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Results of approximating coupled fixed point of Kannan interpolative contraction mappings

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Abstract

This paper presents the results of approximating coupled fixed point of kannan interpolative contraction mappings. Let X be a complete metric space and $T : X \times X \to X$ be a coupled mapping. We proved the existence and uniqueness of a fixed point theory using coupled interpolative Kannan contractions. We also proved the stability of the interpolative Kannan contraction to validate the well-posedness of the conditions. Our results extend some results in the literature.

Keywords: Coupled Fixed Point, Metric Space, Interpolative Kannan contraction, Complete metric space.
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1 Introduction

In the realm of mathematical analysis and its applications, fixed point theory has grown in popularity in recent years. It is well known that the challenge of solving a nonlinear equation is closely related to the approximation of fixed points of a matching constructive type operator. Consequently, there is theoretical and practical interest in approximating fixed points of numerous contractive type operators. In this paper, we established results of approximating coupled fixed point of kannan interpolative contraction mappings.

In the study of the existence and approximation of the solutions of various nonlinear problems, including optimization problems, variation inequality problems, inclusion problems, equilibrium problems, and nonlinear functional equations, fixed point theory provides valuable tools for nonlinear analysis. Aniki *et al.* [1], obtained coupled fixed points theorem for mappings satisfying a contractive condition of integral type in cauchy Spaces. Bhasker and Lakshmikantham [3], presented fixed point theorems results in partially ordered metric spaces and applications. Bota *et al.* [4], introduced fixed points and coupled fixed points in b-metric spaces via graphical contractions. Guo and Lakshmikantham [5], established Coupled fixed points of nonlinear operators with applications. Hammad and Zayed [6], built systems of new generalized contractions by employing two control functions and coupled fixed-point theorems with applications. Kannan [7], introduced Some results on fixed points. Karapinar [8], Revisited the Kannan type contractions via interpolation. Karapinar [9], constructed approach to Interpolative Kannan-Meir-Keeler type contraction. Liu *et al.* [10], deduced common coupled fixed point theorem for Geraghty-type contraction in partially ordered metric spaces. Olatinwo and Tijani [11], extended coupled fixed point concept in higher dimension and applications. Opoitsev [12], constructed heterogeneous and combined concave operators. Rauf and Oyekanmi [13], published some results of coupled iterative fixed point approximation.

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2 Preliminaries

Definition 2.1 (Coupled Fixed Point). [1] Let X be a nonempty set. A pair $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $f : X \times X \to X$ if it is a solution of the system.

$$F(x, y) = x \text{ and } f(x, y) = y.$$

Definition 2.2 (Coupled Generalized Interpolative Kannan Contraction). [8] A self mapping $T : X \times X \to X$ is called a coupled generalized interpolative Kannan contraction if there exist $\lambda \in (0, 1)$, $\gamma \in (0, 1)$ for which

$$d\Big(T(x, y), T(u, v)\Big) \leq \lambda \max\left[d(x, T(x, y))^{\gamma} d(u, T(u, v))^{1-\gamma},\ d(x, T(x, y))^{1-\gamma} d(u, T(u, v))^{\gamma}\right] \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \neq T(x, y)$ and $u \neq T(u, v)$.

Definition 2.3 (Coupled Fixed Point Iterative Procedure). [8] Let (X, d) be a metric space and $T : X \times X \to X$ a mapping. For $(x_0, y_0) \in X \times X$, the sequence $\{(x_n, y_n)\}_{n=0}^{\infty} \subset X \times X$ defined iteratively by

$$x_{n+1} = T(x_n, y_n), \quad y_{n+1} = T(y_n, x_n), \qquad n = 0, 1, 2, \cdots \tag{2.2}$$

is said to be a coupled fixed point iterative procedure.

Definition 2.4. [9] Let (X, d) be a complete metric space. Suppose that

$$fix(T) = \{(x^*, y^*) \in X \times X | T(x^*, y^*) = x^*, \quad T(x^*, y^*) = y^*\} \tag{2.3}$$

is the set of coupled fixed point of T. Let $[(x_n, y_n)]_{n=0}^{\infty} \subset X \times X$ be the sequence generated by an iterative procedure involving T defined by

$$x_{n+1} = f(T, (x_n, y_n)), \quad y_{n+1} = f(T, (y_n, x_n)), \qquad n = 0, 1, 2, \cdots \tag{2.4}$$

where $(x_0, y_0) \in X \in X$ is the initial approximation and f is some function. Suppose $[(x_n, y_n)]_{n=0}^{\infty} \subset X \times X$ converges to a coupled fixed point (x^*, y^*) of T. Let $[(u_n, v_n)]_{n=0}^{\infty}$ be a sequences in $X \times X$ and set

$$\epsilon_n = d\Big(U_{n+1}, f(T, (u_n, v_n))\Big), \quad \delta_n = d\Big(V_{n+1}, f(T, (v_n, u_n))\Big) \quad (n = 0, 1, 2, \cdots).$$

Then, the coupled fixed point iterative procedure is said to be $T-$stable, or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = 0$ implies $\lim_{n\to\infty} U_n = x^*$ and $\lim_{n\to\infty} V_n = y^*$.

