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Jordan generalized derivation on generalized module extension Banach algebras

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Abstract

In this paper, we study the problem of describing the form of Jordan generalized d-derivations on generalized module extension Banach algebras. As an application, we characterize the form of Jordan generalized d-derivations on θ-Lau product of Banach algebras. Finally, we show that under some sufficient and necessary conditions, every Jordan generalized d-derivation on a generalized module extension Banach algebra is the sum of a generalized d-derivation and an antiderivation.

Keywords: generalized Jordan d-derivation, generalized module extension Banach algebra, θ-Lau product of Banach algebras
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1 Introduction and Preliminaries

The notion of a generalized derivation was introduced and studied by Brešar [2]. Also, Generalized derivations on prime rings were somewhat later thoroughly studied by Hvala [7]. Later, many researches introduced generalization of Jordan derivations, like Jordan generalized derivation. Suppose that $\mathcal{A}$ is a Banach algebra, $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule and $d\colon \mathcal{A} \longrightarrow \mathcal{M}$ is a linear map. Then, we say a linear map $f\colon \mathcal{A} \longrightarrow \mathcal{M}$ is a *Jordan generalized d-derivation* if

$$f(a \circ b) = f(a) \circ b + a \circ d(b) \quad (a, b \in \mathcal{A}), \tag{1.1}$$

where, $a \circ b = ab + ba$ and $a \circ m = a \cdot m + m \cdot a$, for all $a, b \in \mathcal{A}, m \in \mathcal{M}$.

Letting $f = d$ in (1.1), one can see easily that f is just the classical Jordan derivation.

Many results have been established on those derivations over some rings and algebras with their applications, for example see [6, 8, 10]. Li and Benkonič in [9] studied extensively the structures and properties of Jordan generalized d-derivations on triangular algebras. We note that every generalized d-derivation is a Jordan generalized d-derivation, but the converse is not true in general. It seems natural to ask under what additional assumptions the converse is true. Characterizing various kind of derivation on some algebra constructions has been the subject of several interesting works. It mainly helps to construct new interesting examples of algebras satisfying preassigned conditions.

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In particular, trivial extension algebras, C*-algebras and triangular algebras, have been used by many authors (see [1, 9] and references therein). Li and Benkonič have shown that any Jordan generalized derivation of a triangular algebra is a generalized derivation [9, Theorem 2.5]. Recently in [1, Theorem 2.19], the authors under some conditions, generalized the result by showing that every Jordan generalized d-derivation on trivial extension algebras can be written as the sum of a generalized d-derivation and an antiderivation.

In this paper, we are interested in describing the Jordan generalized d-derivations, generalized d-derivations and antiderivations on generalized module extension Banach algebra $\mathcal{A} \ltimes \mathcal{M}$. It is worthwhile to mention that, this class of Banach algebras includes a wide and important category of Banach algebras such as the θ-Lau product of Banach algebras, semidirect product of Banach algebras, triangular Banach algebras and classical module extension Banach algebras. Motivated by these facts, we also would like to obtain some sufficient and necessary conditions for Jordan generalized derivation on the generalized module extension Banach algebra $\mathcal{A} \ltimes \mathcal{M}$, can be written as the sum of a generalized derivation and an antiderivation.

To do this, we first characterize the general form of Jordan generalized d-derivations on generalized module extension Banach algebra $\mathcal{A} \ltimes \mathcal{M}$. As an application, we give some results for Jordan generalized d-derivations on the category θ-Lau product of Banach algebras. Finally, we characterize in terms of the form of their components when every Jordan generalized d-derivation on a generalized module extension Banach algebra can be written as a sum of a generalized d-derivation and an antiderivation. Indeed, these facts can be regarded as the generalization of results which are given in [1].

2 results on generalized module extension Banach algebras

The original idea of generalized module extension Banach algebras is due to D'Anne and Fontana in an algebraic view, see [3]. Moreover, some homological and cohomological properties of generalized module extension Banach algebra have been studied in [4, 5, 11, 12].

