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Positive solutions for multi-parameter cyclic $(p_1, \dots, p_n)$-Laplacian systems with combined and falling-zero nonlinearities

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Abstract

We consider a cyclic $n \times n$ system of quasilinear elliptic equations

$$\begin{cases} -\Delta_{p_i} u_i = \lambda_i f_i(u_i) + \mu_i g_i(u_{i+1}), & i = 1, 2, \dots, n-1, \\ -\Delta_{p_n} u_n = \lambda_n f_n(u_n) + \mu_n g_n(u_1), \end{cases}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet boundary conditions, where $\Delta_{p_i} z = \operatorname{div}(|\nabla z|^{p_i - 2} \nabla z)$, $p_i > 1$, and $\lambda_i, \mu_i > 0$. The nonlinearities $f_i, g_i : [0, \infty) \to \mathbb{R}$ are increasing and satisfy a subcritical growth condition at infinity for f_i and a combined sublinear composition condition for the cooperative chain $(g_i)_{i=1}^n$. This setting includes power-type and piecewise power-type nonlinearities. Under these assumptions, we prove the existence of positive weak solutions for all sufficiently large values of the sums $\lambda_i + \mu_i$. Under additional flatness conditions near the origin we obtain at least two distinct positive solutions. We also treat the case where f_i have a falling-zero structure ($f_i > 0$ on $(0, r_i)$, $f_i(r_i) = 0$, $f_i < 0$ on (r_i, ∞)) and derive analogous existence and multiplicity results. The proofs rely on the method of sub- and supersolutions and a three-solution theorem in an ordered Banach space.

Keywords: Multiple parameters, $(p_1, p_2, \dots, p_n)$-Laplacian systems, Combined sublinear effects, Falling zeroes, Sub- and supersolutions.
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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. In this paper, we consider the $n \times n$ elliptic system

$$\begin{cases} -\Delta_{p_1} u_1 = \lambda_1 f_1(u_1) + \mu_1 g_1(u_2), & x \in \Omega, \\ -\Delta_{p_2} u_2 = \lambda_2 f_2(u_2) + \mu_2 g_2(u_3), & x \in \Omega, \\ \quad \vdots \\ -\Delta_{p_n} u_n = \lambda_n f_n(u_n) + \mu_n g_n(u_1), & x \in \Omega, \\ u_1 = \cdots = u_n = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Delta_s z = \operatorname{div}(|\nabla z|^{s-2}\nabla z)$, $s > 1$, and $\lambda_i, \mu_i > 0$ are real parameters.

Scalar p-Laplacian equations and 2×2 systems with multiple parameters and combined nonlinear effects (including semipositone and falling-zero nonlinearities) have been studied extensively; see, for instance, J. Ali and Shivaji [4], Lee–Shivaji–Ye [11], and the references therein. More recently, $n\times n$ systems with combined nonlinear effects (without falling zeroes) have been considered in Ali–Brown–Shivaji [5]. On the other hand, infinite semipositone problems with falling zeros and (p,q)-Laplacian problems of falling-zero type have been investigated in Lee–Shivaji–Ye [12], Afrouzi–Shakeri–Chung [2], and in the more recent papers of Zeditri–Akrout–Guefaifia [18] and Alreshidi–Hai [3], all of which are based on the method of sub- and supersolutions or fixed point arguments in cones. Nonexistence and multiplicity phenomena for related p-Laplacian systems have been analyzed by Abebe and Chhetri [1], while a series of works by Khafagy and co-authors treat degenerate p- and (p,q)-Laplacian systems and their stability properties; see, e.g., [10].

Recent related developments for anisotropic and competing (p,q)-Laplacian problems and systems can be found, for example, in Razani–Figueiredo [19, 20], Motreanu–Razani [21], Razani [22, 23], Razani–Safari [24], Razani–Baraket [25], and Mahshid–Razani [26].

Our first goal is to extend and refine the results in [4, 5, 11] to the cyclic $n\times n$ system (1.1). In particular, we allow the exponents $p_i > 1$ to be possibly different, and we impose a sharp combined sublinear condition at infinity on the chain of cooperative nonlinearities (g_i), which is compatible with power-type growth conditions appearing in many models. Our second goal is to carry over the falling-zero framework of [11] to the $n\times n$ system (1.1), giving explicit parameter thresholds and a refined multiplicity picture.

