

A new approach to Ky Fan type inequalities

Jamal Rooin

Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran

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Abstract

In this article, using a discrete method, we get some new interesting recursive identities concerning arithmetic, geometric, and harmonic means of positive numbers, from which, the most important Ky Fan-type inequalities are handled by induction at once.

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1 Introduction

Throughout this article, let A_n , G_n and H_n be the arithmetic, geometric and harmonic means of $x_1, \dots, x_n > 0$ respectively, i.e.

$$A_n = \frac{1}{n} \sum_{i=1}^n x_i, \quad G_n = \sqrt[n]{\prod_{i=1}^n x_i}, \quad H_n = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}. \quad (1.1)$$

Also, if $x_i \in (0, 1/2]$, we denote by A'_n , G'_n , and H'_n the arithmetic, geometric and harmonic means of $1 - x_1, \dots, 1 - x_n$ respectively, i.e.

$$A'_n = \frac{1}{n} \sum_{i=1}^n (1 - x_i), \quad G'_n = \sqrt[n]{\prod_{i=1}^n (1 - x_i)}, \quad H'_n = \frac{n}{\sum_{i=1}^n \frac{1}{1 - x_i}}. \quad (1.2)$$

When emphasizing, we write $A_n(x_1, \dots, x_n)$ instead of A_n , and so on. The following inequalities are the most important inequalities concerning these means:

$$G_n \leq A_n, \quad (1.3)$$

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}, \quad (1.4)$$

$$A'_n - G'_n \leq A_n - G_n, \quad (1.5)$$

$$\frac{G'_n}{H'_n} \leq \frac{G_n}{H_n}, \quad (1.6)$$

*Corresponding author
Email address: rooin@iasbs.ac.ir (Jamal Rooin)

and

$$\frac{1}{H'_n} - \frac{1}{G'_n} \leq \frac{1}{H_n} - \frac{1}{G_n}. \quad (1.7)$$

In (1.3), all x_i 's are positive, and in the others all belong to $(0, \frac{1}{2}]$. Moreover, equality holds in each of them if and only if $x_1 = \dots = x_n$. In literature, (1.3) and (1.4) are known as AGM and Ky Fan inequalities respectively, and all inequalities above and from the above kinds are referred as Ky Fan type inequalities.

There are several interesting proofs for the AGM inequality (1.3) and more than fifty of them have been mentioned in [5] in order of their appearances, see also [8] and [7] for further recent proofs.

The Ky Fan inequality (1.4), was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [4, p. 5], and from then it has evoked the interest of several mathematicians and in numerous articles new proofs, extensions, refinements and various related results have been published; see the survey paper [3] and the references therein, see also [8, 9, 6] and [10].

Among these remarkable results, the additive analogue of Ky Fan's inequality (1.5) and also the inequality (1.7) are due to H. Alzer, and the inequality (1.6) is due to Wang-Wang; see [1], [2] and [11] respectively, and see also [3].

2 Results

In this section, first using a simple lemma, we establish some new recursive identities concerning arithmetic, geometric and harmonic means of positive numbers, and then, as applications, we prove the inequalities (1.3, 1.4, 1.6, 1.7) by induction on n . Unfortunately, this method is not so straightly applied for the inequality (1.5), which remains a challenging problem yet. Although this method is similar to that of [8], but the obtained recursive identities and results are quite different from each others. All we need is the following elementary lemma.

Lemma 2.1. *If a, b are positive real numbers and $n \geq 2$ is a positive integer, then*

$$\begin{aligned} (n-1)a + b - na^{\frac{n-1}{n}}b^{\frac{1}{n}} &= \left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right)^2 \sum_{k=1}^{n-1} \sum_{l=1}^k a^{\frac{n-2-k+l}{n}} b^{\frac{k-l}{n}} \\ &= \left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right)^2 a^{\frac{n-2}{n}} \sum_{k=1}^{n-1} \left(\frac{1 - \left(\frac{b}{a}\right)^{\frac{k}{n}}}{1 - \left(\frac{b}{a}\right)^{\frac{1}{n}}} \right), \end{aligned} \quad (2.1)$$

where we set $\frac{1 - \left(\frac{b}{a}\right)^{\frac{k}{n}}}{1 - \left(\frac{b}{a}\right)^{\frac{1}{n}}} = k$, if $a = b$.