Definition 2.1. Suppose that $\mathcal{A}$ and $\mathcal{M}$ are Banach algebras such that $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule. Then $\mathcal{M}$ is called an *algebraic Banach $\mathcal{A}$-bimodule* if the left and right actions of $\mathcal{M}$ on $\mathcal{A}$ are compatible, i.e. for all $m \in \mathcal{M}$ and $a, b \in \mathcal{A}$,

$$m \cdot (ab) = (m \cdot a)b, \quad (a \cdot m)b = a(m \cdot b), \quad (ab) \cdot m = a(b \cdot m).$$

The *generalized module extension Banach algebra*, denoted by $\mathcal{A} \ltimes \mathcal{M}$, is defined as the set $\mathcal{A} \times \mathcal{M}$ equipped with the norm

$$\|(a, m)\| = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}},$$

and the multiplication

$$(a, m)(b, n) = (ab, a \cdot n + m \cdot b + mn) \qquad (a, b \in \mathcal{A},\ m, n \in \mathcal{M}).$$

In this section, we describe the general form of Jordan generalized d-derivations on generalized module extension Banach algebra $\mathcal{A} \ltimes \mathcal{M}$. Moreover, we give some results for Jordan generalized d-derivations on the category θ-Lau product of Banach algebras.

Following [1], throughout the paper for the linear mapping $f\colon \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$, we represent

$$f(a, m) = (f_{\mathcal{A}}(a) + h_1(m), f_{\mathcal{M}}(a) + h_2(m)) \quad ((a, m) \in \mathcal{A} \ltimes \mathcal{M}), \tag{2.1}$$

where the linear mappings $f_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \mathcal{A}$, $f_{\mathcal{M}}\colon \mathcal{A} \longrightarrow \mathcal{M}$, $h_1\colon \mathcal{M} \longrightarrow \mathcal{A}$ and $h_2\colon \mathcal{M} \longrightarrow \mathcal{M}$ are given by $f_{\mathcal{A}}(a) = (\pi_{\mathcal{A}} \circ f)(a, 0)$, $f_{\mathcal{M}}(a) = (\pi_{\mathcal{M}} \circ f)(a, 0)$, $h_1(m) = (\pi_{\mathcal{A}} \circ f)(0, m)$ and $h_2(m) = (\pi_{\mathcal{M}} \circ f)(0, m)$, respectively, where the maps $\pi_{\mathcal{A}}: \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A}$ and $\pi_{\mathcal{M}}: \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{M}$ are given by $\pi_{\mathcal{A}}(a, m) = a$ and $\pi_{\mathcal{M}}(a, m) = m$.

Also, we consider the linear map $d : \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ as follows:

$$d(a, m) = (d_{\mathcal{A}}(a) + T(m), d_{\mathcal{M}}(a) + S(m)), \quad ((a, m) \in \mathcal{A} \ltimes \mathcal{M}).$$

In the sequel, we obtain a characterization for Jordan generalized d-derivations on generalized module extension Banach algebras. Indeed, the following result can be regarded as a generalization of [1, Lemma 2.1].

Lemma 2.2. A linear map $f\colon \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ is a Jordan generalized d-derivation if and only if for all $a, b \in \mathcal{A}$ and $m, n \in \mathcal{M}$ the following conditions hold:

1. $f_{\mathcal{A}}(a \circ b) = f_{\mathcal{A}}(a) \circ b + a \circ d_{\mathcal{A}}(b)$,
2. $f_{\mathcal{M}}(a \circ b) = f_{\mathcal{M}}(a) \circ b + a \circ d_{\mathcal{M}}(b)$,
3. $h_1(a \circ n) = a \circ T(n), h_1(m \circ b) = h_1(m) \circ b$ and $h_1(m \circ n) = 0$,
4. $h_2(a \circ n) = f_{\mathcal{A}}(a) \circ n + a \circ S(n) + f_{\mathcal{M}}(a) \circ n$,
5. $h_2(m \circ b) = h_2(m) \circ a + m \circ d_{\mathcal{A}}(b) + m \circ d_{\mathcal{M}}(b)$,
6. $h_2(m \circ n) = h_1(m) \circ n + h_2(m) \circ n + m \circ T(n) + m \circ S(n)$.

