Int. J. Nonlinear Anal. Appl. In Press, 1–8
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2025.33612.5023

Approximation results of non-homogeneous Cauchy problem of semigroup of linear operators

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(Communicated by Abbas Najati)

Abstract

In this paper, results of ω-order reversing partial contraction mapping generated by approximation results of non-homogeneous Cauchy problem were presented. We investigated the concepts of C', strong and respectively C^0-solution. We established that $Z : D(Z) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup of contraction, $\xi \in X$ and $f \in L^2(a, b; X)$. Furthermore, we deduced that the unique C^0-solution is strong and that the class is absolutely continuous on $[a, b]$.

Keywords: ω-$ORCP_n$, Absolutely Continuous, C'-solution, Mild Solution
2020 MSC: Primary 15A60; Secondary 65F35, 65J05

1 Introduction

Suppose $Z : D(Z) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{J(t); t \geq 0\}$, then for each $a \geq 0$ and $\xi_\rho \in D(Z)$, the function $u : [a, +\infty) \to X$, defined by $u(t) = J(t-a)\xi$ for each $t \geq 0$, is the unique solution of the homogeneous Cauchy problem

$$\begin{cases} u' = Zu \\ u(a) = \xi. \end{cases} \tag{1.1}$$

From this reason, it is quite natural to consider that, for each $\xi \in X$, that function $u' = Zu + f$, $u(a) = \xi$ is a solution for (1.1), in a generalized sense. We then consider the non-homogeneous problem

$$\begin{cases} u' = Zu + f \\ u(a) = \xi, \end{cases} \tag{1.2}$$

where Z is the infinitesimal generator of a C_0-semigroup, $\xi \in X$, and $f \in L'(a, b; X)$. Assume that X is a Banach space, $X_n \subseteq X$ is a finite set, H is Hilbert space, $\omega - ORCP_n$ the ω-order reversing partial contraction mapping, M_m be a matrix, $\mathcal{L}(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(Z)$ a resolvent set,

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Received: March 2024 *Accepted:* January 2025

$\sigma(Z)$ a spectrum of Z. This paper consist of results of approximation results of non-homogeneous Cauchy problem of semigroup of linear operators. In [1] and [2], Akinyele *et al.* established differentiable and analytical conclusions on ω-order preserving partial contraction mapping in semigroup of linear operator. They also made a description of ω-order reversing partial contraction mapping as a compact semigroup of linear operator. An operator calculus for infinitesimal semigroup generators was showed by Balakrishnan [3]. Banach [4] introduced and first proposed the idea of Banach spaces. The nonlinear Schrödinger evolution equation was established by Brezis and Gallouet [5]. A resolvent method to the stability operator semigroup was generated by Chill and Tomilov [6]. Davies [7] discovered the spectrum of linear operators. For equations of linear evolution, Engel and Nagel presented the one-parameter semigroup in their paper [8]. Omosowon *et al.* [9] produced some analytical results of semigroup of linear operator with dynamic boundary conditions, as well as introducing dual properties of ω-order reversing partial contraction mapping in semigroup of linear operator in [10]. In their study, Omosowon *et al.* [11] derived the outcomes of a semigroup of linear equations that produced a wave equation. Rauf and Akinyele [12] created ω-order preserving partial contraction mapping and acquired its qualities. Also in [13], Rauf *et al.* established some results of stability and spectra properties on semigroup of linear operator. Vrabie [14] demonstrated a few applications of the C_0-semigroup's findings. Yosida [15] derived several conclusions on the differentiability and representation of a linear operator one-parameter semigroup.

2 Preliminaries

Definition 2.1. (C_0-Semigroup)[14] A C_0-semigroup is a strongly continuous one parameter semigroup of bounded linear operators on a Banach space.