Proof . We have

$$\begin{aligned} (n-1)a + b - na^{\frac{n-1}{n}}b^{\frac{1}{n}} &= (n-1)a^{\frac{n-1}{n}}a^{\frac{1}{n}} + b^{\frac{n-1}{n}}b^{\frac{1}{n}} - na^{\frac{n-1}{n}}b^{\frac{1}{n}} \\ &= (n-1)a^{\frac{n-1}{n}}\left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right) - \left(a^{\frac{n-1}{n}} - b^{\frac{n-1}{n}}\right)b^{\frac{1}{n}} \\ &= (n-1)a^{\frac{n-1}{n}}\left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right) - b^{\frac{1}{n}}\left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right) \sum_{k=1}^{n-1} b^{\frac{k-1}{n}} a^{\frac{n-1-k}{n}} \\ &= \left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right) \left(\sum_{k=1}^{n-1} a^{\frac{k}{n}} a^{\frac{n-1-k}{n}} - \sum_{k=1}^{n-1} b^{\frac{k}{n}} a^{\frac{n-1-k}{n}} \right) \\ &= \left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right) \sum_{k=1}^{n-1} a^{\frac{n-1-k}{n}} \left(a^{\frac{k}{n}} - b^{\frac{k}{n}}\right) \\ &= \left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right)^2 \sum_{k=1}^{n-1} a^{\frac{n-1-k}{n}} \sum_{l=1}^k a^{\frac{l-1}{n}} b^{\frac{k-l}{n}} \\ &= \left(a^{\frac{1}{n}} - b^{\frac{1}{n}}\right)^2 \sum_{k=1}^{n-1} \sum_{l=1}^k a^{\frac{n-2-k+l}{n}} b^{\frac{k-l}{n}}. \end{aligned}$$

This completes the proof. \square

Corollary 2.2. *If $n \geq 2$, then*

$$\begin{aligned} A_n - G_n &= \left(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}} \right) x_n^{\frac{1}{n}} + \frac{\left(x_n^{\frac{1}{n}} - A_{n-1}^{\frac{1}{n}} \right)^2}{n} \sum_{k=1}^{n-1} \sum_{l=1}^k A_{n-1}^{\frac{n-2-k+l}{n}} x_n^{\frac{k-l}{n}} \\ &= \left(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}} \right) x_n^{\frac{1}{n}} + \frac{\left(x_n^{\frac{1}{n}} - A_{n-1}^{\frac{1}{n}} \right)^2}{n} A_{n-1}^{\frac{n-2}{n}} \sum_{k=1}^{n-1} \left(\frac{1 - \left(\frac{x_n}{A_{n-1}} \right)^{\frac{k}{n}}}{1 - \left(\frac{x_n}{A_{n-1}} \right)^{\frac{1}{n}}} \right), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \frac{A_n}{G_n} &= \left(\frac{A_{n-1}}{G_{n-1}} \right)^{\frac{n-1}{n}} + \left(\frac{A_{n-1}}{G_{n-1}} \right)^{\frac{n-2}{n}} \left(\frac{x_n}{G_{n-1}} \right)^{\frac{1}{n}} \frac{\left[1 - \left(\frac{A_{n-1}}{x_n} \right)^{\frac{1}{n}} \right]^2}{n} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \left(\frac{x_n}{A_{n-1}} \right)^{\frac{l}{n}} \\ &= \left(\frac{A_{n-1}}{G_{n-1}} \right)^{\frac{n-1}{n}} + \left(\frac{A_{n-1}}{G_{n-1}} \right)^{\frac{n-2}{n}} \left(\frac{x_n}{G_{n-1}} \right)^{\frac{1}{n}} \frac{\left[1 - \left(\frac{A_{n-1}}{x_n} \right)^{\frac{1}{n}} \right]^2}{n} \sum_{k=1}^{n-1} \left(\frac{1 - \left(\frac{x_n}{A_{n-1}} \right)^{\frac{k}{n}}}{1 - \left(\frac{x_n}{A_{n-1}} \right)^{\frac{1}{n}}} \right), \end{aligned} \tag{2.3}$$

