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NON-ARCHIMEDEAN STABILITY OF CAUCHY-JENSEN TYPE FUNCTIONAL EQUATION

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Abstract. In this paper we investigate the generalized Hyers-Ulam stability of the following Cauchy-Jensen type functional equation

$$Q\Big(\frac{x+y}{2}+z\Big)+Q\Big(\frac{x+z}{2}+y\Big)+Q\Big(\frac{z+y}{2}+x\Big)=2[Q(x)+Q(y)+Q(z)]$$

in non-Archimedean spaces .

1. Introduction

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?*.

If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [35] in 1940. In the next year, Hyres [11] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [30] proved a generalization of Hyres's theorem for additive mappings. The result of Rassias has influenced the development of what is now called the *Hyers-Ulam-Rassias stability problem* for functional equations. In 1994, a generalization of Rassias's theorem was obtained by Găvruta [9] by replacing the bound $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\phi(x,y)$.

The functional equation

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [34] for mappings $f: X\to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [6] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

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The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([1]- [4], [8], [12]-[15], [18]- [26],[28]- [24]).
In 1897, Hensel [10] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [7], [16], [17], [27]).

2. Preliminaries

A *valuation* is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r||s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq \max\{|r|, |s|\}.$$

A field $\mathbb{K}$ is called a *valued field* if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.
Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}$$

for all $r, s \in \mathbb{K}$, then the function $|\cdot|$ is called a *non-Archimedean valuation* and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 2.1. Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if the following conditions hold:
(a) $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$;
(b) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;
(c) the strong triangle inequality holds:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 2.2. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.
(a) A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a *Cauchy sequence* iff, the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.
(b) The sequence $\{x_n\}$ is said to be *convergent* if, for any $\varepsilon > 0$, there are a positive integer N and $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denote by $\lim_{n\to\infty} x_n = x$.
(c) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

Definition 2.3. Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:
(a) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(c) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 2.4. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty \tag{2.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n_0 \geq n_0$*;*
(b) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*
(c) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$*;*
(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

In [25], Nejati introduced the following functional equation:

$$Q\Big(\frac{x+y}{2} + z\Big) + Q\Big(\frac{x+z}{2} + y\Big) + Q\Big(\frac{z+y}{2} + x\Big) = 2[Q(x) + Q(y) + Q(z)]. \tag{2.2}$$

In this paper, we prove the generalized Hyers-Ulam stability of functional equation (2.2) in non-Archimedean spaces.

3. Non-Archimedean Stability of Eq. (2.2): Direct Method

Theorem 3.1. *Let $\zeta : G^3 \to [0, +\infty)$ be a mapping such that*

$$\lim_{n\to\infty} |2|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\Big) = 0 \tag{3.1}$$

for all $x, y, z \in G$ and let for each $x \in G$ the limit

$$\Theta(x) = \lim_{n\to\infty} max\Big\{|2|^k \zeta\Big(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\Big);\ 0 \leq k < n\Big\} \tag{3.2}$$

exists. Suppose that $Q : G \to X$ is a mapping satisfies

$$\begin{aligned} &\Big\|Q\Big(\frac{x+y}{2} + z\Big) + Q\Big(\frac{x+z}{2} + y\Big) \\ &+Q\Big(\frac{z+y}{2} + x\Big) - 2[Q(x) + Q(y) + Q(z)]\Big\| \\ &\leq \zeta(x, y, z). \end{aligned} \tag{3.3}$$

Then

$$\Im(x) := \lim_{n\to\infty} 2^n Q\Big(\frac{x}{2^n}\Big) \tag{3.4}$$

exists for all $x \in G$ and defines an additive mapping $\Im : G \to X$ such that

$$||Q(x) - \Im(x)|| \leq \Theta(x) \tag{3.5}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} max\Big\{|2|^k \zeta\Big(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\Big);\ j \leq k < n + j\Big\} = 0 \tag{3.6}$$

then T is the unique additive mapping satisfying (3.5).