Lemma 2.5. [11] If δ is a real number such that $0 \leq \delta < 1$, and $[b_n]_{n=0}^{\infty}$ is a sequence of positive numbers such $\lim_{n\to\infty} b_n = 0$, then to any sequence of positive number $[a_n]_{n=0}^{\infty}$ satisfying

$$a_{n+1} \leq \delta a_n + b_n, \quad (n = 0, 1, 2, \cdots) \tag{2.5}$$

we have $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [2] If $\psi : R^* \to R^+$ is a subadditive comparison function and $[\epsilon_n]_{n=0}^{\infty}$ is a sequence of positive number such that $\lim_{n\to\infty} \epsilon_n = 0$, then to any sequence of positive number $[U_n]_{n=0}^{\infty}$ satisfying

$$U_{n+1} \leq \sum_{k=0}^{m} \delta_k \psi^k(U_n) + \epsilon_n, \quad n = 0, 1, 2, \cdots \tag{2.6}$$

where $\delta_k \in [0, 1]$, $\quad k = 0, 1, 2, \cdots, m$, $0 \leq \sum_{k=0}^{m} \delta_k \leq 1$, we have $\lim_{n\to\infty} U_n = 0$.

Lemma 2.7. [2] If $\{x_n\}_n \geq 0$ is a sequence of non negative real numbers satisfying

$$x_{n+1} \leq \beta_1 x_n + \beta_2 x_{n-1}, n \geq 1, \tag{2.7}$$

where $\beta_1, \beta_2 \in (0, 1)$ are such that $\beta_1 + \beta_2 \leq 1$, then:

(a) $\{x_n\}_{n\geq 0}$ is convergent;

(b) There exist $L > 0$ and $\theta \in [0, 1]$ such that

$$x_n \leq L \cdot \theta^n, \text{ for all } n \geq 1. \tag{2.8}$$

(c) If $\beta_1 + \beta_2 < 1$, then $x_n \to 0$ as $\to \infty$.

3 Main Results

This section presents the results established by approximating coupled fixed point of kannan interpolative contraction mappings.

Theorem 3.1. Let (X,d) be a complete metric space and $T: X\times X \to X$ be a coupled interpolative Kannan type contraction, a self-map such that there exist $\lambda \in (0,1)$, $\gamma \in (0,1)$ so that

$$d(T(x,y),T(u,v)) \leq \lambda(d(x,T(x,y)))^{\gamma}(d(u,T(u,v)))^{1-\gamma} \tag{3.1}$$

for all $x,y,u,v \in X$ with $x \neq T(x,y)$; $u \neq T(u,v)$. Then, T has a unique fixed point in $X \times X$

Proof . Let x_0, $y_0 \in X$ and construct the sequence $[x_n]$ by $x_{n+1} = T(x_n,y_n)$ for all positive integer n. Taking $x = x_n$, $y = y_n$,

$$\begin{aligned}
d(x_{n+1},x_n) =& d\Big(T(x_n,y_n),T(x_{n-1},y_{n-1})\Big)\\
\leq& \lambda d\Big(x_n,T(x_n,y_n)\Big)^{\gamma}\cdot d\Big(x_{n-1},T(x_{n-1},y_{n-1})\Big)^{1-\gamma}\\
=& \lambda d\Big(x_n,x_{n+1}\Big)^{\gamma}\cdot d\Big(x_{n-1},x_n\Big)^{1-\gamma}\\
d(x_{n+1},x_n) \leq& \lambda d\Big(x_n,x_{n+1}\Big)^{\gamma}\cdot d\Big(x_{n-1},x_n\Big)^{1-\gamma}
\end{aligned} \tag{3.2}$$

This implies

$$d(x_{n+1},x_n)^{1-\gamma} \leq \lambda d\Big(x_{n-1},x_n\Big)^{1-\gamma} \tag{3.3}$$

Similarly,

$$\begin{aligned}
d(x_n,x_{n+1}) &= d\Big(T(x_{n-1},y_{n-1}),T(x_n,y_n)\Big)\\
&\leq \lambda d\Big(x_{n-1},T(x_{n-1},y_{n-1})\Big)^{\gamma}\cdot d\Big(x_n,T(x_n,y_n)\Big)^{1-\gamma}\\
d(x_n,x_{n+1})^{\gamma} &\leq \lambda d\Big(x_{n-1},T(x_{n-1},y_{n-1})\Big)^{\gamma}
\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$d(x_n,x_{n+1}) \leq \lambda d(x_{n-1},T(x_{n-1},y_{n-1})).$$