The method of sub- and supersolutions is a powerful tool for obtaining existence and multiplicity results for nonlinear boundary value problems (ODE and PDE). Its roots go back to Picard's monotone iteration schemes [14, 15], with major contributions due to Scorza Dragoni [16] and Nagumo [13]. A modern abstract framework in ordered Banach spaces was developed by Amann [6], and has been successfully applied to elliptic problems by Cañada–Drábek–Gámez [7, 8] and many others. We work in a setting that fits into Amann's abstract theory and exploit a three-solution result due to Shivaji [17] to derive multiplicity.

1.1 Ordered Banach space setting and sub-supersolutions

Let $X_i := W_0^{1,p_i}(\Omega)\cap C(\overline{\Omega})$, $X := \prod_{i=1}^n X_i$, endowed with the product norm $\|\underline{u}\|_X := \max_{1\le i\le n}\|u_i\|_{X_i}$. We equip X with the usual cone

$$P := \{\underline{u}\in X:\ u_i(x)\ge 0 \text{ for all } x\in\overline{\Omega},\ i=1,\dots,n\},$$

and write $\underline{u}\le\underline{v}$ if $v_i - u_i\in P$ for all i. Then (X,P) is an ordered Banach space and the partial order $\le$ is compatible with the Banach space structure.

We say that $\underline{u} = (u_1,\dots,u_n)\in X$ is a *weak solution* of (1.1) if $u_i = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla u_i|^{p_i-2}\nabla u_i\cdot\nabla\xi\,dx = \int_\Omega \big[\lambda_i f_i(u_i) + \mu_i g_i(u_{i+1})\big]\xi\,dx$$

for all nonnegative test functions $\xi\in C_0^\infty(\Omega)$ and $i=1,\dots,n$, with the convention $u_{n+1} = u_1$.

Definition 1.1. We say that $\underline{\psi} = (\psi_1,\dots,\psi_n)\in X$ is a *subsolution* of (1.1) if $\psi_i = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla \psi_i|^{p_i-2}\nabla \psi_i\cdot\nabla\xi\,dx \le \int_\Omega \big[\lambda_i f_i(\psi_i) + \mu_i g_i(\psi_{i+1})\big]\xi\,dx$$

for all $0\le\xi\in C_0^\infty(\Omega)$ and $i=1,\dots,n$ (with $\psi_{n+1}=\psi_1$). Similarly, $\underline{Z} = (Z_1,\dots,Z_n)\in X$ is called a *supersolution* of (1.1) if $Z_i = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla Z_i|^{p_i-2}\nabla Z_i\cdot\nabla\xi\,dx \ge \int_\Omega \big[\lambda_i f_i(Z_i) + \mu_i g_i(Z_{i+1})\big]\xi\,dx$$

for all $0\le\xi\in C_0^\infty(\Omega)$ and $i=1,\dots,n$.

When $\underline{\psi}\le\underline{Z}$, we say that $(\underline{\psi},\underline{Z})$ is an ordered pair of sub- and supersolutions. We will frequently use the following standard results (see Amann [6], Cañada–Drábek–Gámez [7, 8] and Shivaji [17]).

Lemma 1.2 (Existence between sub- and supersolutions). Assume that there exist a subsolution $\underline{\psi} \in X$ and a supersolution $\underline{Z} \in X$ of (1.1) such that $\underline{\psi} \le \underline{Z}$. Then there exists at least one weak solution $\underline{u} \in X$ of (1.1) satisfying $\underline{\psi} \le \underline{u} \le \underline{Z}$. Moreover, there exists a *minimal* solution $\underline{u}^{\min}$ and a *maximal* solution $\underline{u}^{\max}$ in the order interval

$$[\underline{\psi}, \underline{Z}] := \{\underline{u} \in X : \ \underline{\psi} \le \underline{u} \le \underline{Z}\}.$$