where we set $\frac{1 - \left(\frac{x_n}{A_{n-1}} \right)^{\frac{k}{n}}}{1 - \left(\frac{x_n}{A_{n-1}} \right)^{\frac{1}{n}}} = k$, if $x_n = A_{n-1}$.

Proof . Taking $a = A_{n-1}$ and $b = x_n$ in Lemma 2.1, and taking into account that $A_n = \frac{1}{n}[(n-1)A_{n-1} + x_n]$ and $G_n = G_{n-1}^{\frac{n-1}{n}} x_n^{\frac{1}{n}}$, we have

$$\begin{aligned} A_n - G_n &= \frac{1}{n}[(n-1)A_{n-1} + x_n] - G_{n-1}^{\frac{n-1}{n}} x_n^{\frac{1}{n}} \\ &= \frac{1}{n} \left[(n-1)A_{n-1} + x_n - nA_{n-1}^{\frac{n-1}{n}} x_n^{\frac{1}{n}} \right] + \left(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}} \right) x_n^{\frac{1}{n}} \\ &= \left(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}} \right) x_n^{\frac{1}{n}} + \frac{\left(x_n^{\frac{1}{n}} - A_{n-1}^{\frac{1}{n}} \right)^2}{n} \sum_{k=1}^{n-1} \sum_{l=1}^k A_{n-1}^{\frac{n-2-k+l}{n}} x_n^{\frac{k-l}{n}}. \end{aligned}$$

The identity (2.3) is achieved in the same manner. \square

Corollary 2.3. *If $n \geq 2$, then*

$$\begin{aligned} \frac{1}{H_n} - \frac{1}{G_n} &= \left(\frac{1}{H_{n-1}^{\frac{n-1}{n}}} - \frac{1}{G_{n-1}^{\frac{n-1}{n}}} \right) \frac{1}{x_n^{\frac{1}{n}}} + \frac{\left[1 - \left(\frac{x_n}{H_{n-1}} \right)^{\frac{1}{n}} \right]^2}{n x_n^{\frac{2}{n}} H_{n-1}^{\frac{n-2}{n}}} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \left(\frac{H_{n-1}}{x_n} \right)^{\frac{l}{n}} \\ &= \left(\frac{1}{H_{n-1}^{\frac{n-1}{n}}} - \frac{1}{G_{n-1}^{\frac{n-1}{n}}} \right) \frac{1}{x_n^{\frac{1}{n}}} + \frac{\left[1 - \left(\frac{x_n}{H_{n-1}} \right)^{\frac{1}{n}} \right]^2}{n x_n^{\frac{2}{n}} H_{n-1}^{\frac{n-2}{n}}} \sum_{k=1}^{n-1} \left(\frac{1 - \left(\frac{H_{n-1}}{x_n} \right)^{\frac{k}{n}}}{1 - \left(\frac{H_{n-1}}{x_n} \right)^{\frac{1}{n}}} \right), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \frac{G_n}{H_n} &= \left(\frac{G_{n-1}}{H_{n-1}}\right)^{\frac{n-1}{n}} + \left(\frac{G_{n-1}}{H_{n-1}}\right)^{\frac{n-2}{n}} \left(\frac{G_{n-1}}{x_n}\right)^{\frac{1}{n}} \frac{\left[1 - \left(\frac{x_n}{H_{n-1}}\right)^{\frac{1}{n}}\right]^2}{n} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \left(\frac{H_{n-1}}{x_n}\right)^{\frac{l}{n}} \\ &= \left(\frac{G_{n-1}}{H_{n-1}}\right)^{\frac{n-1}{n}} + \left(\frac{G_{n-1}}{H_{n-1}}\right)^{\frac{n-2}{n}} \left(\frac{G_{n-1}}{x_n}\right)^{\frac{1}{n}} \frac{\left[1 - \left(\frac{x_n}{H_{n-1}}\right)^{\frac{1}{n}}\right]^2}{n} \sum_{k=1}^{n-1} \left(\frac{1 - \left(\frac{H_{n-1}}{x_n}\right)^{\frac{k}{n}}}{1 - \left(\frac{H_{n-1}}{x_n}\right)^{\frac{1}{n}}}\right), \end{aligned} \quad (2.5)$$