Therefore,

$$d(x_n,x_{n+1}) \leq \lambda d(x_{n-1},x_n) \quad \text{for all} \quad n \geq 1.$$

Next, we show that the sequence is a Cauchy for $m,n \in \mathbb{N}$ with $m > n$. That is,

$$\begin{aligned}
d(x_n,x_m) &\leq d\Big(x_n,x_{n+1}\Big) + d\Big(x_{n+1},x_{n+2}\Big) + d\Big(x_{n+2},x_{n+3}\Big) + \cdots + d\Big(x_{m-1},x_m\Big)\\
&\leq \lambda^n d\Big(x_0,T(x_0,y_0)\Big) + \lambda^{n+1} d\Big(x_0,T(x_0,y_0)\Big) + \cdots + \lambda^{m-1} d\Big(x_0,T(x_0,y_0)\Big)\\
&\leq \left[\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{m-1}\right] d\Big(x_0,T(x_0,y_0)\Big)\\
&\leq \frac{\lambda^n}{1-\lambda} d\Big(x_0,T(x_0,y_0)\Big)\\
d(x_n,x_m) &< \lambda d\Big(x_0,T(x_0,y_0)\Big).
\end{aligned}$$

Hence, the sequence is a Cauchy sequence. Since X is complete, then X_n converges to a limit l, therefore exist a limit such that the sequence converges to the limit say, for a given $(l,v) \in X \times X$.

$$\begin{aligned}
l &= T(l,v)\\
d(x_{n+1},l) &= d\Big(T(x_n,y_n),T(l,v)\Big)\\
&\leq \lambda d\Big(x_n,T(x_n,y_n)\Big)^{\gamma}\cdot d\Big(l,T(l,v)\Big)^{1-\gamma}\\
d(x_{n+1},l) &= 0\\
\lim_{n\to\infty} x_{n+1} &= l
\end{aligned}$$

Similarly, for

$$\lim_{n\to\infty} y_n = v.$$

Hence, l is the fixed point. Suppose there exist another fixed point $q = T(q,u)$. Then

$$\begin{aligned} d(l,q) &= d\Big(T(l,v),T(q,u)\Big) \\ &\leq \lambda d\Big(l,T(l,v)\Big)^{\gamma}\cdot d\Big(q,T(q,u)\Big)^{1-\gamma} \\ d(l,q) &= 0. \end{aligned}$$

Therefore $l = q$. Hence, the fixed point is unique. □

Theorem 3.2. Let F be a let (X,d) be a complete metric space and T is a coupled self mapping on X such that

$$d\Big(T(x,y),T(u,v)\Big) \leq \lambda\Big(d(x,T(x,y))\Big)^{\gamma}\Big(d(y,T(u,v))\Big)^{\beta}$$

for all $x,y\in X$ with $x\neq Tx$ and $y\neq Ty$, and where $\lambda\in[0,1]$ and $\gamma,\beta\in[0,1]$ such that $\gamma+\beta<1$ if there exist $x\in X$ such that $d(x,T(x,y))>0$, then T has a fixed point in X. Let F be a bounded, closed and convex subset of a Banach space X and let $E : F\times F\to F$ be weakly nonexpansive and demicompact operator. Then the set of coupled fixed points of E is nonempty and the double iterative algorithm $\{(x_n,x_n)\}_{n=0}^{\infty}$ given by x_0 in F and

$$x_{n+1} = \lambda x_n + (1-\lambda)E(x_n,x_n), \quad n\geq 0 \tag{3.5}$$

where $\lambda\in(0,1)$, converges (strongly) to a coupled fixed point of E.

Proof . Let $x_0,y_0\in X$, and let the sequence $x_{n+1} = T(x_n,y_n)$ for all positive integer n. Taking $x = x_n, \quad y = y_n$

$$\begin{aligned} d(x_n,x_{n+1}) &= d\Big(T(x_{n-1},y_{n-1}),T(x_n,y_n)\Big) \\ &\leq \lambda\Big(d(x_{n-1},T(x_{n-1},y_{n-1}))\Big)^{\gamma}\cdot\Big(d(x_n,T(x_n,y_n))\Big)^{\beta} \end{aligned}$$

Since $\beta+\gamma<1 \therefore \quad \beta<1-\gamma$ and $\gamma<1-\beta$

$$\Big(d(x_n,x_{n+1})\Big)^{1-\beta} \leq \lambda\Big(d(x_{n-1},T(x_n,y_n))\Big)^{\gamma} \leq \lambda\Big(d(x_{n-1},T(x_{n-1},y_{n-1}))\Big)^{1-\beta}.$$