Lemma 1.3 (Three-solution theorem). Assume that there exist subsolutions $\underline{\psi}, \underline{\omega} \in X$, supersolutions $\underline{\zeta}, \underline{Z} \in X$ and $\underline{\psi}, \underline{\omega}$ are strict in the sense that the inequalities in the definition of subsolutions are strict on a set of positive measure, that is, componentwise: for every $i = 1, \dots, n$ the corresponding inequality is strict on a subset of Ω of positive measure, and that

$$\underline{\psi} \le \underline{\zeta} \le \underline{Z}, \qquad \underline{\psi} \le \underline{\omega} \le \underline{Z}, \qquad \underline{\omega} \not\le \underline{\zeta}.$$

Then (1.1) has at least three distinct weak solutions $\underline{u}^{(i)} = (u_1^{(i)}, \dots, u_n^{(i)})$, $i = 1, 2, 3$, such that

$$\underline{u}^{(1)} \in [\underline{\psi}, \underline{\zeta}], \qquad \underline{u}^{(2)} \in [\underline{\omega}, \underline{Z}],$$

and

$$\underline{u}^{(3)} \in [\underline{\psi}, \underline{Z}] \setminus \big([\underline{\psi}, \underline{\zeta}] \cup [\underline{\omega}, \underline{Z}]\big).$$

The proofs of these lemmas rely on monotone iterations (starting from $\underline{\psi}$ and from $\underline{Z}$) combined with the order-preserving property of the solution operator associated with (1.1) in the cooperative case (nondecreasing g_i). For completeness we indicate the iterative scheme in our proofs.

The rest of the paper is organized as follows. In Section 2 we treat (1.1) under combined sublinear conditions at infinity without sign assumptions on $f_i(0)$ and $g_i(0)$. We give a sharp asymptotic interpretation of the key hypothesis (H3) and derive an explicit lower bound for the parameters $\lambda_i + \mu_i$ ensuring existence of positive solutions, together with refined multiplicity statements. Section 3 is devoted to nonlinearities with falling zeroes. Finally, in Section 4 we present classes of nonlinearities satisfying our hypotheses.

2 The system without sign conditions on $f_i(0)$ and $g_i(0)$

In this section we impose the following hypotheses:

(H1) $f_i, g_i : [0, \infty) \to \mathbb{R}$ are increasing C^1 functions such that $f_i(x) \to \infty$ and $g_i(x) \to \infty$ as $x \to \infty$ for $i = 1, 2, \dots, n$.

(H2) For each i,

$$\lim_{x \to \infty} \frac{f_i(x)}{x^{p_i - 1}} = 0.$$

(H3) For every $M_1, \dots, M_{n-1} > 0$,

$$\lim_{x \to \infty} \frac{G_{M_1, \dots, M_{n-1}}(x)}{x^{p_1 - 1}} = 0,$$

where

$$G_{M_1, \dots, M_{n-1}}(x) := g_1^{[M_1]} \circ \Big[(g_2^{[M_2]})^{\frac{1}{p_2 - 1}} \circ \dots \circ (g_{n-1}^{[M_{n-1}]})^{\frac{1}{p_{n-1} - 1}} \circ (g_n(x))^{\frac{1}{p_n - 1}}\Big],$$
$$g_i^{[M]}(x) := g_i(Mx).$$

2.1 Asymptotic interpretation of (H3)

We first clarify the meaning of (H3) in the important case of power-like nonlinearities.

Lemma 2.1. Assume that for each $i = 1, \dots, n$ there exist constants $\beta_i > 0$ and $c_i > 0$ such that

$$\lim_{x \to \infty} \frac{g_i(x)}{c_i x^{\beta_i}} = 1.$$

Then (H3) holds if and only if

$$\prod_{i=1}^{n} \beta_i < \prod_{i=1}^{n} (p_i - 1). \tag{2.1}$$

Proof . We compute the asymptotic growth of $G_{M_1,\dots,M_{n-1}}(x)$ as $x \to \infty$. Fix $M_1, \dots, M_{n-1} > 0$ and set

$$h_n(x) := (g_n(x))^{\frac{1}{p_n-1}}, \qquad h_i(x) := \left(g_i^{[M_i]}\big(h_{i+1}(x)\big)\right)^{\frac{1}{p_i-1}}, \quad i = n-1, \dots, 2.$$

Thus $G_{M_1,\dots,M_{n-1}}(x) = g_1^{[M_1]}\big(h_2(x)\big)$. Since $g_n(x) \sim c_n x^{\beta_n}$, we have

$$h_n(x) \sim c_n^{\frac{1}{p_n-1}} x^{\frac{\beta_n}{p_n-1}} \quad \text{as } x \to \infty.$$

