu_n < \mu_{n+1} \leq Q\mu$ and $\mu = \sup\{\mu_n\}$, we get $\mu \leq Q\mu$.

Construct a sequence $\{\nu_n\}$ as $\nu_0 = \mu$, $\nu_{n+1} = Q\nu_n$, for all $n \geq 0$. Since, $\nu_0 \leq Q\nu_0$, aruguing as above we obtain that $\{\nu_n\}$ is non-decreasing sequence and $\lim_{n\to\infty} \nu_n = \nu$ for some $\nu \in U$. So, we have $\nu = \sup\{\nu_n\}$. Since $\mu_n \leq \mu = \nu_0 \leq Q\mu = Q\nu_0 \leq \nu_n \leq \nu$ for all n. Using 3.8, we have

$$\begin{aligned}
\varrho(\mu_{n+1},\nu_{n+1}) &= \varrho(Q\mu_n, Q\nu_n) \\
&\leq \max\left\{\frac{\varrho(\nu_n,Q\nu_n)[1+\varrho(\mu_n,Q\mu_n)]}{1+\varrho(\mu_n,\nu_n)}, \varrho(\mu_n,\nu_n)\right\} \\
&\quad -\phi\left(\max\left\{\frac{\varrho(\nu_n,Q\nu_n)[1+\varrho(\mu_n,Q\mu_n)]}{1+\varrho(\mu_n,\nu_n)}, \varrho(\mu_n,\nu_n)\right\}\right) \\
&= \max\left\{\frac{\varrho(\nu_n,\nu_{n+1})[1+\varrho(\mu_n,\mu_{n+1})]}{1+\varrho(\mu_n,\nu_n)}, \varrho(\mu_n,\nu_n)\right\} \\
&\quad -\phi\left(\max\left\{\frac{\varrho(\nu_n,\nu_{n+1})[1+\varrho(\mu_n,\mu_{n+1})]}{1+\varrho(\mu_n,\nu_n)}, \varrho(\mu_n,\nu_n)\right\}\right).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\varrho(\mu,\nu) \leq \max\{0, \varrho(\mu,\nu)\} - \phi\left(\max\{0,\varrho(\mu,\nu)\}\right) < \varrho(\mu,\nu)$$

which is a contradiction. Therefore, $\mu = \nu$. We have $\mu \leq Q\mu \leq \mu$, therefore $Q\mu = \mu$. Hence, μ is a fixed point. □

Corollary 3.3. Let $(U, \leq)$ be a partially ordered set, and suppose that there is a metric ϱ such that (U,ϱ) be a complete metric space. Let $Q : U \longrightarrow U$ be a non-decreasing mappping such that

$$\varrho(Q\mu, Q\nu) \leq k \max\left\{\frac{\varrho(\nu,Q\nu)[1+\varrho(\mu,Q\mu)]}{1+\varrho(\mu,\nu)}, \varrho(\mu,\nu)\right\}$$

for all $\mu,\nu \in$ U with $\mu \leq \nu$, where k $\in (0,1)$. Also, assume either Q is continuous or Q has the property 3.7. If there exist $\mu_0 \in U$ with $\mu_0 \leq Q\mu_0$, then Q has a fixed point.

Proof . In Theorem 3.1, taking $\phi(t) = (1-k)t$, for all $t \in [0,\infty)$, we get Corollary 3.3. □

Remark 3.4. For $\beta_1, \beta_2 > 0$, $\beta_1 + \beta_2 < 1$ and for all $\mu,\nu \in$ U, we have

$$\begin{aligned}
\varrho(Q\mu, Q\nu) &\leq \beta_1 \frac{\varrho(\nu,Q\nu)[1+\varrho(\mu,Q\mu)]}{1+\varrho(\mu,\nu)} + \beta_2\varrho(\mu,\nu) \\
&\leq (\beta_1+\beta_2)\max\left\{\frac{\varrho(\nu,Q\nu)[1+\varrho(\mu,Q\mu)]}{1+\varrho(\mu,\nu)}, \varrho(\mu,\nu)\right\} \\
&= k\max\left\{\frac{\varrho(\nu,Q\nu)[1+\varrho(\mu,Q\mu)]}{1+\varrho(\mu,\nu)}, \varrho(\mu,\nu)\right\}
\end{aligned}$$

where k $= \beta_1 + \beta_2 \in (0,1)$. Therefore, Corollary 3.3 is a generalization of Theorem 3.1.