Definition 2.2. (ω-$ORCP_n$)[12] A transformation $\alpha \in P_n$ (partial transformation semigroup) is called ω-order preserving partial contraction mapping if for all $x, y \in \mathrm{D}(\alpha) : x \le y \implies \alpha x \ge \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3. (classical or C'-solution)[14] The function $u : [a, b] \to X$ is called classical, or C'-solution of the problem (1.2), if u is continuous on $[a, b]$, continuously differentiable on $(a, b]$, $u(t) \in D(Z)$ for each $t \in (a, b]$ and it satisfied $u(t) = Zu(t) + f(t)$ for each $t \in [a, b]$ and $u(a) = \xi$.

Definition 2.4. (Absolutely Continuous)[14] The function $u : [a, b] \to X$ is called absolutely continuous, or strong solution, of the problem (1.2), if u is absolutely continuous on $[a, b]$, $u' \in L^1(a, b; X)$, $u(t) \in D(Z)$, that is for $t \in (a,)$, and it satisfies $u'(t) = Zu(t) + f(t)$ almost everywhere for $t \in (a, b)$ and $u(a) = \xi$.

Definition 2.5. (C^0 or mild solution)[14] The function $u : [a, b] \to X$, defined by

$$u(t) = J(t-a)\xi + \int_a^t J(t-s)f(s)ds \tag{2.1}$$

is called C^0 or mild solution of problem (1.2).

Example 2.6. For every 2×2 matrix in $[M_m(\mathbb{R}^n)]$. Suppose that

$$Z = \begin{pmatrix} 2 & 0 \\ \Delta & 2 \end{pmatrix}$$

and let $J(t) = e^{tZ}$, where Δ is any vector, then we have

$$e^{tZ} = \begin{pmatrix} e^{2t} & I \\ e^{\Delta t} & e^{2t} \end{pmatrix}.$$

Example 2.7. For every 3×3 matrix in $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(Z)$, where $\rho(Z)$ is a resolvent set on X. Suppose that we have

$$Z = \begin{pmatrix} 2 & 2 & I \\ 2 & 2 & 2 \\ \Delta & 2 & 2 \end{pmatrix}$$

and let $J(t) = e^{tZ_\lambda}$, then we have

$$e^{tZ_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & I \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{\Delta t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 2.8. Let $X = C_{ub}(\mathbb{N} \cup \{0\})$ be the space of all bounded and uniformly continuous functions from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\| \cdot \|_\infty$ and let $\{J(t); t \in \mathbb{R}_+\} \subseteq L(X)$ be defined by

$$[J(t)f](s) = f(t+s)$$

For each $f \in X$ and each $t, s \in \mathbb{R}_+$, one may easily verify that $\{J(t); t \in \mathbb{R}_+\}$ satisfies in Examples 1 and 2.

Example 2.9. Let $Z : D(Z) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup $\{J(t);\ t \geq 0\}$, for which there exists $\eta \in X$ such that $J(t)\eta \notin D(Z)$ for each $t \geq 0$. Let us define $f(s) = J(s)\eta$ for each $s \in [0, T]$ and let us observe that f is continuous. On the other hand, the problem (1.2) with $\xi = 0$ has no solution. Indeed, if u is a strong solution of the problem (1.2), in view that each classical solution of (1.2) is a strong solution of the same problem, but not conversely, it is then given by

$$u(t) = \int_0^t J(t-s)J(s)\eta ds = tJ(t)\eta$$

for each $t \in [0, T]$, function which clearly is not almost everywhere differentiable on $[0, T]$.

Example 2.10. Let H be a real Hilbert space, whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and let $Z : D(Z) \subseteq H \to H$ where $Z \in \omega - ORCP_n$ such that $-Z$ generates a C_0-semigroup of contractions on H. Assume that Z is self-adjoint and invertible with compact inverse Z^{-1}. Then by positive definite properties of Hilbert space, there exists a sequence of positive numbers $\mu_k > 0$, $\mu_{k+1} \leq \mu_k$ for $k \in \mathbb{N}$ and an orthonormal basis $\{e_k; k \in \mathbb{N}\}$ of H such that

$$Z^{-1}e_k = \mu_k e_k \tag{2.2}$$

for $k \in \mathbb{N}$ and $Z \in \omega - ORCP_n$. Let $\lambda_k = \mu_k^{-1}$, and observe that $e_k \in D(Z)$ and