Suppose inductively that for some $i+1 \in \{3, \dots, n\}$,

$$h_{i+1}(x) \sim C_{i+1} x^{E_{i+1}} \quad \text{as } x \to \infty,$$

with $C_{i+1} > 0$ and $E_{i+1} > 0$. Then

$$g_i^{[M_i]}\big(h_{i+1}(x)\big) = g_i(M_i h_{i+1}(x)) \sim c_i\big(M_i h_{i+1}(x)\big)^{\beta_i} \sim \tilde{C}_i x^{\beta_i E_{i+1}},$$

and hence

$$h_i(x) = \left(g_i^{[M_i]}\big(h_{i+1}(x)\big)\right)^{\frac{1}{p_i-1}} \sim C_i x^{E_i}, \qquad E_i := \frac{\beta_i E_{i+1}}{p_i - 1}.$$

Starting from $E_n = \frac{\beta_n}{p_n-1}$, we obtain

$$E_2 = \frac{\beta_2 \cdots \beta_n}{\prod_{j=2}^n (p_j - 1)}.$$

Finally,

$$G_{M_1,\dots,M_{n-1}}(x) = g_1^{[M_1]}\big(h_2(x)\big) \sim c_1\big(M_1 h_2(x)\big)^{\beta_1} \sim C x^E, \qquad E := \beta_1 E_2 = \frac{\beta_1 \cdots \beta_n}{\prod_{j=2}^n (p_j - 1)}.$$

Hence

$$\frac{G_{M_1,\dots,M_{n-1}}(x)}{x^{p_1-1}} \sim C x^{E-(p_1-1)} \quad \text{as } x \to \infty,$$

which tends to 0 if and only if $E < p_1 - 1$, i.e.

$$\frac{\beta_1 \cdots \beta_n}{\prod_{j=2}^n (p_j - 1)} < p_1 - 1 \quad \Longleftrightarrow \quad \prod_{i=1}^n \beta_i < \prod_{i=1}^n (p_i - 1).$$

□

Remark 2.2. In particular, if $g_i(x) = b_i x^{\beta_i}$ for $x \geq 0$, with $b_i > 0$ and $\beta_i \geq 0$, then (H3) holds if and only if (2.1) holds. Thus (H3) is a natural and essentially sharp combined sublinearity condition at infinity for the chain $g_1, \dots, g_n$.

2.2 Existence of a positive solution and explicit parameter thresholds

We now prove the existence of at least one positive weak solution of (1.1) for large values of the combined parameters $\lambda_i + \mu_i$. As a byproduct we obtain an explicit (though nonoptimal) lower bound for these parameters.

Theorem 2.3. Let (H1)–(H3) hold. Then there exists a constant $\Lambda_* > 0$, explicitly computable in terms of Ω, (p_i), the nonlinearities (f_i, g_i) and the eigenpairs of $-\Delta_{p_i}$, such that if

$$\lambda_i + \mu_i \geq \Lambda_*, \qquad i = 1, \dots, n,$$

then problem (1.1) has a positive weak solution $\underline{u} \in X$.

Proof . Step 1. Subsolutions. For each i, let $\sigma_i > 0$ be the first eigenvalue of $-\Delta_{p_i}$ on $W_0^{1,p_i}(\Omega)$, and $\phi_i > 0$ a corresponding eigenfunction normalized by $\|\phi_i\|_{L^{p_i}(\Omega)} = 1$. Thus

$$\begin{cases} -\Delta_{p_i}\phi_i = \sigma_i \phi_i^{p_i-1}, & x \in \Omega, \\ \phi_i = 0, & x \in \partial\Omega. \end{cases}$$

Standard regularity implies $\phi_i \in C^{1,\alpha}(\overline{\Omega})$, $\phi_i > 0$ in Ω and $|\nabla\phi_i| \neq 0$ on $\partial\Omega$. Since $|\nabla\phi_i|^{p_i} - \sigma_i\phi_i^{p_i}$ is continuous and strictly negative on $\partial\Omega$, there exist $\delta > 0$ and $m > 0$ such that

$$|\nabla\phi_i|^{p_i} - \sigma_i\phi_i^{p_i} \le -m \quad \text{on} \quad \Omega_\delta := \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) \le \delta\} \tag{2.2}$$

for all i. Moreover, there exists $\eta > 0$ such that

$$\phi_i(x) \ge \eta > 0 \quad \text{for all } x \in \Omega \setminus \Omega_\delta, \ i = 1, \dots, n. \tag{2.3}$$