Proof. Putting $x = y = z$ in (3.11), we get

$$\Big\|2Q\Big(\frac{x}{2}\Big) - Q(x)\Big\| \leq \zeta\Big(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\Big) \tag{3.7}$$

for all $x \in G$. Replacing x by $\frac{x}{2^n}$ in (3.7), we obtain

$$\Big\|2^{n+1}Q\Big(\frac{x}{2^{n+1}}\Big) - 2^n Q\Big(\frac{x}{2^n}\Big)\Big\| \leq |2|^n \zeta\Big(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\Big) \tag{3.8}$$

It follows from (3.1) and (3.8) that the sequence $\left\{2^nQ\left(\frac{x}{2^n}\right)\right\}_{n\geq 1}$ is a Cauchy sequence. Since X is complete, so $\left\{2^nQ\left(\frac{x}{2^n}\right)\right\}_{n\geq 1}$ is convergent. Set

$$\Im(x) := \lim_{n\to\infty} 2^nQ\left(\frac{x}{2^n}\right).$$

Using induction one can show that

$$\left\|2^nQ\left(\frac{x}{2^n}\right) - Q(x)\right\| \leq max\left\{|2|^k\zeta\left(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\right);\ 0 \leq k < n\right\}. \tag{3.9}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (3.9), and using (3.2), one obtains (3.5). By (3.1) and (3.11), we get

$$\begin{aligned}
& \left\|\Im\left(\frac{x+y}{2}+z\right)+\Im\left(\frac{x+z}{2}+y\right)+\Im\left(\frac{z+y}{2}+x\right)-2[\Im(x)+\Im(y)+\Im(z)]\right\| \\
= & \lim_{n\to\infty}\left\|2^n\left[Q\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^n}\right)+Q\left(\frac{x+z}{2^{n+1}}+\frac{y}{2^n}\right)+Q\left(\frac{z+y}{2^{n+1}}+\frac{x}{2^n}\right)\right]\right. \\
- & \left. 2^{n+1}\left[Q\left(\frac{x}{2^n}\right)+Q\left(\frac{y}{2^n}\right)+Q\left(\frac{z}{2^n}\right)\right]\right\| \\
\leq & \lim_{n\to\infty}|2|^n\zeta\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) \\
= & \ 0
\end{aligned}$$

for all $x, y, z \in G$. Therefore the function $\Im : G \to X$ satisfies (2.2). To prove the uniqueness property of $\Im$, let $\Re : G \to X$ be another function satisfying (3.5). Then

$$\begin{aligned}
\left\|\Im(x)-\Re(x)\right\| &= \lim_{n\to\infty}|2|^n\left\|\Im\left(\frac{x}{2^n}\right)-\Re\left(\frac{x}{2^n}\right)\right\| \\
&\leq \lim_{k\to\infty}|2|^n max\left\{\left\|\Im\left(\frac{x}{2^n}\right)-Q\left(\frac{x}{2^n}\right)\right\|,\left\|Q\left(\frac{x}{2^n}\right)-\Re\left(\frac{x}{2^n}\right)\right\|\right\} \\
&\leq \lim_{j\to\infty}\lim_{n\to\infty} max\left\{|2|^k\zeta\left(\frac{x}{2^k},\frac{x}{2^k},\frac{x}{2^k}\right);\ j\leq k<n+j\right\} \\
&= 0
\end{aligned}$$

for all $x \in G$. Therefore $\Im = \Re$, and the proof is complete. □

Corollary 3.2. *Let $\xi : [0, \infty) \to [0, \infty)$ be a mapping satisfying*

$$\xi(|2|^{-1}t) \leq \xi(|2|^{-1})\xi(t)\ (t \geq 0) \quad \xi(|2|^{-1}) < |2|^{-1}. \tag{3.10}$$

Let $\kappa > 0$ and $Q : G \to X$ be a mapping satisfying

$$\begin{aligned}
&\left\|Q\left(\frac{x+y}{2}+z\right)+Q\left(\frac{x+z}{2}+y\right)\right. \\
&\left.+Q\left(\frac{z+y}{2}+x\right)-2[Q(x)+Q(y)+Q(z)]\right\| \\
&\leq \kappa(\xi(|x|)+\xi(|y|)+\xi(|z|)).
\end{aligned} \tag{3.11}$$

for all $x, y, z \in G$. Then there exists a unique additive mapping $\Im : G \to X$ such that

$$||Q(x) - \Im(x)|| \leq 3\kappa\xi(|x|). \tag{3.12}$$

Proof. Defining $\zeta : G^3 \to [0,\infty)$ by $\zeta(x,y,z) := \kappa(\xi(|x|)+\xi(|y|)+\xi(|z|))$, then, we have

$$\lim_{n\to\infty} |2|^n \zeta\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) \le \lim_{n\to\infty} (|2|\xi(|2|^{-1}))^n \zeta(x,y,z) = 0 \tag{3.13}$$

for all $x,y,z \in G$. The last equality comes form fact that $|2|\xi(|2|^{-1}) < 1$. On the other hand