$$Ze_k = \lambda_k e_k$$

for $k \in \mathbb{N}$ and $Z \in \omega - ORCP_n$. We also have that $\lim_{k \to \infty} \lambda_k = +\infty$. Assume that $\alpha > 0$ and defined $Z_\alpha : D(Z\alpha) \subseteq H \to H$ by

$$D(Z_\alpha) = \left\{ u \in H; u = \sum_{k=1}^{\infty} u_k e_k,\ \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 < +\infty \right\}$$

$$Z_\alpha u = \sum_{k=1}^{\infty} \lambda_k^\alpha u_k e \quad for \quad u \in D(Z_\alpha) \quad and \quad Z_\alpha \in \omega - ORCP_n.$$

A simple calculation shows that $Z_\alpha = Z^\alpha$. We notice that $D(Z^\alpha)$, endowed with the natural inner product $\langle \cdot, \cdot \rangle_\alpha : D(Z^\alpha) \times D(Z^\alpha) \to \mathbb{R}$, defined by

$$\langle u, v \rangle_\alpha = \sum_{k=1}^{\infty} \lambda_k^{2\alpha} \langle u_k, v_k \rangle$$

for each $u, v \in D(Z^\alpha)$, $u = \sum_{k=1}^\infty u_k e_k$, $v = \sum_{k=1}^\infty v_k e_k$ and $Z \in \omega - ORCP_n$ is a real Hilbert space. In addition, with respect to this inner product, the family $\{\lambda_k^{-\alpha} e_k; k \in \mathbb{N}\}$ is an orthonormal basis in $D(Z^\alpha)$.

3 Main Results

This section presents approximation results of non-homogeneous Cauchy problem of semigroup of linear operators generated by ω-$ORCP_n$:

Theorem 3.1. Suppose that $Z : D(Z) \subseteq H \to H$, (where H is Hilbert space) is a linear self-adjoint operator which generates a C_0-semigroup of contractions for all $Z \in \omega - ORCP_n$. Let $u \in W^{1,2}(0, J, H)$ with $u(t) \in D(Z)$ almost everywhere for $t \in [a, b]$, and $Zu \in L^2(a, b; H)$. Then the function $t \mapsto \frac{1}{2}\langle Zu(t), u(t) \rangle$ is absolutely continuous on $[a, b]$ and

$$\frac{d}{dt}\left(\frac{1}{2} \langle Zu(t), u(t) \rangle \right) = \langle Zu(t), u'(t) \rangle. \tag{3.1}$$

Proof . Let $\lambda > 0$ and let $Z_\lambda \in L(H)$ (which is a linear space on H) be the Yosida approximation of Z. Obviously, Z_λ is a self-adjoint operator. Suppose that H is reflexive, then each function $u \in W^{1,p}(a,b;H)$ is almost everywhere differentiable on (a,b) and for each $t \in [a,b]$, we have

$$u(t) = u(a) + \int_a^t u'(s)ds, \tag{3.2}$$

So that it is absolutely continuous on $[0,J]$, almost everywhere differentiable on $(0,J)$, its derivative belongs to $L^2(a,b;H)$, and u is given by

$$u(t) = u(0) + \int_0^t u'(s)ds. \tag{3.3}$$

Then we have

$$\frac{d}{dt}\left(\frac{1}{2}\langle Z_\lambda u(t), u(t)\rangle\right) = \frac{1}{2}\langle Z_\lambda u(t), u'(t)\rangle + \frac{1}{2}\langle Z_\lambda u'(t), u(t)\rangle = \langle Z_\lambda u(t), u'(t)\rangle. \tag{3.4}$$

Integrating (3.4) from s to t, we obtain

$$\frac{1}{2}\langle Z_\lambda u(t), u(t)\rangle - \frac{1}{2}\langle Z_\lambda u(s), u(s)\rangle = \int_s^t \langle Z_\lambda u(\eta), u'(\eta)\rangle d\eta. \tag{3.5}$$