Let $e_i \in W_0^{1,p_i}(\Omega) \cap C(\overline{\Omega})$ solve

$$\begin{cases} -\Delta_{p_i} e_i = 1, & x \in \Omega, \\ e_i = 0, & x \in \partial\Omega, \end{cases}$$

and set $E_* := \max_{1\le i\le n} \|e_i\|_{L^\infty(\Omega)}$. Choose $K_0 \ge 0$ such that $f_i(x) \ge -K_0$ and $g_i(x) \ge -K_0$ for all $x \ge 0$ and i. For each i define

$$\psi_i(x) := \alpha_i(\lambda_i + \mu_i)^{\frac{1}{p_i-1}} \phi_i(x)^{\frac{p_i}{p_i-1}}, \qquad \alpha_i := \left(\frac{K_0}{m}\right)^{\frac{1}{p_i-1}} \left(\frac{p_i - 1}{p_i}\right)^{\frac{p_i}{p_i-1}}.$$

Then $\psi_i \in W_0^{1,p_i}(\Omega) \cap C(\overline{\Omega})$, $\psi_i \ge 0$ and a direct computation yields

$$|\nabla\psi_i|^{p_i-2}\nabla\psi_i = \frac{(\lambda_i + \mu_i)K_0}{m} \phi_i |\nabla\phi_i|^{p_i-2}\nabla\phi_i.$$

For $0 \le \xi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \int_\Omega |\nabla\psi_i|^{p_i-2}\nabla\psi_i \cdot \nabla\xi \, dx &= \frac{(\lambda_i + \mu_i)K_0}{m} \int_\Omega \Big(|\nabla\phi_i|^{p_i-2}\nabla\phi_i \cdot \nabla(\phi_i\xi) - |\nabla\phi_i|^{p_i}\xi \Big) \, dx \\ &= \frac{(\lambda_i + \mu_i)K_0}{m} \int_\Omega \big(\sigma_i\phi_i^{p_i} - |\nabla\phi_i|^{p_i}\big)\xi \, dx. \end{aligned}$$

Using (2.2)–(2.3) and the silently–assumed normalization $\|\phi_i\|_{L^\infty} = 1$, which implies $\sigma_i\phi_i^{p_i} - |\nabla\phi_i|^{p_i} \le \sigma_i$ on $\Omega \setminus \Omega_\delta$, we obtain

$$\int_\Omega |\nabla\psi_i|^{p_i-2}\nabla\psi_i \cdot \nabla\xi \, dx \le -(\lambda_i + \mu_i)K_0 \int_{\Omega_\delta} \xi \, dx + \frac{(\lambda_i + \mu_i)K_0}{m} \int_{\Omega\setminus\Omega_\delta} \sigma_i\xi \, dx.$$

By (H1), for each $L > 0$ and i there exists $T_i(L) > 0$ such that $f_i(s) \ge L$ and $g_i(s) \ge L$ for all $s \ge T_i(L)$. Let $L_* := \frac{K_0}{m} E_*$ and set $T_i := T_i(L_*)$. For $x \in \Omega \setminus \Omega_\delta$ we have

$$\psi_i(x) \ge \alpha_i(\lambda_i + \mu_i)^{1/(p_i-1)} \eta^{p_i/(p_i-1)}.$$

Impose

$$\lambda_i + \mu_i \ge \Lambda_{*,i}^{\mathrm{sub}} := \left(\frac{T_i}{\alpha_i \eta^{\frac{p_i}{p_i-1}}} \right)^{p_i-1},$$

so that $\psi_i(x) \ge T_i$ on $\Omega \setminus \Omega_\delta$. Then $f_i(\psi_i) \ge L_*$ and $g_i(\psi_{i+1}) \ge L_*$ there, hence

$$\lambda_i f_i(\psi_i) + \mu_i g_i(\psi_{i+1}) \ge (\lambda_i + \mu_i)L_* = \frac{(\lambda_i + \mu_i)K_0}{m} E_*.$$