Now, we will give an example where it can be proved that assumption in Theorem 3.1 do not guarantee the uniqueness of the fixed point.

Example 3.5. Let $U = \{(1,2),(2,1)\} \subset R^2$ and consider the usual order given by

$$(\mu,\nu) \leq (\omega,\chi) \Longleftrightarrow \mu \leq \omega,\ \nu \leq \chi.$$

Hence, $(U,\leq)$ is a partially ordered set in which distinct non-comparable elements. Besides (U,ϱ_2) is a complete metric space, where ϱ_2 is the Euclidean distance. The identity map $Q(\mu,\nu) = (\mu,\nu)$ is obviously non-decreasing and continuous and assumption (2.1) of Theorem 3.1 is satified because elements in U are only comparable to themselves. Moreover, (1,2) $\leq$ Q(1,2) and Q has two fixed point in U.

Now, for the uniqueness of the fixed point, we suppose that for $\mu,\nu \in U$, there exists a lower bound or an upper bound. Nieto et al. [11] proved that it is equivalent to following condition:

$$\text{For } \mu,\nu \in U, \text{ their exist } \omega \in U \text{ which is comparable to } \mu \text{ and } \nu. \tag{3.9}$$

Moreover, the condition 3.9 is a sufficient condition for the uniqueness of the fixed point.

Theorem 3.6. Adding condition 3.9 to the assumption of Theorem 3.1 and Theorem 3.2. We obtain uniqueness of the fixed point of Q.

Proof . Suppose that their exist $\mu,\nu \in U$ which are two fixed point of Q. Now, we have two different cases.

Case 1. If $\mu \neq \nu$, μ and ν are comparable. Then using 3.1, we have

$$\begin{aligned}
\varrho(\mu,\nu) &= \varrho(Q\mu,Q\nu) \\
&\leq \max\left\{\frac{\varrho(\nu,Q\nu)[1+\varrho(\mu,Q\mu)]}{1+\varrho(\mu,\nu)},\varrho(\mu,\nu)\right\} - \phi\left(\max\left\{\frac{\varrho(\nu,Q\nu)[1+\varrho(\mu,Q\mu)]}{1+\varrho(\mu,\nu)},\varrho(\mu,\nu)\right\}\right) \\
&= \max\left\{\frac{\varrho(\nu,\nu)[1+\varrho(\mu,\mu)]}{1+\varrho(\mu,\nu)},\varrho(\mu,\nu)\right\} - \phi\left(\max\left\{\frac{\varrho(\nu,\nu)[1+\varrho(\mu,\mu)]}{1+\varrho(\mu,\nu)},\varrho(\mu,\nu)\right\}\right).
\end{aligned}$$

Therefore,

$$\varrho(\mu,\nu) \leq \max\{0,\varrho(\mu,\nu)\} - \phi\left(\max\{0,\varrho(\mu,\nu)\}\right) < \varrho(\mu,\nu)$$

which is a contradiction. Therfore, $\mu = \nu$.