Proof . First, suppose that the linear map $f : \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ is a Jordan generalized d-derivation. So, for every $(a, m), (b, n) \in \mathcal{A} \ltimes \mathcal{M}$, we have

$$f((a, m) \circ (b, n)) = f(a, m) \circ (b, n) + (a, m) \circ d(b, n). \tag{2.2}$$

On the other hand,

$$\begin{aligned} f(a, m) \circ (b, n) =& (f_{\mathcal{A}}(a) + h_1(m), f_{\mathcal{M}}(a) + h_2(m)) \circ (b, n) \\ =&(f_{\mathcal{A}}(a) \circ b + h_1(m) \circ b, f_{\mathcal{A}}(a) \circ n + h_1(m) \circ n \\ &+ f_{\mathcal{M}}(a) \circ b + h_2(m) \circ b + f_{\mathcal{M}}(a) \circ n + h_2(m) \circ n), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} (a, m) \circ d(b, n) =&(a, m) \circ (d_{\mathcal{A}}(b) + T(n), d_{\mathcal{M}}(b) + S(n)) \\ =&(a \circ d_{\mathcal{A}}(b) + a \circ T(n), a \circ d_{\mathcal{M}}(b) + a \circ S(n) \\ &+ m \circ d_{\mathcal{A}}(b) + m \circ T(n) + m \circ d_{\mathcal{M}}(b) + m \circ S(n)). \end{aligned} \tag{2.4}$$

By adding the equations (2.3) and (2.4), the right hand side of the equation (2.2) is equal to (Γ, Π), where

$$\Gamma = f_{\mathcal{A}}(a) \circ b + a \circ d_{\mathcal{A}}(b) + h_1(m) \circ b + a \circ T(n), \tag{2.5}$$

and

$$\begin{aligned} \Pi =&f_{\mathcal{A}}(a) \circ n + h_1(m) \circ n + f_{\mathcal{M}}(a) \circ b + h_2(m) \circ b \\ &+ f_{\mathcal{M}}(a) \circ n + h_2(m) \circ n + a \circ d_{\mathcal{M}}(b) + a \circ S(n) \\ &+ m \circ d_{\mathcal{A}}(b) + m \circ T(n) + m \circ d_{\mathcal{M}}(b) + m \circ S(n). \end{aligned} \tag{2.6}$$

Also, the left hand side of the equation (2.2) is as follows:

$$\begin{aligned} f((a, m) \circ (b, n)) =&f(a \circ b, a \circ n + m \circ b + m \circ n) \\ =&(f_{\mathcal{A}}(a \circ b) + h_1(a \circ n) + h_1(m \circ b) + h_1(m \circ n), \\ &f_{\mathcal{M}}(a \circ b) + h_2(a \circ n) + h_2(m \circ b) + h_2(m \circ n)). \end{aligned} \tag{2.7}$$

Now, by choosing $m = n = 0$ in the first and second component of (2.5) and (2.7), we have

$$f_{\mathcal{A}}(a \circ b) = f_{\mathcal{A}}(a) \circ b + a \circ d_{\mathcal{A}}(b).$$

Also, put $m = n = 0$ in the second component of (2.6) and (2.7). Then we obtain

$$f_{\mathcal{M}}(a \circ b) = f_{\mathcal{M}}(a) \circ b + a \circ d_{\mathcal{M}}(b).$$