where we set $\frac{1 - \left(\frac{H_{n-1}}{x_n}\right)^{\frac{k}{n}}}{1 - \left(\frac{H_{n-1}}{x_n}\right)^{\frac{1}{n}}} = k$, if $x_n = H_{n-1}$.

Proof . Clearly for each k ,

$$A_k\left(\frac{1}{x_1}, \dots, \frac{1}{x_k}\right) = \frac{1}{H_k(x_1, \dots, x_k)},$$

and

$$G_k\left(\frac{1}{x_1}, \dots, \frac{1}{x_k}\right) = \frac{1}{G_k(x_1, \dots, x_k)}.$$

Now, changing the roles of x_i 's by $\frac{1}{x_i}$'s, the identities (2.4) and (2.5) follow from (2.2) and (2.3) respectively by replacing $A_n, G_n, A_{n-1}, G_{n-1}$ and x_n , by $\frac{1}{H_n}, \frac{1}{G_n}, \frac{1}{H_{n-1}}, \frac{1}{G_{n-1}}$ and $\frac{1}{x_n}$ respectively. \square

Now, we prove Inequalities (1.3,1.4) and (1.6,1.7) by induction on n . If $n = 1$, there is nothing to prove. Suppose $n \geq 2$ and the assertions hold for $n - 1$. If $x_1 = \dots = x_n$, then obviously equality holds in each of these inequalities. Let not all x_i 's be equal.

For proving (1.3) and (1.4), arrange x_i 's so that $x_n = \max_{1 \leq i \leq n} x_i$. Now, using the induction hypothesis

$$A_{n-1} \geq G_{n-1},$$

and considering $x_n > A_{n-1}$, the AGM inequality (1.3) follows from (2.2) with strict inequality. By Corollary (2.2), we have

$$\frac{A'_n}{G'_n} = \left(\frac{A'_{n-1}}{G'_{n-1}}\right)^{\frac{n-1}{n}} + \left(\frac{A'_{n-1}}{G'_{n-1}}\right)^{\frac{n-2}{n}} \left(\frac{1-x_n}{G'_{n-1}}\right)^{\frac{1}{n}} \frac{\left[1 - \left(\frac{A'_{n-1}}{1-x_n}\right)^{\frac{1}{n}}\right]^2}{n} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \left(\frac{1-x_n}{A'_{n-1}}\right)^{\frac{l}{n}}, \quad (2.6)$$

where $A'_{n-1} = \frac{\sum_{i=1}^{n-1} (1-x_i)}{n-1}$ and $G'_{n-1} = \prod_{i=1}^{n-1} (1-x_i)^{\frac{1}{n-1}}$. But,

$$\frac{x_n}{G_{n-1}} > 1 > \frac{1-x_n}{G'_{n-1}}, \quad \frac{x_n}{A_{n-1}} > 1 > \frac{1-x_n}{A'_{n-1}}, \quad 1-x_n \geq x_n,$$