Since $Z : D(Z) \subseteq H \to H$ is a linear operator, we can assume that for each $\lambda > 0$ we have $(I - \lambda Z)^{-1} \in L(H)$ and

$$\|(I - \lambda Z)^{-1}\|_{L(H)} \leq 1. \tag{3.6}$$

Moreover, let us observe that if Z is densely defined for $\lambda > 0$, $\lambda I - Z$ is invertible with continuous inverse and

$$\|(\lambda I - Z)^{-1}\|_{L(H)} \leq \frac{1}{\lambda}$$

for all $Z \in \omega - ORCP_n$. Then

$$\lim_{\lambda\to\infty} \lambda R(\lambda; Z)x = x \tag{3.7}$$

for $x \in H$ and $Z \in \omega - ORCP_n$,

$$Z_\lambda x = \lambda^2 R(\lambda; Z)x - \lambda x \tag{3.8}$$

for each $x \in H$, we have

$$\lim_{\lambda\to\infty} Z_\lambda x = Zx \tag{3.9}$$

for each $x \in D(Z)$ and $Z \in \omega - ORCP_n$. By Lebesque's dominated convergence theorem, we can pass to the limit in (3.5). Then we have

$$\frac{1}{2}\langle Zu(t), u(t)\rangle - \frac{1}{2}\langle Zu(s), u(s)\rangle = \int_s^t \langle Zu(\eta), u'(\eta)\rangle d\eta$$

for each $a \leq s \leq t \leq b$. Therefore, the function $t \mapsto \frac{1}{2}\langle Zu(t), u(t)\rangle$ is absolutely continuous on $[a,b]$ and (3.1) holds. Then the proof is completed.
□

Theorem 3.2. Let $Z : D(Z) \subseteq H \to H$ be infinitesimal generator of a C_0-semigroup of contractions $\{J(t); t \geq 0\}$. If $Z \in \omega - ORCP_n$ and self-adjoint, then for each $\xi \in X$ and $f \in L^2(a,b;X)$, the unique C^0-solution of (1.2) is strong. Moreover, the function $t \mapsto (t-a)^{1/2}u'(t)$ belongs to $L^2(a,b;H)$, the function $t \mapsto \frac{1}{2}\langle Zu(t), u(t)\rangle$ belongs to $L^1(a,b)$ and for each $c \in (a,b)$, is absolutely continuous on $[c,b]$. If $\xi \in D(Z)$ and $Z \in \omega - ORCP_n$, then for each $f \in L^2(a,b;H)$, the unique C^0-solution of (1.2) is srong and satisfies $u' \in L^2(a,b;H)$, and the function $t \mapsto \frac{1}{2}\langle Zu(t), u(t)\rangle$ is absolutely continuous on $[a,b]$.

Proof . Since $Z : D(Z) \subseteq H \mapsto H$ is the infinitesimal generator of a C_0-semigroup of contractions $\{J(t); t \geq 0\}$, $Z \in \omega - ORCP_n$, $\xi \in H$ and $u(t) = J(t)\xi$ for $t \geq 0$. If Z is self-adjoint, then $u \in C([0,+\infty); H) \cap C((0,+\infty); D(Z) \cap C'((0,+\infty); H)$ and u is the unique solution of the Cauchy problem

$$\begin{cases} u' = Zu \\ u(0) = \xi \end{cases} \tag{3.10}$$

in this space and

$$\|Zu(t)\| \leq \frac{1}{t\sqrt{2}}\|\xi\|$$

for each $t > 0$ and $Z \in \omega - ORCP_n$. The function $t \mapsto \sqrt{t}\|Zu(t)\|$ belongs to $L^2(0, +\infty)$ and

$$\int_0^\infty s\|Zu(s)\|^2 ds \leq \frac{1}{4}\|\xi\|^2 \tag{3.11}$$

and suppose

$$\xi \in D(Z), \quad then \quad \|Zu(t)\|^2 \leq \frac{1}{2t}\langle -Z\xi, \xi\rangle$$

for each $t > 0$ and

$$t \mapsto Zu(t) \in L^2(0, +\infty), \quad and \quad \int_0^t \|Zu(s)\|^2 ds \leq \frac{1}{2}\langle -Z\xi, \xi\rangle.$$