Therefore

$$\Big(d(x_n,x_{n+1})\Big)^{1-\beta} \leq \lambda\Big(d(x_{n-1},T(x_{n-1},y_{n-1}))\Big)^{1-\beta}$$

and

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \lambda^{\frac{1}{1-\beta}} d(x_{n-1},x_n) \\ &\leq \lambda d(x_{n-1},x_n) \end{aligned}$$

for all integer $n\geq 1$,

$$\begin{aligned} \Big(d(x_{n+1},x_n)\Big) &= d\Big(T(x_n,y_n),T(x_{n-1},y_{n-1})\Big) \\ &\leq \lambda\Big(d(x_n,T(x_n,y_n))\Big)^{\gamma}\cdot\Big(d(x_{n-1},T(x_{n-1},y_{n-1}))\Big)^{\beta} \\ \Big(d(x_{n+1},x_n)\Big)^{1-\beta} &\leq \lambda\Big(d(x_{n-1},T(x_{n-1},y_{n-1}))\Big)^{\beta} \\ &\leq \lambda\Big(d(x_{n-1},T(x_{n-1},y_{n-1}))\Big)^{1-\gamma} \\ d\Big(x_{n+1},x_n\Big) &\leq \lambda^{\frac{1}{1-\gamma}} d\Big(x_{n-1},T(x_{n-1},y_{n-1})\Big) \\ &\leq \lambda d\Big(x_{n-1},T(x_{n-1},y_{n-1})\Big) \\ &\leq \lambda^2 d\Big(x_{n-2},T(x_{n-2},y_{n-2})\Big) \\ &\;\;\vdots \\ &\leq \lambda^n d\Big(x_0,T(x_0,y_0)\Big) \end{aligned}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. For all $m, n \in N$ If $m < n$

$$\begin{aligned} d(x_m, x_n) &\leq d\Big(x_m, x_{m+1}\Big) + d\Big(x_{m+1}, x_{m+2}\Big) + \cdots + d\Big(x_{n-1}, x_n\Big) \\ &\leq \lambda^m d(x_0, T(x_0, y_0)) + \lambda^{m+1} d(x_0, T(x_0, y_0)) + \lambda^{m+2} d(x_0, T(x_0, y_0)) \\ &\quad + \cdots + \lambda^{n-1} d(x_0, T(x_0, y_0)) \\ &\leq \left[\lambda^m + \lambda^{m+1} + \lambda^{m+2} + \cdots + \lambda^{n-1}\right] d\Big(x_0, T(x_0, y_0)\Big) \\ &\leq \frac{\lambda^m}{1-\lambda^m} d\Big(x_0, T(x_0, y_0)\Big) \end{aligned}$$

If $m > n$

$$\begin{aligned} d(x_n, x_m) &\leq d\Big(x_n, x_{n+1}\Big) + d\Big(x_{n+1}, x_{n+2}\Big) + \cdots + d\Big(x_{m-1}, x_m\Big) \\ &\leq \lambda^n d(x_0, T(x_0, y_0)) + \lambda^{n+1} d(x_0, T(x_0, y_0)) + \cdots + \lambda^{n-1} d(x_0, T(x_0, y_0)) \\ &\leq \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n-1}\right] d\Big(x_0, T(x_0, y_0)\Big) \\ &\leq \frac{\lambda^n}{1-\lambda} d\Big(x_0, T(x_0, y_0)\Big) \\ &\leq d\Big(x_0, T(x_0, y_0)\Big) \end{aligned}$$

Hence, the Cauchy sequence. Since X is complete, then x_n converges. Suppose there exists, $(l, v) \in X \times X$, then the sequence and a limit, say $(l, v) \in X \times X$ such that $l = T(l, v)$. Therefore

$$\begin{aligned} d(x_{n+1}, l) &= d\Big(T(x_n, y_n), T(l, v)\Big) \\ &\leq \lambda\Big(d(x_n, T(x_n, y_n))\Big)^{1-\gamma} \cdot \Big(d(l, T(l, v))\Big)^{\beta} \end{aligned}$$

since $T(l, v) = l$

$$d(x_{n+1}, l) = 0$$
$$\lim_{n\to\infty} x_{n+1} = l$$

Hence, l is the fixed point of T. Suppose there exist another fixed point of T. Say q such that $q = T(q, u)$, then

$$\begin{aligned} d(l, q) &= d\Big(T(l, v), T(q, u)\Big) \\ &\leq \lambda\Big(d(l, T(l, v))\Big)^{\gamma} \cdot \Big(d(q, T(q, u))\Big)^{\beta} \\ d(l, q) &= 0 \end{aligned}$$