Case 2. If μ is not comparable to ν, then their exist $\omega \in U$ which is comparable to μ and ν. Monotonicity implies that $Q^n\omega$ is comparable to $Q^n\mu = \mu$ and $Q^n\nu = \nu$ for n = 0,1,2,3,.... If their exist $n_0 \geq 1$ such that $Q^{n_0}\omega = \mu$, then as μ is a fixed point, the sequence $\{\ Q^n\omega : n \geq n_0\}$ is constant and consequently, $\lim_{n\to\infty} Q^n\omega = \mu$. On the other hand, if $Q^n\omega \neq \mu$, for $n \geq 1$, using the contractive condition, we obtain for $n \geq 2$.

$$\begin{aligned}
\varrho(Q^n\omega,\mu) &= \varrho(Q^n\omega,Q^n\mu) \\
&= \varrho(Q(Q^{n-1}\omega),Q(Q^{n-1}\mu)) \\
&\leq \max\left\{\frac{\varrho(Q^{n-1}\mu,Q^n\mu)[1+\varrho(Q^{n-1}\omega,Q^n\omega)]}{1+\varrho(Q^{n-1}\omega,Q^{n-1}\mu)},\varrho(Q^{n-1}\omega,Q^{n-1}\mu)\right\} \\
&\quad -\phi\left(\max\left\{\frac{\varrho(Q^{n-1}\mu,Q^n\mu)[1+\varrho(Q^{n-1}\omega,Q^n\omega)]}{1+\varrho(Q^{n-1}\omega,Q^{n-1}\mu)},\varrho(Q^{n-1}\omega,Q^{n-1}\mu)\right\}\right).
\end{aligned}$$

Therefore,

$$\varrho(Q^n\omega,\mu) \leq \max\{0,\varrho(Q^{n-1}\omega,\mu)\} - \phi\left(\max\{0,\varrho(Q^{n-1}\omega,\mu)\}\right) < \varrho(Q^{n-1}\omega,\mu) \tag{3.10}$$

which implies that $\varrho(Q^n\omega,\mu) < \varrho(Q^{n-1}\omega,\mu)$ for all $n \geq 1$, that is, $\varrho(Q^n\omega,\mu)$ is a decreasing sequence of postive real numbers, there exist $\gamma \geq 0$ such that $\lim_{n\to\infty} \varrho(Q^n\omega,\mu) = \gamma$. We shall prove that $\gamma = 0$. Assume to the contrary, that $\gamma > 0$. Applying limit as n $\to \infty$ and using the properties of the function ϕ in 3.10, we obtain

$$\gamma = \lim_{n\to\infty} \varrho(Q^n\omega,\mu) \leq \gamma - \liminf_{n\to\infty} \phi(\varrho(Q^{n-1}\omega,\mu)) = \gamma - \phi(\gamma) < \gamma.$$

which is a contradiction. Therefore, $\gamma = 0$, that is $Q^n\omega = \mu$ as $n \to \infty$. Using a simila argument, we can show that $\lim_{n\to\infty} Q^n\omega = \nu$. Also, uniqueness of the limit implies $\mu = \nu$. Hence, Q has a unique fixed point.

□

Now, we present an example where Theorem 3.1 can be applied but it cannot be hold on Theorem 1.2.

Example 3.7. Let $U = \{0, 1, 2, 3, 4, ...\}$ and let $\varrho : U \times U \to [0, \infty)$ be given by

$$\varrho(\mu, \nu) = \begin{cases} 0, & \text{if } \mu = \nu \\ \mu + \nu, & \text{if } \mu \neq \nu. \end{cases}$$

Then (U, ϱ) be a complete metric space. Let Q : U → U be defined by

$$Q\mu = \begin{cases} 0, & \text{if } \mu = 0 \\ \mu - 1, & \text{if } \mu \neq 0, \end{cases}$$

and let $\phi : [0, \infty) \to [0, \infty)$ be defined by $\phi(t) = 1$ for all $t > 0$. Now, we discuss the following cases:

Case 1. If $\mu = 0$, $\nu \neq 0$, we have

$$\varrho(Q\mu, Q\nu) = \varrho(Q0, Q\nu) = \varrho(0, \nu - 1) = \nu - 1$$

$$\begin{aligned} M(\mu, \nu) &= \max\left\{\frac{\varrho(\nu, Q\nu)[1 + \varrho(0, Q0)]}{1 + \varrho(0, \nu)}, \varrho(0, \nu)\right\} \\ &= \max\left\{\frac{\varrho(\nu, \nu - 1)[1 + \varrho(0, Q0)]}{1 + \varrho(0, \nu)}, \varrho(0, \nu)\right\} \\ &= \max\left\{\frac{2\nu - 1}{1 + \nu}, \nu\right\} \\ &= \nu. \end{aligned}$$

Thus, $M(\mu, \nu) - \phi M(\mu, \nu) = \nu - 1 = \varrho(Q\mu, Q\nu)$.

Case 2. If $\nu > \mu$ *and* $\mu, \nu \neq 0$, we have

$$\varrho(Q\mu, Q\nu) = \varrho(\mu - 1, \nu - 1) = \mu + \nu - 2.$$

Now,

$$\begin{aligned} M(\mu, \nu) &= \max\left\{\frac{\varrho(\nu, Q\nu)[1 + \varrho(\mu, Q\mu)]}{1 + \varrho(\mu, \nu)}, \varrho(\mu, \nu)\right\} \\ &= \max\left\{\frac{\varrho(\nu, \nu - 1)[1 + \varrho(\mu, \mu - 1)]}{1 + \varrho(\mu, \nu)}, \varrho(\mu, \nu)\right\} \\ &= \max\left\{\frac{(2\nu - 1)(2\mu)}{(1 + \mu + \nu)}, (\mu + \nu)\right\} \\ &= \mu + \nu. \end{aligned}$$

Thus, $M(\mu, \nu) - \phi M(\mu, \nu) = \mu + \nu - 1 > \varrho(Q\mu, Q\nu)$.

Case 3. If $\mu = \nu$, then trivially

$$\varrho(Q\mu, Q\nu) \leq M(\mu, \nu) - \phi(M(\mu, \nu)), for\ all\ \mu, \nu \in U$$

Hence, all the condition of Theorem 3.1 are satisfied and 0 is the unique fixed point of Q. But the contrative condition appearing in Theorem 1.2 is not satisfied. For example taking $\mu = 0$ and $\nu =$ n+1, we have

$$\begin{aligned} \varrho(Q(0), Q(n + 1)) &\leq \beta_1 \frac{\varrho(n + 1, Q(n + 1))[1 + \varrho(0, Q0))]}{1 + \varrho(0, n + 1)} + \beta_2 \varrho(0, n + 1) \\ n &\leq \beta_1 \frac{2n + 1}{n + 2} + \beta_2(n + 1). \end{aligned}$$

Applying n → ∞ in this inequality, we get $\beta_2 \geq 1$ which is a contradiction. Hence, Theorem 3.1 is a proper generalization of Theorem 1.2.

4 Application to boundary value problem

In this section as an application we present an example, where Theorems 3.2 and 3.6 can be applied. We prove an existence and uniqueness of solution for the following first order periodic boundary value problem:

$$\mu'(p) = q(p, \mu(p)), \quad p \in I = [0, P] \text{ and } \mu(0) = \mu(P) \tag{4.1}$$

where $P > 0$ and $q : I \times R \longrightarrow R$ is a continuous function. Let $C(I)$ denote the space of all continuous functions defined on I. Obiviously, this space with the metric given by $\varrho(\mu, \nu) = \sup\{|\mu(p) - \nu(p)| : p \in I\}$ for $\mu, \nu \in C(I)$ is a complete metric space. On $C(I)$, define a partial order $\leq$ is given by

$$\mu, \nu \in C(I); \mu(p) \leq \nu(p) \text{ for } p \in I.$$

Now, we recall the following definitions:

Definition 4.1. A function $\beta \in C^1(I)$ is called a lower solution of 4.1, if

$$\beta'(p) \leq q(p, \beta(p)), \quad p \in I, \quad \beta(0) \leq \beta(P)$$

Definition 4.2. A function $\beta \in C^1(I)$ is called a upper solution of 4.1, if

$$\beta'(p) \geq q(p, \beta(p)), \quad p \in I, \quad \beta(0) \geq \beta(P)$$

Theorem 4.3. In addition to the problem (3.1), suppose that there exist $\lambda > 0$ such that for all x,y $\in$ R with y $\geq$ x

$$0 \leq q(p, y) + \lambda y - [q(p, x) + \lambda x] \leq \lambda \ln(y - x + 1) \tag{4.2}$$

Then the existence of a lower solution or an upper solution of problem 4.1 ensures the existence and uniqueness of a solution of problem 4.1.

Proof . Problem 4.1 can be rewritten as

$$\begin{cases} \mu'(p) + \lambda\mu(p) = q(p, \mu(p)) + \lambda\mu(p), & p \in I \\ \mu(0) = \mu(P). \end{cases} \tag{4.3}$$

The problem 4.3 is equivalent to the integral equation

$$\mu(p) = \int_0^P G(p, s)[q(s, \mu(s)) + \lambda\mu(s)]ds$$

where the Green function $G(p, s)$ is given by

$$G(p, s) = \begin{cases} \dfrac{e^{\lambda(P+s-p)}}{e^{\lambda P} - 1}, & 0 \leq s < p \leq P, \\ \dfrac{e^{\lambda(s-p)}}{e^{\lambda P} - 1}, & 0 \leq p < s \leq P, \end{cases}$$

Define a function B : C(I) $\longrightarrow$ C(I) by

$$(B\mu)(p) = \int_0^P G(p, s)[q(s, \mu(s)) + \lambda\mu(s)]ds.$$

Note that if $\mu \in C(I)$ is a fixed point of B then $\mu \in C(I)$ is a solution of 4.1. Now, we check that hypothesis in Theorem 3.2 and Theorem 3.6 are satisfied. The mapping B is non-decreasing since, by hypothesis, for $\mu \geq \nu$.

$$q(p, \mu) + \lambda\mu \geq q(p, \nu) + \lambda\nu$$

which implies that G(p,s) > 0 for (p,s) ∈ I × I, that

$$\begin{aligned}(B\mu)(p) &= \int_0^P G(p,s)[q(s,\mu(s)) + \lambda\mu(s)]ds \\ &\geq \int_0^P G(p,s)[q(s,\nu(s)) + \lambda\nu(s)]ds = (B\nu)(p)\end{aligned}$$

for p ∈ I, Besides, for $\mu \geq \nu$

$$\begin{aligned}\varrho(B\mu, B\nu) &= \sup_{p\in 1} |(B\mu)(p) - (B\nu)(p)| \\ &\leq \sup_{p\in 1} \int_0^P G(p,s)[q(s,\mu(s)) + \lambda\mu(s) - q(s,\nu(s)) - \lambda\nu(s)]ds \\ &\leq \sup_{p\in 1} \int_0^P G(p,s).\lambda\ \ln(\mu(s) - \nu(s) + 1)ds\end{aligned}$$

As the function $\Psi(x) = \ln(x+1)$ is nondecreasing and $\mu \geq \nu$, then $\ln(\mu(s) - \nu(s) + 1) \leq \ln(\| \mu - \nu \| + 1)$, and hence we get