$$\begin{aligned}\Theta(x) &= \lim_{n\to\infty} max\Big\{|2|^k \zeta\Big(\frac{x}{2^k},\frac{x}{2^k},\frac{x}{2^k}\Big);\ 0 \le k < n\Big\} \\ &= \zeta(x,x,x) \\ &= 3\kappa\xi(|x|)\end{aligned} \tag{3.14}$$

exists for all $x \in G$. Also

$$\begin{aligned}&\lim_{j\to\infty}\lim_{n\to\infty} max\Big\{|2|^k \zeta\Big(\frac{x}{2^k},\frac{x}{2^k},\frac{x}{2^k}\Big);\ j \le k < n+j\Big\} \\ &= \lim_{j\to\infty} |2|^j \zeta\Big(\frac{x}{2^j},\frac{x}{2^j},\frac{x}{2^j}\Big) \\ &= 0.\end{aligned} \tag{3.15}$$

Applying Theorem (3.1), we get desired result. □

Theorem 3.3. *Let $\zeta : G^3 \to [0,+\infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{\zeta(2^n x, 2^n y, 2^n z)}{|2|^n} = 0 \tag{3.16}$$

for all $x,y,z \in G$ and let for each $x \in G$ the limit

$$\Theta(x) = \lim_{n\to\infty} max\Big\{\frac{\zeta(2^k x, 2^k x, 2^k x)}{|2|^k};\ 0 \le k < n\Big\} \tag{3.17}$$

exists. Suppose that $f : G \to X$ is a mapping satisfies

$$\begin{aligned}&\Big\|Q\Big(\frac{x+y}{2}+z\Big)+Q\Big(\frac{x+z}{2}+y\Big) \\ &+Q\Big(\frac{z+y}{2}+x\Big)-2[Q(x)+Q(y)+Q(z)]\Big\| \\ &\le \zeta(x,y,z).\end{aligned} \tag{3.18}$$

Then

$$T(x) := \lim_{n\to\infty} \frac{Q(2^n x)}{2^n} \tag{3.19}$$

exists for all $x \in G$ and defines an additive mapping $\Im : G \to X$, such that

$$||Q(x) - \Im(x)|| \le \frac{1}{|2|}\Theta(x) \tag{3.20}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} max\Big\{\frac{\zeta(2^k x, 2^k x, 2^k x)}{|2|^k};\ j \le k < n+j\Big\} = 0, \tag{3.21}$$

then T is the unique mapping satisfying (3.20).

Proof. Putting $x = y = z$ in (3.18), we get

$$\Big\| Q(x) - \frac{Q(2x)}{2} \Big\| \leq \frac{\zeta(x,x,x)}{|2|} \tag{3.22}$$

for all $x \in G$. Replacing x by $2^n x$ in (3.22), we obtain

$$\Big\| \frac{Q(2^n x)}{2^n} - \frac{Q(2^{n+1}x)}{2^{n+1}} \Big\| \leq \frac{\zeta(2^n x, 2^n x, 2^n x)}{|2|^{n+1}}. \tag{3.23}$$

It follows from (3.16) and (3.23) that the sequence $\Big\{ \frac{Q(2^n x)}{2^n} \Big\}_{n \geq 1}$ is convergent. Set $\Im(x) := \lim_{n\to\infty} \frac{Q(2^n x)}{2^n}$. On the other hand, it follows from (3.23) that

$$\begin{aligned} \Big\| \frac{Q(2^p x)}{2^p} - \frac{Q(2^q x)}{2^q} \Big\| &= \Big\| \sum_{k=p}^{q-1} \frac{Q(2^k x)}{2^k} - \frac{Q(2^{k+1}x)}{2^{k+1}} \Big\| \\ &\leq max\Big\{ \Big\| \frac{Q(2^k x)}{2^k} - \frac{Q(2^{k+1}x)}{2^{k+1}} \Big\| \ ; \ p \leq k < q \Big\} \\ &\leq \frac{1}{|2|} max\Big\{ \frac{\zeta(2^k x, 2^k x, 2^k x)}{|2|^k} ; p \leq k < q \Big\} \end{aligned}$$

for all $x \in G$ and all non-negative integers p, q with $q > p \geq 0$. Letting $p = 0$ and passing the limit $q \to \infty$ in the last inequality and using (3.17), we obtain (3.20). The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $\xi : [0,\infty) \to [0,\infty)$ be a mapping satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t) \ (t \geq 0), \qquad \xi(|2|) < |2|. \tag{3.24}$$