Hence, fixed point of T is unique. □

Theorem 3.3. Let (X, d) be a complete metric space and T is a coupled self mapping i.e. $TX \times X \to X$ such that

$$d\Big(T(x, y), T(u, v)\Big) \leq \lambda \max\left[\Big(d(x, T(x, y))\Big)^{\gamma} \cdot \Big(d(u, T(u, v))\Big)^{1-\gamma}, \Big(d(u, T(u, v))\Big)^{\gamma} \cdot \Big(d(x, T(x, y))\Big)^{1-\gamma}\right]$$

for all $x, y, u, v \in X$ with $x \neq T(x, y)$ and $u \neq T(u, v)$. $\gamma \in (0, 1)$ and $\lambda \in (0, 1)$. Then T has a fixed point in X.

Proof . Let $(x_0, y_0) \in X \times X$ and define $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$, then

$$\begin{aligned} d(x_n, x_{n+1}) &= d\Big(T(x_{n-1}, y_{n-1}), T(x_n, y_n)\Big) \\ &\leq \lambda \max\Big[\Big(d(x_{n-1}, T(x_{n-1}, y_{n-1}))\Big)^{\gamma} \cdot \Big(d(x_n, T(x_n, y_n))\Big)^{1-\gamma}, \\ &\qquad \Big(d(x_{n-1}, T(x_{n-1}, y_{n-1}))\Big)^{1-\gamma} \cdot \Big(d(x_n, T(x_n, y_n))\Big)^{\gamma}\Big] \end{aligned}$$

Case 1 if $x_n < x_{n+1}$

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \lambda\Big(d(x_{n-1}, T(x_{n-1}, y_{n-1}))\Big)^{1-\gamma} \cdot \Big(d(x_n, T(x_n, y_n))\Big)^{\gamma} \\
d(x_n, x_{n+1})^{1-\gamma} &\leq \lambda\Big(d(x_{n-1}, T(x_{n-1}, y_{n-1}))\Big)^{1-\gamma} \\
d(x_n, x_{n+1}) &\leq \lambda^{\frac{1}{1-\gamma}} d\Big(x_{n-1}, T(x_{n-1}, y_{n-1})\Big) \\
d(x_n, x_{n+1}) &\leq \lambda d\Big(x_{n-1}, T(x_{n-1}, y_{n-1})\Big) \\
&\leq \lambda^2 d\Big(x_{n-2}, T(x_{n-2}, y_{n-2})\Big) \\
&\leq \lambda^3 d\Big(x_{n-3}, T(x_{n-3}, y_{n-3})\Big) \\
&\vdots \\
d(x_n, x_{n+1}) &\leq \lambda^n d\Big(x_0, T(x_0, y_0)\Big)
\end{aligned}$$

Case 2 if $x_n > x_{n+1}$

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \lambda\Big(d(x_{n-1}, T(x_{n-1}, y_{n-1}))\Big)^{\gamma} \cdot \Big(d(x_n, T(x_n, y_n))\Big)^{1-\gamma} \\
\Big(d(x_n, x_{n+1})\Big)^{\gamma} &\leq \lambda\Big(d(x_{n-1}, T(x_{n-1}, y_{n-1}))\Big)^{\gamma} \\
d(x_n, x_{n+1}) &\leq \lambda^{\frac{1}{\gamma}} d\Big(x_{n-1}, T(x_{n-1}, y_{n-1})\Big) \\
&\leq \lambda d\Big(x_{n-1}, T(x_{n-1}, y_{n-1})\Big) \\
&\leq \lambda^2 d\Big(x_{n-2}, T(x_{n-2}, y_{n-2})\Big) \\
&\leq \lambda^3 d\Big(x_{n-3}, T(x_{n-3}, y_{n-3})\Big) \\
&\vdots \\
&\leq \lambda^n d\Big(x_0, T(x_0, y_0)\Big)
\end{aligned}$$

Hence, both cases converges. **Case 1** If $m < n$

$$\begin{aligned}
d(x_m, x_n) &\leq d\Big(x_m, x_{m+1}\Big) + d\Big(x_{m+1}, x_{m+2}\Big) + d\Big(x_{m+2}, x_{m+3}\Big) + \cdots + d\Big(x_{n-1}, x_n\Big) \\
&\leq \lambda^m d(x_0, T(x_0, y_0)) + \lambda^{m+1} d(x_0, T(x_0, y_0)) + \lambda^{m+2} d(x_0, T(x_0, y_0)) + \cdots + \lambda^{n-1} d(x_0, T(x_0, y_0)) \\
&\leq \left[\lambda^m + \lambda^{m+1} + \lambda^{m+2} + \cdots + \lambda^{n-1}\right] d\Big(x_0, T(x_0, y_0)\Big) \\
&\leq \frac{\lambda^m}{1-\lambda} d\Big(x_0, T(x_0, y_0)\Big)
\end{aligned}$$

$$\frac{\lambda^m}{1-\lambda} d\Big(x_0, T(x_0, y_0)\Big) \to 0 \quad \text{as} \quad m, n \to \infty.$$