Then we for each $\xi \in X$ and $t > 0$, we have $J(t)\xi \in D(Z)$. This means that the function

$$u(t) = J(t-a)\xi + \int_a^t J(t-s)f(s)ds \tag{3.12}$$

is almost everywhere differentiable on $[a, b]$ and satisfies

$$u'(t) = Zu(t) + f(t) \tag{3.13}$$

almost everywhere for $t \in [a, b]$. Let us take the inner product of both sides in (3.13) by $(t-a)u'(t)$. Since Z is self-adjoint and generates a C_0-semigroup, by virtue of Theorem 3.1, we obtain

$$(t-a)\|x'(t)\|^2 = (t-a)\frac{1}{2}\frac{d}{dt}(\langle Zu(t), u(t)\rangle) + (t-a)\langle f(t), u'(t)\rangle. \tag{3.14}$$

Integrating from a to b in (3.14), we get

$$\int_a^b (t-a)\|u'(t)\|^2 dt = \frac{1}{2}(t-a)\langle Zu(t), u(t)\rangle\Big|_a^b - \frac{1}{2}\int_a^b \langle Zu(t), u(t)\rangle dt + \int_a^b (t-a)\langle f(t), u'(t)\rangle dt.$$

Recalling that since $Z : D(Z) \subseteq H \to H$ is the infinitesimal generator of a C_0-semigroup of contractions. Then, we have

$$\langle Zx, x\rangle \leq 0,$$

for all $x \in D(Z)$, $Z \in \omega - ORCP_n$ and using

$$\langle f(t), u'(t)\rangle \leq \frac{1}{2}\|f(t)\|^2 + \frac{1}{2}\|u'(t)\|^2,$$

we deduce

$$\int_a^b (t-a)\|u'(t)\|^2 dt \leq \int_a^b (t-a)\|f(t)\|^2 dt - \int_a^b \langle Zu(t), u(t)\rangle dt$$

but

$$\begin{aligned} \langle Zu(t), u(t)\rangle &= \langle u'(t) - f(t), u(t)\rangle \\ &= \frac{1}{2}\frac{d}{dt}(\|u(t)\|^2) - \langle f(t), u(t)\rangle, \end{aligned} \tag{3.15}$$

and therefore we have

$$\begin{aligned} \int_a^b |\langle Zu(t), u(t)\rangle| dt &= -\int_a^b \langle Zu(t), u(t)\rangle dt \\ &\leq \frac{1}{2}\|\xi\|^2 + \int_a^b \|f(t)\|\|u(t)\| dt, \end{aligned}$$

and

$$\int_a^b (t-a)\|u'(t)\|^2 dt \le \int_a^b (t-a)\|f(t)\|^2 dt + \frac{1}{2}\|\xi\|^2 + \int_a^b \|f(t)\|\|u(t)\| dt.$$

So, $t \mapsto \langle Zu(t), u(t)\rangle$ is in $L^1(a,b)$. For the fact that the semigroup generated by Z is of contractions, then we have

$$\|u(t)\| \le \|\xi\| + \int_a^b \|f(s)\| ds,$$

and accordingly

$$\int_a^b (t-a)\|u'(t)\|^2 dt \le \int_a^b (t-a)\|f(t)\|^2 + \frac{1}{2}\left(\|\xi\| + \int_a^b \|f(t)\| dt\right)^2. \tag{3.16}$$

From (3.16), we deduce the first part of the conclusion. In order to prove the second, let us consider $\xi \in D(Z)$, and let us observe that in this case $u' \in L^2(a,b;H)$. Indeed taking the inner product of both sides in (1.1) by u' and integrating over $[a,b]$, we obtain

$$\int_a^b \|u'(t)\|^2 dt = \frac{1}{2}\langle Zu(b), u(b)\rangle - \frac{1}{2}\langle Z\xi, \xi\rangle + \int_a^b \langle f(t), u'(t)\rangle dt. \tag{3.17}$$