By choosing $m = 0$ in the first component of (2.5) and (2.7), we have

$$\Gamma = f_{\mathcal{A}}(a) \circ b + a \circ d_{\mathcal{A}}(b) + a \circ T(n), \quad f((a, m) \circ (b, n)) = f_{\mathcal{A}}(a \circ b) + h_1(a \circ n).$$

Hence, we get $h_1(a \circ n) = a \circ T(n)$. Also, by choosing $n = 0$ in the first component of (2.5) and (2.7), we conclude that

$$\Gamma = f_{\mathcal{A}}(a) \circ b + a \circ d_{\mathcal{A}}(b) + h_1(m) \circ b, \quad f((a, m) \circ (b, n)) = f_{\mathcal{A}}(a \circ b) + h_1(m \circ b).$$

Moreover, by choosing $a = 0 = b$ in the first component of (2.5) and (2.7), it follows that $\Gamma = 0$ and so

$$f((0,m)\circ(0,n)) = h_1(m\circ n) = 0.$$

Furthermore, by choosing $m = 0$ in the second component of (2.6) and (2.7), we have

$$\Pi = f_{\mathcal{A}}(a)\circ n + f_{\mathcal{M}}(a)\circ n + a\circ S(n) + f_{\mathcal{M}}(a)\circ b + a\circ d_{\mathcal{M}}(b).$$

On the other hand,

$$f((a,m)\circ(b,n)) = f_{\mathcal{M}}(a\circ b) + h_2(a\circ n).$$

That means

$$h_2(a\circ n) = f_{\mathcal{A}}(a)\circ n + a\circ S(n) + f_{\mathcal{M}}(a)\circ n.$$

Now, put $n = 0$ in the second component of (2.6) and (2.7). Hence

$$\Pi = f_{\mathcal{M}}(a)\circ b + h_2(m)\circ b + a\circ d_{\mathcal{M}}(b) + m\circ d_{\mathcal{A}}(b) + m\circ d_{\mathcal{M}}(b).$$

Also, we obtain that $f((a,m)\circ(b,n)) = f_{\mathcal{M}}(a\circ b) + h_2(m\circ b)$. Consequently,

$$h_2(m\circ b) = h_2(m)\circ b + m\circ d_{\mathcal{A}}(b) + m\circ d_{\mathcal{M}}(b).$$

By choosing $a = 0 = b$ in the second component of (2.6) and (2.7), we have

$$\Pi = h_1(m)\circ n + h_2(m)\circ n + m\circ T(n) + m\circ S(n).$$

Moreover, we get

$$f((a,m)\circ(b,n)) = h_2(m\circ n).$$

So, we conclude that

$$h_2(m\circ n) = h_1(m)\circ n + h_2(m)\circ n + m\circ T(n) + m\circ S(n).$$

The converse is also trivial. □

Remark 2.3. Considering M as a Banach $\mathcal{A}$-bimodule with the trivial multiplication, it is straightforward to check that $\mathcal{A}\ltimes\mathcal{B}$ is exactly the trivial extension Banach algebra and so Lemma 2.2 can be regarded as a generalization of [1, Lemma 2.1]. By this fact, we note that the condition (5) in [1, Lemma 2.1] should be modified as follows:

$$m\circ T(n) + h_1(m)\circ n = 0 \qquad (m,n\in\mathcal{M}).$$

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\theta\in\Delta(A)$, where $\Delta(A)$ be the character space of $\mathcal{A}$. We denote the θ-Lau product of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}\times_\theta\mathcal{B}$ with the norm

$$\|(a,b)\| = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}},$$

and the following multiplication:

$$(a_1,b_1)(a_2,b_2) = (a_1a_2, \theta(a_1)b_2 + \theta(a_2)b_1 + b_1b_2) \qquad (a_1,a_2\in\mathcal{A},\ b_1,b_2\in\mathcal{B}).$$

As a consequence, we give the following characterization for Jordan generalized d-derivation on θ-Lau product of Banach algebras.