Let $\kappa > 0$ and $f : G \to X$ be a mapping satisfying

$$\begin{aligned} &\Big\| Q\Big(\frac{x+y}{2} + z\Big) + Q\Big(\frac{x+z}{2} + y\Big) \\ &+ Q\Big(\frac{z+y}{2} + x\Big) - 2[Q(x) + Q(y) + Q(z)] \Big\| \\ &\leq \kappa(\xi(|x|).\xi(|y|).\xi(|z|)). \end{aligned} \tag{3.25}$$

for all $x, y, z \in G$. Then, there exists a unique additive mapping $\Im : G \to X$ such that

$$||Q(x) - \Im(x)|| \leq \frac{\kappa \xi^3(|x|)}{|2|}. \tag{3.26}$$

Proof. Define $\zeta : G^3 \to [0,\infty)$ by $\zeta(x,y,z) := \kappa(\xi(|x|).\xi(|y|).\xi(|z|))$ and apply Theorem 3.3 to get the result. □

4. Non-Archimedean Stability of Eq.(2.2): Fixed Point Method

Theorem 4.1. *Let $\zeta : X^3 \to [0,\infty)$ be a mapping such that there exists an $L < 1$ with*

$$\zeta(x,y,z) \leq \frac{L}{|2|} \zeta(2x, 2y, 2z) \tag{4.1}$$

for all $x, y, z \in X$. *Let* $Q : X \to Y$ *be a mapping satisfying*

$$\begin{aligned} &\Big\|Q\Big(\frac{x+y}{2}+z\Big)+Q\Big(\frac{x+z}{2}+y\Big) \\ &+Q\Big(\frac{z+y}{2}+x\Big)-2[Q(x)+Q(y)+Q(z)]\Big\| \\ &\leq \zeta(x,y,z) \end{aligned} \tag{4.2}$$

for all $x, y, z \in X$. *Then there is a unique additive mapping* $R : X \to Y$ *such that*

$$\|Q(x)-R(x)\| \leq \frac{L}{|2|-|2|L}\zeta(x,x,x). \tag{4.3}$$

Proof. Putting $x = y = z$ in (4.2), we have

$$\Big\|2Q\Big(\frac{x}{2}\Big)-Q(x)\Big\| \leq \zeta\Big(\frac{x}{2},\frac{x}{2},\frac{x}{2}\Big) \tag{4.4}$$

for all $x \in X$. Consider the set

$$S := \{g : X \to Y\} \tag{4.5}$$

and the generalized metric d in S defined by

$$d(f,g) = \inf\Big\{\mu \in \mathbb{R}^+ : \|g(x)-h(x)\| \leq \mu\zeta(x,x,x), \forall x \in X\Big\}, \tag{4.6}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [20], Lemma 2.1). Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\Big(\frac{x}{2}\Big) \tag{4.7}$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$\|g(x)-h(x)\| \leq \epsilon\zeta(x,x,x) \tag{4.8}$$

for all $x \in X$ and so

$$\begin{aligned} \|Jg(x)-Jh(x)\| &= \Big\|2g\Big(\frac{x}{2}\Big)-2h\Big(\frac{x}{2}\Big)\Big\| \\ &\leq |2|\epsilon\zeta\Big(\frac{x}{2},\frac{x}{2},\frac{x}{2}\Big) \\ &\leq |2|\epsilon\frac{L}{|2|}\zeta(x,x,x) \end{aligned}$$

for all $x \in X$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg,Jh) \leq Ld(g,h) \tag{4.9}$$

for all $g, h \in S$. It follows from (4.4) that

$$d(Q,JQ) \leq \frac{L}{|2|}. \tag{4.10}$$

By Theorem 2.4, there exists a mapping $R : X \to Y$ satisfying

(1) R is a fixed point of J, that is,

$$R\Big(\frac{x}{2}\Big) = \frac{1}{2}R(x) \tag{4.11}$$

for all $x \in X$. The mapping R is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g,h) < \infty\}. \tag{4.12}$$

This implies that R is a unique mapping satisfying (4.11) such that there exists $\mu \in (0, \infty)$ satisfying

$$\|Q(x) - R(x)\| \leq \mu\zeta(x, x, x) \tag{4.13}$$

for all $x \in X$.

(2) $d(J^n Q, R) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n Q\Big(\frac{x}{2^n}\Big) = Q(x) \tag{4.14}$$

for all $x \in X$.