From (3.17), the fact Z is dissipative, and from

$$\int_a^b \langle f(t), u'(t)\rangle dt \le \frac{1}{2}\left(\int_a^b \|f(t)\|^2 dt + \int_a^b \|u'(t)\|^2 dt\right),$$

we deduce

$$\int_a^b \|u'(t)\|^2 dt \le -\langle Z\xi, \xi\rangle + \int_a^b \|f(t)\|^2 dt.$$

Then it follows that $u' \in L^2(a,b;H)$. Consequently, $Zu \in L^2(a,b;H)$ too, and the conclusion follows from Theorem 3.1 and this achieved the proof. □

Theorem 3.3. Assume $Z : D(Z) \subseteq X \to X$ (where X is a Banach space) is the infinitesimal generator of a C_0-semigroup of type (M,ω), let $Z \in \omega - ORCP_n$, $f \in L^1(a,b;X)$ and $\xi \in X$. Then there exist a sequence $(\xi_n)_{n=\mathbb{N}}$ in $D(Z)$ and sequence $(f_n)_{n\in\mathbb{N}}$ in $C^1([a,b];X)$ such that

$$\begin{cases} \lim_{n\to\infty} \xi_n = \xi \text{ strongly in } X \\ \lim_{n\to\infty} f_n = f \text{ strongly in } L^1(a,b;X) \\ \lim_{n\to\infty} u_n(t) = u(t) \text{ uniformly on } [a,b], \end{cases}$$

where u_n is the unique classical solution of (1.2) corresponding to ξ_n and f_n, and u is the unique C^0-solution of problem (1.2).

Proof . Suppose that $Z : D(Z) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{J(t; t \ge 0)\}$. Then $D(Z)$ is dense in X and Z is a closed operator for each $\xi \in X$ there exists $(\xi_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \xi_n = \xi. \tag{3.18}$$

Moreover, since $C^1([a,b];X)$ is dense in $L^1(a,b;X)$, for each $f \in L^1(a,b;X)$, there exists $(f_n)_{n\in\mathbb{N}}$, such that in $C^1([a,b];X)$

$$\lim_{n\to\infty} f_n = f. \tag{3.19}$$

Since operator Z is a infinitesimal generator of a C_0-semigroup of contractions, so that f is of class C^1 on $[a,b]$, then for each $\xi \in D(Z)$, the problem (1.2) has a unique classical solution. Then for each $n \in \mathbb{N}$, the Cauchy problem

$$\begin{cases} u_n' = Au_n + f_n \\ u_n(a) = \xi_n \end{cases}$$

has a unique classical solution given by the variation of constants formula

$$u_n(t) = J(t-a)\xi_n + \int_a^t J(t-s)f_n(s)ds.$$

But the semigroup is of type (M,ω), which is bounded, and thus

$$\|J(t)\|_{L(X)} \leq Me^{\omega t}$$

for each $t \in \mathbb{R}^+$. We then have

$$\begin{aligned}\|u(t) - u_n(t)\| &\leq \|J(t-a)(\xi - \xi_n)\| + \int_a^t \|J(t-s)f(s) - f_n(s)\| \\ &\leq \|J(t-a)\|_{L(X)}\|\xi - \xi_n\| + \int_a^t \|J(t-s)\|_{L(X)}\|f(s) - f_n(s)\|ds \\ &\leq Me^{\omega(t-a)}\|\xi - \xi_n\| + M\int_a^t e^{\omega(t-s)}\|f(s) - f_n(s)\|ds \\ &\leq Me^{\|\omega\|(b-a)}(\|\xi - \xi_n\| + \int_a^b \|f(s) - f_n(s)\|ds).\end{aligned}$$

Hence the proof is completed. □

Conclusion

It has been demonstrated in this study that results of approximation results of non-homogeneous Cauchy problem of semigroup of linear operators generated by partial contraction mapping with ω-order reversing has been established.

Acknowledgment

The authors acknowledge the management of the university of Ilorin for providing us with a suitable research laboratory and library to enable us carried out this research.

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