Theorem 2.5. A linear map $f\colon\mathcal{A}\times_\theta B\longrightarrow\mathcal{A}\times_\theta\mathcal{B}$ is a Jordan generalized d-derivation if and only if for all $a,b\in\mathcal{A}$ and $b_1,b_2\in\mathcal{B}$ the following conditions hold:

1. $f_{\mathcal{A}}(a\cdot b) = f_{\mathcal{A}}(a)\cdot b + a\cdot d_{\mathcal{A}}(b)$,
2. $f_{\mathcal{M}}(a\cdot b) = f_{\mathcal{M}}(a)\cdot b + a\cdot d_{\mathcal{M}}(b)$,
3. $h_1 = T, 2\theta(a)h_1(b_1) = h_1(b_1)a + ah_1(b_1)\, and\, h_1(b_1b_2 + b_2b_1) = 0$,
4. $h_2(2\theta(a)b_1) = 2\theta(f_{\mathcal{A}}(a))b_1 + 2\theta(a)S(b_1) + f_{\mathcal{M}}(a)\circ b_1$,

5. $h_2(2\theta(a)b_1) = 2\theta(a)h_2(b_1) + 2\theta((d_{\mathcal{A}}(a))b_1 + b_1 \circ d_{\mathcal{M}}(a)$,
6. $h_2(b_1 \circ b_2) = 2\theta(h_1(b_1))b_2 + h_2(b_1) \circ b_2 + 2\theta(T(b_2))b_1 + b_1 \circ S(b_2)$.

Proof . First, we can regard the θ-Lau product of Banach algebras $\mathcal{A} \times_\theta \mathcal{B}$ as the generalized module extension Banach algebra $\mathcal{A} \ltimes \mathcal{B}$, where $\mathcal{B}$ is an algebraic Banach $\mathcal{A}$-bimodule with the following module actions:

$$a \cdot b = b \cdot a = \theta(a)b, \qquad (a \in \mathcal{A}, b \in \mathcal{B}).$$

Now, the result immediately follows from Lemma 2.2. □

In the case where $\mathcal{B}$ is a commutative Banach algebra endowed with a bounded approximate identity, we have the following result.

Theorem 2.6. Let $\mathcal{B}$ be a commutative Banach algebra with a bounded approximate identity. Then a linear map $f\colon \mathcal{A} \times_\theta B \longrightarrow \mathcal{A} \times_\theta \mathcal{B}$ is a Jordan generalized d-derivation if and only if for all $a, b \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$ the following conditions hold:

1. $f_{\mathcal{A}}(a \cdot b) = f_{\mathcal{A}}(a) \cdot b + a \cdot d_{\mathcal{A}}(b)$,
2. $f_{\mathcal{M}}(a \cdot b) = f_{\mathcal{M}}(a) \cdot b + a \cdot d_{\mathcal{M}}(b)$,
3. $h_1 = T = 0$,
4. $h_2(2\theta(a)b_1) = 2\theta(f_{\mathcal{A}}(a))b_1 + 2\theta(a)S(b_1) + f_{\mathcal{M}}(a) \circ b_1$,
5. $h_2(2\theta(a)b_1) = 2\theta(a)h_2(b_1) + 2\theta((d_{\mathcal{A}}(a))b_1 + b_1 \circ d_{\mathcal{M}}(a)$,
6. $h_2(b_1 \circ b_2) = h_2(b_1) \circ b_2 + b_1 \circ S(b_2)$.