$$\begin{aligned}\varrho(B\mu, B\nu) &\leq \sup_{p\in 1} \int_0^P G(p,s) \cdot \lambda\ \ln(\mu(s) - \nu(s) + 1)ds \\ &\leq \ln(\| \mu - \nu \| + 1) \cdot \lambda . \sup_{p\in 1} \int_0^P G(p,s)ds \\ &= \ln(\| \mu - \nu \| + 1) \cdot \lambda \cdot \sup_{p\in 1} \frac{1}{e^{\lambda P} - 1} \left(\frac{1}{\lambda} e^{\lambda(P+s-p)}]_0^p + \frac{1}{\lambda} e^{\lambda(s-p)}]_p^P \right) \\ &= \ln(\| \mu - \nu \| + 1) \cdot \lambda \cdot \frac{1}{\lambda(e^{\lambda P} - 1)} (e^{\lambda P} - 1) = \ln(\|\mu - \nu\| + 1) \\ &= M(\mu,\nu) - (M(\mu,\nu) - \ln(\|\mu - \nu\| + 1)),\end{aligned}$$

where $M(\mu,\nu) = \max\left\{ \frac{(\|\nu - Q\nu\|)[1 + (\|\mu - Q\mu\|)]}{1 + (\|\mu - \nu\|)}, (\|\mu - \nu\|) \right\}$. Moreover,

$$\varrho(B\mu, B\nu) \leq M(\mu,\nu) - (M(\mu,\nu) - \ln(M(\mu,\nu) + 1)).$$

Now, putting $\phi(t) = t - \ln(t+1)$, then $\phi : [0,\infty) \longrightarrow [0,\infty)$ is continuous and nondecreasing. Hence, above inequality can be written as,

$$\varrho(B\mu, B\nu) \leq M(\mu,\nu) - \phi(M(\mu,\nu))$$

Finally, let β(p) be a lower solution for 4.1 and we will show that $\beta \leq B\beta$. Now,

$$\beta'(p) + \lambda\beta(p) \leq q(p, \beta(p)) + \lambda\beta(p)$$

for $p \in I$. Multiplying by $e^{\lambda p}$ we get

$$(\beta(p)e^{\lambda p})' \leq [q(p,\beta(p)) + \lambda\beta(p)]e^{\lambda p}, \text{ for } p \in I$$

and this gives us

$$\beta(p)e^{\lambda p} \leq \beta(0) + \int_0^P [q(s,\beta(s)) + \lambda\beta(s)]e^{\lambda s}ds, for\ p \in I \tag{4.4}$$

which implies that

$$\beta(0)e^{\lambda p} \leq \beta(P)e^{\lambda p} \leq \beta(0) + \int_0^P [q(s,\beta(s)) + \lambda\beta(s)]e^{\lambda s}ds$$

and so

$$\beta(0) \leq \int_0^P \frac{e^{\lambda s}}{e^{\lambda P} - 1}[q(s,\beta(s)) + \lambda\beta(s)]ds.$$

From this inequality and 4.4 we obtain

$$\beta(p)e^{\lambda p} \leq \int_0^p \frac{e^{\lambda(P+s)}}{e^{\lambda P}-1}[q(s,\beta(s))+\lambda\beta(s)]ds + \int_p^P \frac{e^{\lambda s}}{e^{\lambda P}-1}[q(s,\beta(s))+\lambda\beta(s)]ds$$

and consequently,

$$\beta(p) \leq \int_0^p \frac{e^{\lambda(P+s-p)}}{e^{\lambda P}-1}[q(s,\beta(s))+\lambda\beta(s)]ds + \int_p^P \frac{e^{\lambda(s-p)}}{e^{\lambda P}-1}[q(s,\beta(s))+\lambda\beta(s)]ds.$$

Hence,

$$\beta(p) \leq \int_0^P G(p,s)[q(s,\beta(s))+\lambda\beta(s)]ds = (B\beta)(p),\ p \in I.$$

Using our Theorem 3.2 and 3.6, we have B has a unique fixed point. □

5 Conclusion

In this article, we establish results for a generalized weak contractive condition based on rational-type expressions, which extend and generalize several prominent theorems in the literature, specifically within the context of metric spaces endowed with a partial order. An example is provided to illustrate the existence and uniqueness of fixed points for this class of mappings. Moreover, the article includes an application to a first-order periodic boundary value problem arising in ordinary differential equations, further highlighting the significance of the presented results.

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