On Ω_δ use only $f_i(\psi_i), g_i(\psi_{i+1}) \ge -K_0$ to obtain

$$\lambda_i f_i(\psi_i) + \mu_i g_i(\psi_{i+1}) \ge -(\lambda_i + \mu_i)K_0.$$

Combining these inequalities shows that for all $\lambda_i + \mu_i \geq \Lambda_{*,i}^{\mathrm{sub}}$,

$$\int_\Omega |\nabla\psi_i|^{p_i-2}\nabla\psi_i \cdot \nabla\xi \, dx \leq \int_\Omega \big[\lambda_i f_i(\psi_i) + \mu_i g_i(\psi_{i+1})\big]\xi \, dx$$

for every $0 \leq \xi \in C_0^\infty(\Omega)$. Let

$$\Lambda_*^{\mathrm{sub}} := \max_{1\leq i\leq n} \Lambda_{*,i}^{\mathrm{sub}}.$$

Then $\underline{\psi}$ is a subsolution whenever $\lambda_i + \mu_i \geq \Lambda_*^{\mathrm{sub}}$.

Step 2. Supersolutions. Define for a parameter $C > 0$:

$$Z_1 := Ce_1, \qquad Z_k := \big((\lambda_k + \mu_k)g_k(\|Z_{k+1}\|_\infty)\big)^{\frac{1}{p_k-1}} e_k, \; k = 2, \ldots, n-1,$$

and

$$Z_n := \big((\lambda_n + \mu_n)g_n(\|Z_1\|_\infty)\big)^{\frac{1}{p_n-1}} e_n.$$

Then

$$-\Delta_{p_1} Z_1 = C^{p_1-1}, \qquad -\Delta_{p_k} Z_k = (\lambda_k + \mu_k)g_k(\|Z_{k+1}\|_\infty).$$

By (H2)–(H3), for each fixed set of parameters (λ_i, μ_i),

$$\frac{\lambda_i f_i(\|Z_i(C)\|_\infty) + \mu_i g_i(\|Z_{i+1}(C)\|_\infty)}{\|Z_i(C)\|_\infty^{p_i-1}} \to 0 \quad \text{as } C \to \infty.$$

Hence there exists $C_* > 0$ such that for all $C \geq C_*$,

$$C^{p_1-1} \geq \lambda_1 f_1(\|Z_1\|_\infty) + \mu_1 g_1(\|Z_2\|_\infty),$$

and similarly

$$(\lambda_k + \mu_k)g_k(\|Z_{k+1}\|_\infty) \geq \lambda_k f_k(\|Z_k\|_\infty) + \mu_k g_k(\|Z_{k+1}\|_\infty)$$

for $k = 2, \ldots, n$, with $Z_{n+1} = Z_1$. From these inequalities, for every $0 \leq \xi \in C_0^\infty(\Omega)$ we obtain

$$\int_\Omega |\nabla Z_i|^{p_i-2}\nabla Z_i \cdot \nabla\xi \, dx \geq \int_\Omega \big[\lambda_i f_i(Z_i) + \mu_i g_i(Z_{i+1})\big]\xi \, dx,$$

so $\underline{Z}$ is a supersolution. Increasing C further if needed we may ensure $Z_i \geq \psi_i$ for all i.

Step 3. Conclusion. Let $\Lambda_* := \Lambda_*^{\mathrm{sub}}$ and assume $\lambda_i + \mu_i \geq \Lambda_*$. Then $(\underline{\psi}, \underline{Z})$ is an ordered pair of sub- and supersolutions, and Lemma 1.2 yields a weak solution $\underline{u}$ with $\underline{\psi} \leq \underline{u} \leq \underline{Z}$. By the strong maximum principle and Hopf's boundary lemma for each p_i-Laplacian, every component u_i is strictly positive in Ω. □

2.3 Refined multiplicity and minimal positive solutions

We next refine multiplicity results in the spirit of [4].