This indicate $d\Big(x_m, x_n\Big)$ can be made arbitrarily small by large m and n, hence $d\Big(x_m, x_n\Big)$ is a Cauchy sequence.
Case 2 If $m > n$. Following the above prove

$$d(x_m, x_n) \leq \frac{\lambda^n}{1-\lambda} d\Big(x_0, T(x_0, y_0)\Big)$$

we conclude the sequence is a Cauchy sequence. The sequence converges to some $\rho^* \in T$ such that

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = \rho^* = T(\rho^*, u)$$

$$\begin{aligned} d(x_{n+1},\rho^*) &\leq d\Big(T(x_n,y_n),T(\rho^*,u)\Big) \\ &\leq \lambda \max\Big[\Big(d(x_n,T(x_n,y_n))\Big)^{\gamma}\cdot\Big(d(\rho^*,T(\rho^*,u))\Big)^{1-\gamma},\Big(d(\rho^*,T(\rho^*,u))\Big)^{\gamma}\cdot\Big(d(x_n,T(x_n,y_n))\Big)^{1-\gamma}\Big]. \end{aligned}$$

Therefore $d(x_{n+1},\rho^*)=0$ and so $\lim_{n\to\infty}=\rho^*$. Suppose there exist another fixed point of q^* such that $q^*=T(q^*,v)$

$$\begin{aligned} d(\rho^*,q^*) &= d\Big(T(\rho^*,u),T(q^*,u)\Big) \\ &\leq \lambda \max\Big[\Big(d(\rho^*,T(\rho^*,u))\Big)^{\gamma}\cdot\Big(d(q^*,T(q^*,v))\Big)^{1-\gamma},\Big(d(q^*,T(q^*,v))\Big)^{\gamma}\cdot\Big(d(\rho^*,T(\rho^*,u))\Big)^{1-\gamma}\Big] \end{aligned}$$

$$d(\rho^*,q^*)=0$$

Hence

$$\rho^*=q^*.$$

We conclude that the fixed point is unique. □

Theorem 3.4
Let (X,d) be a complete metric space and $T:X\times X\to X$ be a mapping satisfying coupled interpolative Kannan contractive condition

$$d\Big(T(x,y),T(u,v)\Big)\leq\lambda\Big(d\Big(T(x,y),T(u,v)\Big)\Big)^{\gamma}\cdot\Big(d\Big(u,T(u,v)\Big)\Big)^{1-\gamma}$$

for all $x,y,u,v\in X$, $x\neq u$, $\gamma\in(0,1)$ and $\lambda\in(0,1)$. Suppose T has a coupled fixed point (x^*,y^*). For $(x_0,y_0)\in X\times X$, let $[(x_n,y_n)]\subset X\times X$ be the coupled fixed point iterative procedure defined by (Olatinwo, 2012). Then, the coupled fixed point iterative procedure is stable with respect to T.

Proof . Let $[x_n]$, $[y_n]\subset X\times X$, $\epsilon_n=d\Big(U_{n+1},T(u_n,v_n)\Big)$ and $\delta_n=d\Big(V_{n+1},T(v_n,u_n)\Big)$. Let $\lim\limits_{n\to\infty}\epsilon_n=\lim\limits_{n\to\infty}\delta_n=0$. We shall prove that $\lim\limits_{n\to\infty}U_n=x^*$ and $\lim\limits_{n\to\infty}V_n=y^*$. Using the stated contractive condition, we have

$$\begin{aligned} d\Big(U_{n+1},x^*\Big) &\leq d\Big(U_{n+1},T(u_n,v_n)\Big)+d\Big(T(u_n,v_n),x^*\Big) \\ &= d\Big(T(u_n,v_n),T(x^*,y^*)\Big)+\epsilon_n \\ &\leq \lambda\Big(d\Big(u_n,T(u_n,v_n)\Big)\Big)^{\gamma}\cdot\Big(d\Big(x^*,T(x^*,y^*)\Big)\Big)^{1-\gamma}+\epsilon_n \\ &= \lambda\Big(d\Big(u_n,T(u_n,v_n)\Big)\Big)^{\gamma}\cdot\Big(d\Big(x^*,x^*\Big)\Big)^{1-\gamma}+\epsilon_n \\ &\leq \epsilon_n. \end{aligned} \tag{3.6}$$