(3) $d(Q, R) \leq \frac{d(Q,JQ)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, C) \leq \frac{L}{|2| - |2|L}. \tag{4.15}$$

This implies that the inequality (4.3) holds.

By (3.45),

$$\begin{aligned}
&\Big\|R\Big(\frac{x+y}{2} + z\Big) + R\Big(\frac{x+z}{2} + y\Big) + R\Big(\frac{z+y}{2} + x\Big) - 2[R(x) + R(y) + R(z)]\Big\| \\
\leq\ & \lim_{n\to\infty} |2|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\Big) \\
\leq\ & \lim_{n\to\infty} |2|^n . \frac{L^n}{|2|^n}\zeta(x, y, z)
\end{aligned}$$

for all $x, y, z \in X$ and $n \in \mathbb{N}$. So

$$\Big\|R\Big(\frac{x+y}{2} + z\Big) + R\Big(\frac{x+z}{2} + y\Big) + R\Big(\frac{z+y}{2} + x\Big) - 2[R(x) + R(y) + R(z)]\Big\| = 0$$

for all $x, y, z \in X$. Thus, the mapping $R : X \to Y$ is additive, as desired. □

Corollary 4.2. *Let $\theta \geq 0$ and r be a real number with $0 < r < 1$. Let $Q : X \to Y$ be a mapping satisfying*

$$\begin{aligned}
&\Big\|Q\Big(\frac{x+y}{2} + z\Big) + Q\Big(\frac{x+z}{2} + y\Big) \\
&+ Q\Big(\frac{z+y}{2} + x\Big) - 2[Q(x) + Q(y) + Q(z)]\Big\| \\
&\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned} \tag{4.16}$$

for all $x, y, z \in X$. Then

$$R(x) = \lim_{n\to\infty} 2^n Q\Big(\frac{x}{2^n}\Big) \tag{4.17}$$

exists for all $x \in X$ and $R : X \to Y$ is a unique additive mapping such that

$$\|Q(x) - R(x)\| \leq \frac{3|2|\theta\|x\|^r}{|2|^{r+1} - |2|^2} \tag{4.18}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking

$$\zeta(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{4.19}$$

for all $x, y, z \in X$. In fact, if we choose $L = |2|^{1-r}$, then we get the desired result. □

Theorem 4.3. *Let $\zeta : X^3 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\zeta(2x,2y,2z) \le |2| L\zeta(x,y,z) \tag{4.20}$$

for all $x,y,z \in X$. Let $Q : X \to Y$ be a mapping satisfying

$$\begin{aligned}&\Big\|Q\Big(\frac{x+y}{2}+z\Big)+Q\Big(\frac{x+z}{2}+y\Big)\\&+Q\Big(\frac{z+y}{2}+x\Big)-2[Q(x)+Q(y)+Q(z)]\Big\|\\&\le \zeta(x,y,z)\end{aligned} \tag{4.21}$$

for all $x,y,z \in X$. Then, there is a unique additive mapping $R : X \to Y$ such that

$$\|Q(x)-R(x)\| \le \frac{1}{|2|-|2|L}\zeta(x,x,x). \tag{4.22}$$

Proof. It follows from (3.22) that

$$\Big\|Q(x)-\frac{Q(2x)}{2}\Big\| \le \frac{\zeta(x,x,x)}{|2|} \tag{4.23}$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $\theta \ge 0$ and r be a real number with $r > \frac{1}{3}$. Let $Q : X \to Y$ be a mapping satisfying*

$$\begin{aligned}&\Big\|Q\Big(\frac{x+y}{2}+z\Big)+Q\Big(\frac{x+z}{2}+y\Big)\\&+Q\Big(\frac{z+y}{2}+x\Big)-2[Q(x)+Q(y)+Q(z)]\Big\|\\&\le \theta(\|x\|^r.\|y\|^r.\|z\|^r)\end{aligned} \tag{4.24}$$

for all $x,y,z \in X$. Then

$$R(x) = \lim_{n\to\infty}\frac{Q(2^n x)}{2^n} \tag{4.25}$$

exists for all $x \in X$ and $R : X \to Y$ is a unique additive mapping such that

$$\|Q(x)-R(x)\| \le \frac{\theta\|x\|^{3r}}{|2|-|2|^{3r}} \tag{4.26}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.3 by taking

$$\zeta(x,y,z) = \theta(\|x\|^r.\|y\|^r.\|z\|^r) \tag{4.27}$$

for all $x,y,z \in X$. In fact, if we choose $L = |2|^{3r-1}$, then we get the desired result. □

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