and

$$\begin{aligned} 1 - \left(\frac{A_{n-1}}{x_n}\right)^{\frac{1}{n}} &= \frac{1 - \frac{A_{n-1}}{x_n}}{\sum_{k=0}^{n-1} \left(\frac{A_{n-1}}{x_n}\right)^{\frac{k}{n}}} = \frac{x_n - A_{n-1}}{x_n \sum_{k=0}^{n-1} \left(\frac{A_{n-1}}{x_n}\right)^{\frac{k}{n}}} \\ &> \frac{x_n - A_{n-1}}{(1-x_n) \sum_{k=0}^{n-1} \left(\frac{A'_{n-1}}{1-x_n}\right)^{\frac{k}{n}}} = \frac{\frac{A'_{n-1}}{1-x_n} - 1}{\sum_{k=0}^{n-1} \left(\frac{A'_{n-1}}{1-x_n}\right)^{\frac{k}{n}}} = \left(\frac{A'_{n-1}}{1-x_n}\right)^{\frac{1}{n}} - 1 > 0. \end{aligned}$$

Now, considering the induction hypothesis

$$\frac{A_{n-1}}{G_{n-1}} \geq \frac{A'_{n-1}}{G'_{n-1}},$$

the Ky Fan's inequality (1.4) follows with strict inequality by comparing (2.3) with (2.6).

For proving of (1.6) and (1.7), arrange x_i 's so that $x_n = \min_{1 \leq i \leq n} x_i$. By Corollary (2.3), we have

$$\frac{G'_n}{H'_n} = \left(\frac{G'_{n-1}}{H'_{n-1}}\right)^{\frac{n-1}{n}} + \left(\frac{G'_{n-1}}{H'_{n-1}}\right)^{\frac{n-2}{n}} \left(\frac{G'_{n-1}}{1-x_n}\right)^{\frac{1}{n}} \frac{\left[1 - \left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{1}{n}}\right]^2}{n} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \left(\frac{H'_{n-1}}{1-x_n}\right)^{\frac{l}{n}} \quad (2.7)$$

and

$$\frac{1}{H'_n} - \frac{1}{G'_n} = \left(\frac{1}{H'_{n-1} \frac{n-1}{n}} - \frac{1}{G'_{n-1} \frac{n-1}{n}}\right) \frac{1}{(1-x_n)^{\frac{1}{n}}} + \frac{\left[1 - \left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{1}{n}}\right]^2}{n(1-x_n)^{\frac{2}{n}} H'_{n-1} \frac{n-2}{n}} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \left(\frac{H'_{n-1}}{1-x_n}\right)^{\frac{l}{n}}, \quad (2.8)$$

where $H'_{n-1} = \frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{1-x_i}}$ and $G'_{n-1} = \prod_{i=1}^{n-1} (1-x_i)^{\frac{1}{n-1}}$. But,

$$\frac{G_{n-1}}{x_n} > 1 > \frac{G'_{n-1}}{1-x_n}, \quad \frac{H_{n-1}}{x_n} > 1 > \frac{H'_{n-1}}{1-x_n}, \quad (2.9)$$

and

$$\begin{aligned} 1 - \left(\frac{x_n}{H_{n-1}}\right)^{\frac{1}{n}} &= \frac{1 - \frac{x_n}{H_{n-1}}}{\sum_{k=0}^{n-1} \left(\frac{x_n}{H_{n-1}}\right)^{\frac{k}{n}}} \\ &= \frac{\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{x_i - x_n}{x_i}}{\sum_{k=0}^{n-1} \left(\frac{x_n}{H_{n-1}}\right)^{\frac{k}{n}}} \\ &> \frac{\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{x_i - x_n}{1-x_i}}{\sum_{k=0}^{n-1} \left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{k}{n}}} \\ &= \frac{\frac{1-x_n}{H'_{n-1}} - 1}{\sum_{k=0}^{n-1} \left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{k}{n}}} \\ &= \left(\frac{1-x_n}{H'_{n-1}}\right)^{\frac{1}{n}} - 1 > 0. \end{aligned} \quad (2.10)$$

Now, considering the induction hypothesis

$$\frac{G_{n-1}}{H_{n-1}} \geq \frac{G'_{n-1}}{H'_{n-1}},$$

by comparing (2.7) and (2.5), we get (1.6) with strict inequality.