Similarly

$$\begin{aligned} d\Big(V_{n+1},y^*\Big) &\leq d\Big(V_{n+1},T(v_n,u_n)\Big)+d\Big(T(v_n,u_n),y^*\Big) \\ &= d\Big(T(v_n,u_n),y^*\Big)+\delta_n \\ &= d\Big(T(v_n,u_n),T(y^*,x^*)\Big)+\delta_n \\ &\leq \lambda\Big(d\Big(v_n,T(v_n,u_n)\Big)\Big)^{\gamma}\cdot\Big(d\Big(y^*,T(y^*,x^*)\Big)\Big)^{1-\gamma}+\delta_n \\ &= \lambda\Big(d\Big(v_n,T(v_n,u_n)\Big)\Big)^{\gamma}\cdot\Big(d\Big(y^*,y^*\Big)\Big)^{1-\gamma}+\delta_n \\ &\leq \delta_n. \end{aligned} \tag{3.7}$$

Adding (3.6) and (3.7) we have

$$d\Big(u_{n+1},x^*\Big)+d\Big(v_{n+1},y^*\Big)\leq\epsilon_n+\delta_n \tag{3.8}$$

by Theorm 3.1 we have $\Big[\epsilon_n + \delta_n\Big] = 0$. Hence we have

$$\lim_{n\to\infty}\Big[d\Big(u_{n+1}, x^*\Big) + d\Big(v_{n+1}, y^*\Big)\Big] = 0$$

That is,
$$\lim_{n\to\infty} d\Big(u_{n+1}, x^*\Big) = 0 \quad\Leftrightarrow\quad u_n = x^*$$
$$\lim_{n\to\infty} d\Big(v_{n+1}, y^*\Big) = 0 \quad\Leftrightarrow\quad v_n = y^*$$

□

Theorem 3.4. Let (X, d) be a complete metric space and $T : X \times X \to X$ a mapping satisfying a generalized interpolative Kannan contractive condition

$$d\Big(T(x,y), T(u,v)\Big) \le \lambda \max\Big[d\Big(x, T(x,y),\Big)^{\gamma} \cdot d\Big(u, T(u,v)\Big)^{1-\gamma}, d\Big(x, T(x,y)\Big)^{1-\gamma} \cdot d\Big(u, T(u,v),\Big)^{\gamma}\Big]$$

for all $x, y, u, v \in X$, $x \neq u$, $\lambda \in (0,1)$ and $\gamma \in (0,1)$. Suppose T has a coupled fixed point iterative procedure defined in Theorem 4.4. Then, the coupled fixed point iterative procedure is $T-$stable.

Proof . Let (x_n), $(y_n) \subset X \times X$, $\epsilon_n = d\Big(u_{n+1}, T(u_n, v_n)\Big)$ and $\delta_n = d\Big(v_{n+1}, T(v_n, u_n)\Big)$. Let $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = 0$. We shall prove that $\lim_{n\to\infty} U_n = x^*$ and $\lim_{n\to\infty} V_n = y^*$. Using the stated contraction, we have

$$\begin{aligned}
d\Big(u_{n+1}, x^*\Big) &\le d\Big(u_{n+1}, T(u_n, v_n)\Big) + d\Big(T(u_n, v_n), x^*\Big)\\
&= d\Big(T(u_n, v_n), x^*\Big) + \epsilon_n\\
&\le \lambda \max\Big[d\Big(u_n, T(u_n, v_n)\Big)^{\gamma} \cdot d\Big(x^*, T(x^*, y^*)\Big)^{1-\gamma}, d\Big(u_n, T(u_n, v_n)\Big)^{1-\gamma} \cdot d\Big(x^*, T(x^*, y^*)\Big)^{\gamma}\Big] + \epsilon_n \quad (3.9)\\
&= \lambda \max\Big[d\Big(u_n, T(u_n, v_n)\Big)^{\gamma} \cdot d\Big(x^*, x^*\Big)^{1-\gamma}, d\Big(u_n, T(u_n, v_n)\Big)^{1-\gamma} \cdot d\Big(x^*, x^*\Big)^{\gamma}\Big] + \epsilon_n\\
&\le \epsilon_n.
\end{aligned}$$

Similarly

$$\begin{aligned}
d\Big(v_{n+1}, y^*\Big) &\le d\Big(v_{n+1}, T(v_n, u_n)\Big) + d\Big(T(v_n, u_n), y^*\Big)\\
&= d\Big(T(v_n, u_n), y^*\Big) + \delta_n\\
&\le \lambda \max\Big[d\Big(v_n, T(v_n, u_n)\Big)^{\gamma} \cdot d\Big(y^*, T(y^*, x^*)\Big)^{1-\gamma}, d\Big(v_n, T(v_n, u_n)\Big)^{1-\gamma} \cdot d\Big(y^*, T(y^*, x^*)\Big)^{\gamma}\Big] + \delta_n \quad (3.10)\\
&= \lambda \max\Big[d\Big(v_n, T(v_n, u_n)\Big)^{\gamma} \cdot d\Big(y^*, y^*\Big)^{1-\gamma}, d\Big(v_n, T(v_n, u_n)\Big)^{1-\gamma} \cdot d\Big(y^*, y^*\Big)^{\gamma}\Big] + \delta_n\\
&\le \delta_n.
\end{aligned}$$

Summing equations (3.9) and (3.10), we have

$$d\Big(u_{n+1}, x^*\Big) + d\Big(v_{n+1}, y^*\Big) \le \epsilon_n + \delta_n.$$

.