Proof . Since $\mathcal{B}$ have a bounded approximate identity, by the Cohen's factorization theorem, we conclude that $\mathcal{B}^2 = \mathcal{B}$. Now, since $\mathcal{B}$ is commutative, it follows from Theorem 2.5 that $h_1 = T = 0$ and the proof is complete. □

3 Decomposition of Jordan Generalized Derivations

Recently in [1, Theorem 2.19], the authors under some conditions, show that every Jordan generalized d-derivation on trivial extension algebras can be written as the sum of a generalized d-derivation and an antiderivation. In this section, we would like to obtain some sufficient and necessary conditions for which a Jordan generalized derivation on generalized module extension Banach algebras $\mathcal{A} \ltimes \mathcal{M}$, can be written as the sum of a generalized derivation and an antiderivation. We note that the following results obtain with a similar way of Lemma 2.2 and so left the proof to the reader.

Lemma 3.1. A linear map $f\colon \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ is a generalized d-derivation if and only if for all $a, b \in \mathcal{A}$ and $m, n \in \mathcal{M}$ the following conditions hold:

1. $f_{\mathcal{A}}(ab) = f_{\mathcal{A}}(a)b + ad_{\mathcal{A}}(b)$,
2. $f_{\mathcal{M}}(ab) = f_{\mathcal{M}}(a) \cdot b + a \cdot d_{\mathcal{M}}(b)$,
3. $h_1(a \cdot n) = aT(n), h_1(m \cdot b) = h_1(m)b$ and $h_1(mn) = 0$,
4. $h_2(a \cdot n) = f_{\mathcal{A}}(a) \cdot n + a \cdot S(n) + f_{\mathcal{M}}(a)n$,
5. $h_2(m \cdot b) = h_2(m) \cdot b + m \cdot d_{\mathcal{A}}(b) + md_{\mathcal{M}}(b)$,
6. $h_2(mn) = h_1(m) \cdot n + h_2(m)n + m \cdot T(n) + mS(n)$.

Lemma 3.2. A linear map $f\colon \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ is an antiderivation if and only if for all $a, b \in \mathcal{A}$ and $m, n \in \mathcal{M}$ the following conditions hold:

1. $f_{\mathcal{A}}(ab) = f_{\mathcal{A}}(b)a + bf_{\mathcal{A}}(a)$,
2. $f_{\mathcal{M}}(ab) = f_{\mathcal{M}}(b) \cdot a + b \cdot f_{\mathcal{M}}(a)$,
3. $h_1(a \cdot n) = h_1(n)a, h_1(m \cdot b) = bh_1(m)$ *and* $h_1(mn) = 0$,
4. $h_2(a \cdot n) = h_2(n) \cdot a + n \cdot f_{\mathcal{A}}(a) + nf_{\mathcal{M}}(a)$,
5. $h_2(m \cdot b) = f_{\mathcal{M}}(b)m + b \cdot h_2(m) + f_{\mathcal{A}}(b) \cdot m$,
6. $h_2(mn) = h_1(n) \cdot m + h_2(n)m + n \cdot h_1(m) + nh_2(m)$.

Now, we are ready to give the main result of this section.

Theorem 3.3. Every Jordan generalized derivation on $\mathcal{A} \ltimes \mathcal{M}$ can be written as the sum of a generalized derivation and an antiderivation if and only if the following conditions hold:

1. Every Jordan generalized $d_{\mathcal{M}}$-derivation $g\colon \mathcal{A} \longrightarrow \mathcal{M}$ with the property
$$g(a) \circ n = 0, \quad m \circ d_{\mathcal{M}}(b) = 0 \qquad (a, b \in \mathcal{A}, m, n \in \mathcal{M}),$$
is a sum of a generalized derivation and an antiderivation,
2. Every linear map $h\colon \mathcal{M} \longrightarrow \mathcal{A}$ such that, for all $a \in \mathcal{A}$, $m, n \in \mathcal{M}$, $h(a \circ m) = a \circ h(m)$, $h(m \circ n) = 0$ and $m \circ h(n) + h(m) \circ n = 0$, is a sum of a $\mathcal{K}$ and $\mathcal{K}'$ that satisfy the following properties
$$\mathcal{K}(a \cdot n) = h(n)a, \quad \mathcal{K}(m \cdot b) = b\mathcal{K}(m), \quad \mathcal{K}(mn) = 0,$$
$$\mathcal{K}'(a \cdot n) = ah(n), \quad \mathcal{K}'(m \cdot b) = \mathcal{K}'(m)b, \quad \mathcal{K}'(mn) = 0.$$
3. Every Jordan generalized derivation f on $\mathcal{A} \ltimes \mathcal{M}$ of the form $f(a, m) = (f_{\mathcal{A}}(a), h_2(m))$ can be written as the sum of a generalized derivation and an antiderivation.