Finally, by the mean value theorem and the induction hypothesis

$$\frac{1}{H_{n-1}} - \frac{1}{G_{n-1}} \geq \frac{1}{H'_{n-1}} - \frac{1}{G'_{n-1}} \geq 0,$$

we have

$$\begin{aligned} \left(\frac{1}{H_{n-1} \frac{n-1}{n}} - \frac{1}{G_{n-1} \frac{n-1}{n}}\right) \frac{1}{x_n^{\frac{1}{n}}} &= \left(\frac{n-1}{n}\right) \left(\frac{1}{H_{n-1}} - \frac{1}{G_{n-1}}\right) \left(\frac{1}{x_n \eta_{n-1}}\right)^{\frac{1}{n}} \\ &\geq \left(\frac{n-1}{n}\right) \left(\frac{1}{H'_{n-1}} - \frac{1}{G'_{n-1}}\right) \left(\frac{1}{(1-x_n) \eta'_{n-1}}\right)^{\frac{1}{n}} \\ &= \left(\frac{1}{H'_{n-1} \frac{n-1}{n}} - \frac{1}{G'_{n-1} \frac{n-1}{n}}\right) \frac{1}{(1-x_n)^{\frac{1}{n}}}, \end{aligned}$$

where $\eta_{n-1} \in [\frac{1}{G_{n-1}}, \frac{1}{H_{n-1}}]$ and $\eta'_{n-1} \in [\frac{1}{G'_{n-1}}, \frac{1}{H'_{n-1}}]$, since

$$x_n \eta_{n-1} \leq \frac{x_n}{H_{n-1}} < 1 < \frac{1-x_n}{G'_{n-1}} \leq (1-x_n) \eta'_{n-1}.$$

So, considering the second inequality in (2.9),

$$1-x_n > x_n, \quad H'_{n-1} \geq \frac{1}{2} \geq H_{n-1},$$

and (2.10), comparing (2.4) and (2.8), we get strict inequality in (1.7). This completes the proof.

Remark 2.4. It is noted that the identity (2.3) gives us the exact error of the Popoviciu inequality

$$\frac{A_n}{G_n} \geq \left(\frac{A_{n-1}}{G_{n-1}} \right)^{\frac{n-1}{n}},$$

as a finite series of nonnegative terms in the case of equal weights; see [5] for details. Also, by the mean value theorem, there exists $\theta_{n-1} \in [G_{n-1}, A_{n-1}]$, such that

$$\left(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}} \right) x_n^{\frac{1}{n}} = \frac{n-1}{n} (A_{n-1} - G_{n-1}) \left(\frac{x_n}{\theta_{n-1}} \right)^{\frac{1}{n}}.$$

So, if $x_n \geq A_{n-1}$, then $\frac{x_n}{\theta_{n-1}} \geq 1$, and (2.2) gives us a refinement to the Rado inequality [5]:

$$A_n - G_n \geq \frac{n-1}{n} (A_{n-1} - G_{n-1}),$$

as follows

$$\begin{aligned} A_n - G_n &\geq \frac{n-1}{n} (A_{n-1} - G_{n-1}) + \frac{\left(x_n^{\frac{1}{n}} - A_{n-1}^{\frac{1}{n}} \right)^2}{n} A_{n-1}^{\frac{n-2}{n}} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \left(\frac{x_n}{A_{n-1}} \right)^{\frac{l}{n}} \\ &\geq \frac{n-1}{n} (A_{n-1} - G_{n-1}) + \frac{(n-1) \left(x_n^{\frac{1}{n}} - A_{n-1}^{\frac{1}{n}} \right)^2}{2} A_{n-1}^{\frac{n-2}{n}}. \end{aligned}$$

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