By Theorem 3.1

$$\lim_{n\to\infty}\Big[\epsilon_n + \delta_n\Big] = 0 \quad \text{which implies that}$$
$$\lim_{n\to\infty} d\Big(u_{n+1}, x^*\Big) = 0 \quad\Leftrightarrow\quad u_n = x^*$$
$$\lim_{n\to\infty} d\Big(v_{n+1}, y^*\Big) = 0 \quad\Leftrightarrow\quad v_n = y^*$$

Hence, the iterative scheme is stable with respect to T. □

Theorem 3.5. Let (X, d) be a complete metric space and $T : X \times X \to X$ a mapping satisfying a coupled relabled $(X, \gamma, \beta)-$interpolative Kannan contraction

$$d\Big(T(x,y), T(u,v)\Big) \leq \lambda d\Big(x, T(x,y),\Big)^{\gamma} \cdot d\Big(u, T(u,v)\Big)^{\beta}$$

for all $x, y, u, v \in X$, $0 \leq \lambda < 1$, $0 < \gamma$, $\beta \leq 1$ with $0 < \gamma + \beta < 1$.

Proof . Let $\{x_n\}$, $\{y_n\} \subset X$, $\epsilon_n = d\Big(u_{n+1}, T(u_n, v_n)\Big)$ and $\delta_n = d\Big(v_{n+1}, T(v_n, u_n)\Big)$. Assume that $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = 0$. Then, we shall prove that

$$\lim_{n\to\infty} u_n = x^* \quad \text{and} \quad \lim_{n\to\infty} v_n = y^*.$$

By triangle inequality and contractive condition we have

$$\begin{aligned} d\Big(u_{n+1}, x^*\Big) &\leq d\Big(u_{n+1}, T(u_n, v_n)\Big) + d\Big(T(u_n, v_n), x^*\Big) \\ &\leq d\Big(T(u_n, v_n), x^*\Big) + \epsilon_n \\ &= d\Big(T(u_n, v_n), T(x^*, y^*)\Big) + \epsilon_n \\ &\leq \lambda d\Big(u_n, T(u_n, v_n)\Big)^{\gamma} \cdot d\Big(x^*, T(x^*, y^*)\Big)^{\beta} + \epsilon_n \\ &= \lambda d\Big(u_n, T(u_n, v_n)\Big)^{\gamma} \cdot d\Big(x^*, x^*\Big)^{\beta} + \epsilon_n \\ &\leq \epsilon_n \end{aligned} \tag{3.11}$$

Similarly

$$\begin{aligned} d\Big(v_{n+1}, y^*\Big) &\leq d\Big(v_{n+1}, T(v_n, u_n)\Big) + d\Big(T(v_n, u_n), y^*\Big) \\ &\leq d\Big(T(v_n, u_n), y^*\Big) + \delta_n \\ &= d\Big(T(v_n, u_n), T(y^*, x^*)\Big) + \delta_n \\ &\leq \lambda d\Big(v_n, T(v_n, u_n)\Big)^{\gamma} \cdot d\Big(y^*, T(y^*, x^*)\Big)^{\beta} + \delta_n \\ &= \lambda d\Big(v_n, T(v_n, u_n)\Big)^{\gamma} \cdot d\Big(y^*, y^*\Big)^{\beta} + \delta_n \\ &\leq \delta_n \end{aligned} \tag{3.12}$$

Adding equation (3.11) and (3.12) we have

$$d\Big(u_{n+1}, x^*\Big) + d\Big(v_{n+1}, y^*\Big) \leq \epsilon_n + \delta_n.$$

By Theorem 3.1,

$$\lim_{n\to\infty} \Big[\epsilon_n + \delta_n\Big] = 0 \quad \text{i.e.} \lim_{n\to\infty} \epsilon_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \delta_n = 0.$$

Therefore the iterative scheme is $T-$stable. □

Conclusion

In this paper, results of coupled fixed point theorem of Kannan Interpolative Contraction Mappings has been established by showing the existence and uniqueness of a fixed point theory as well as its stability.

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