Proof . First, Let f be a Jordan generalized derivation on $\mathcal{A} \ltimes \mathcal{M}$. It suffices to show that the condition (1), (2) are hold.

For showing the clause (1), we suppose that $g\colon \mathcal{A} \longrightarrow \mathcal{M}$ is a Jordan generalized $d_{\mathcal{M}}$-derivation. Then by setting $f_{\mathcal{A}} = h_1 = h_2 = S = d_{\mathcal{A}} = T = 0$, $f_{\mathcal{M}} = g$ and $d_{\mathcal{M}} = d$, in Lemma 2.2 we conclude that $(0, g)\colon \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ is a Jordan generalized derivation. Hence, there exist a generalized derivation $(\delta_{\mathcal{A}} + \mathcal{K}', \delta_{\mathcal{M}} + \mathcal{L}')$ and an antiderivation $(D_{\mathcal{A}} + \mathcal{K}, D_{\mathcal{M}} + \mathcal{L})$ such that, for all $a \in \mathcal{A}$,$m \in \mathcal{M}$,
$$\begin{aligned} f(a, m) = (0, g(a)) =&(\delta_{\mathcal{A}}(a) + \mathcal{K}'(m), \delta_{\mathcal{M}}(a) + \mathcal{L}'(m)) + (D_{\mathcal{A}}(a) + \mathcal{K}(m), D_{\mathcal{M}}(a) + \mathcal{L}(m)) \\ =&(\delta_{\mathcal{A}}(a) + D_{\mathcal{A}}(a) + \mathcal{K}(m) + \mathcal{K}'(m), \delta_{\mathcal{M}}(a) + D_{\mathcal{M}}(a) + \mathcal{L}(m) + \mathcal{L}'(m)). \end{aligned}$$

By setting $a = 0$, we have
$$(\mathcal{K}(m) + \mathcal{K}'(m), \mathcal{L}(m) + \mathcal{L}'(m)) = (0, 0).$$

Hence, for every $m \in \mathcal{M}$ we get
$$\mathcal{L}(m) + \mathcal{L}'(m) = 0 \Longrightarrow \mathcal{L} + \mathcal{L}' = 0.$$

So, for all $a \in \mathcal{A}$, we obtain that
$$g(a) = \delta_{\mathcal{M}}(a) + D_{\mathcal{M}}(a) \Longrightarrow g = \delta_{\mathcal{M}} + D_{\mathcal{M}},$$
in which, $\delta_{\mathcal{M}}\colon \mathcal{A} \longrightarrow \mathcal{M}$ is a generalized derivation and $D_{\mathcal{M}}\colon \mathcal{A} \longrightarrow \mathcal{M}$ is an antiderivation.

For the clause (2), assume that $h\colon \mathcal{M} \longrightarrow \mathcal{A}$ is a linear mapping such that for all $a \in \mathcal{A}$, $m, n \in \mathcal{M}$,
$$h(a \circ m) = a \circ h(m), \quad h(m \circ n) = 0, \quad m \circ h(n) + h(m) \circ n = 0.$$

Then by setting $f_{\mathcal{A}} = f_{\mathcal{M}} = h_2 = d_{\mathcal{A}} = d_{\mathcal{M}} = S = 0$ and also $T = h_1 = h$, using Lemma 2.2, one can see that the map $(h, 0)\colon \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ is a Jordan generalized derivation. Hence, there is a generalized derivation $(\delta_{\mathcal{A}} + \mathcal{K}', \delta_{\mathcal{M}} + \mathcal{L}')$ and an antiderivation $(\mathcal{L} + D_{\mathcal{M}}, \mathcal{K} + D_{\mathcal{A}})$ such that for every $a \in \mathcal{A}$ and $m \in \mathcal{M}$ we get
$$\begin{aligned} f(a, m) =&(h(m), 0) = (\delta_{\mathcal{A}}(a) + \mathcal{K}'(m), \delta_{\mathcal{M}}(a) + \mathcal{L}'(m)) + (D_{\mathcal{A}}(a) + \mathcal{K}(m), D_{\mathcal{M}}(a) + \mathcal{L}(m)) \\ =&(\delta_{\mathcal{A}}(a) + D_{\mathcal{A}}(a) + \mathcal{K}(m) + \mathcal{K}'(m), \delta_{\mathcal{M}}(a) + D_{\mathcal{M}}(a) + \mathcal{L}(m) + \mathcal{L}'(m)). \end{aligned}$$

By setting $m = 0$, we have
$$(\delta_{\mathcal{A}}(a) + D_{\mathcal{A}}(a), \delta_{\mathcal{M}}(a) + D_{\mathcal{M}}(a)) = (0, 0).$$

Hence, for every $a \in \mathcal{A}$ we conclude that
$$\delta_{\mathcal{A}}(a) + D_{\mathcal{A}}(a) = 0 = \delta_{\mathcal{M}}(a) + D_{\mathcal{M}}(a) \Longrightarrow h = \mathcal{K} + \mathcal{K}'.$$

Conversely, Let $f\colon \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ be a Jordan generalized d-derivation by the form
$$f(a, m) = (f_{\mathcal{A}}(a) + h_1(m), f_{\mathcal{M}}(a) + h_2(m)).$$

Using Lemma 2.2, the mapping h_1 can play the role of h in the assumption (2), and so $h_1 = \delta + \beta$, where $\delta\colon \mathcal{M} \longrightarrow \mathcal{A}$ is an $\mathcal{A}$-antihomomorphism and $\beta\colon \mathcal{M} \longrightarrow \mathcal{A}$ is an $\mathcal{A}$-homomorphism. Also, $f_{\mathcal{M}}$ can play the role of g in the assumption (1). Hence $f_{\mathcal{M}} = f_1 + f_2$, in which f_1 is a generalized derivation and f_2 is an antiderivation. Therefore, it is possible to write

$$f(a,m) = (f_{\mathcal{A}}(a) + h(m), g(a) + h_2(m)).$$

On the other hand, it follows from Lemma 2.2 that the linear map $(a,m) \longmapsto (f_{\mathcal{A}}(a), h_2(m))$ is a Jordan generalized derivation on $A \ltimes \mathcal{M}$. Then, by the condition (3), it can be written as the sum of a generalized derivation Θ and an antiderivation Δ, i.e, $(f_{\mathcal{A}}(a), h_2(m)) = \Theta(a,m) + \Delta(a,m)$, for all $(a,m) \in \mathcal{A} \ltimes \mathcal{M}$. Therefore,

$$\begin{aligned} f(a,m) =& (f_{\mathcal{A}}(a) + h_1(m), f_{\mathcal{M}}(a) + h_2(m)) \\ =& (f_{\mathcal{A}}(a), h_2(m)) + (h_1(m), f_{\mathcal{M}}(a)) \\ =& \Theta(a,m) + \Delta(a,m) + (\delta(m) + \beta(m), f_1(a) + f_2(a)) \\ =& \underbrace{(\delta(m), f_2(a)) + \Delta(a,m)}_{\text{antiderivation}} + \underbrace{(\beta(m), f_1(a)) + \Theta(a,m)}_{\text{generalized derivation}}. \end{aligned}$$

□

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