Theorem 2.4. Assume (H1)–(H3). Suppose in addition that for each i the functions f_i and g_i are C^{k_i+1} in a neighborhood of 0, where $k_i := [p_i - 1]$ (the integer part of $p_i - 1$), and this smoothness/vanishing assumption implies $f_i(s), g_i(s) = o(s^{p_i-1})$ as $s \to 0^+$, which is needed in Step 3 to build a small strict supersolution.

$$f_i(0) = g_i(0) = 0, \qquad f_i^{(k)}(0) = g_i^{(k)}(0) = 0, \quad k = 1, \ldots, k_i.$$

Then there exists $\widetilde{\Lambda}_* > 0$ such that if $\lambda_i + \mu_i \geq \widetilde{\Lambda}_*$ for all i, problem (1.1) has at least two distinct positive weak solutions. Moreover, there is a minimal positive solution $\underline{u}^{\min}$ in $P \setminus \{0\}$.

Proof . Step 1. Zero solution and large supersolution. Under the assumptions near zero, $\underline{0}$ is a weak solution of (1.1): $f_i(0) = g_i(0) = 0$ implies the right-hand side vanishes. Hence $\underline{\psi}^0 := \underline{0}$ is a subsolution.

From Theorem 2.3 we already know how to construct a large supersolution $\underline{Z}$ for any (λ_i, μ_i). Choose such a $\underline{Z}$ with $\underline{0} \leq \underline{Z}$.

Step 2. Strict positive subsolution above zero. Consider the modified problem

$$\begin{cases} -\Delta_{p_i}\omega_i = \lambda_i\big(f_i(\omega_i)-1\big)+\mu_i\big(g_i(\omega_{i+1})-1\big), & i=1,\ldots,n-1,\\ -\Delta_{p_n}\omega_n = \lambda_n\big(f_n(\omega_n)-1\big)+\mu_n\big(g_n(\omega_1)-1\big), \end{cases}$$

with homogeneous Dirichlet boundary conditions. The nonlinearities $\tilde f_i(s) := f_i(s)-1$ and $\tilde g_i(s) := g_i(s)-1$ satisfy (H1)–(H3), so by Theorem 2.3 there exists Λ'_* such that for $\lambda_i+\mu_i \ge \Lambda'_*$ this problem admits a positive solution $\underline{\omega}$.

For such (λ_i,μ_i), $\underline{\omega}$ is a strict subsolution of (1.1) since

$$\begin{aligned} \int_\Omega |\nabla\omega_i|^{p_i-2}\nabla\omega_i\cdot\nabla\xi\,dx &= \int_\Omega \big[\lambda_i\tilde f_i(\omega_i)+\mu_i\tilde g_i(\omega_{i+1})\big]\xi\,dx \\ &= \int_\Omega \big[\lambda_i f_i(\omega_i)+\mu_i g_i(\omega_{i+1})\big]\xi\,dx - (\lambda_i+\mu_i)\int_\Omega \xi\,dx \\ &< \int_\Omega \big[\lambda_i f_i(\omega_i)+\mu_i g_i(\omega_{i+1})\big]\xi\,dx. \end{aligned}$$

Step 3. Small strict supersolution. Let ϕ_i be the first eigenfunctions as before. There exist constants $c_i>0$ such that

$$\phi_{i+1}\le c_i\phi_i,\quad i=1,\ldots,n-1,\qquad \phi_1\le c_n\phi_n.$$

For each i set

$$H_{p_i}(x) := \sigma_i x^{p_i-1} - \lambda_i f_i(x) - \mu_i g_i(c_i x).$$

By the smoothness and flatness assumptions at the origin, there exists an integer $k_i=[p_i-1]$ such that

$$H_{p_i}(0)=H_{p_i}^{(k)}(0)=0\quad \text{for all } k=1,\ldots,k_i,$$

and the first nonzero term in the expansion of H_{p_i} at 0 is the term $\sigma_i x^{p_i-1}$ with positive coefficient. In particular, there exists $a_0>0$ such that $H_{p_i}(x)>0$ for all $x\in(0,a_0]$. Fix $0<\varepsilon\le a_0$ and define $\zeta_i := \varepsilon\phi_i$. Then for $0\le\xi\in C_0^\infty(\Omega)$,

$$\begin{aligned} \int_\Omega |\nabla\zeta_i|^{p_i-2}\nabla\zeta_i\cdot\nabla\xi\,dx &= \varepsilon^{p_i-1}\int_\Omega |\nabla\phi_i|^{p_i-2}\nabla\phi_i\cdot\nabla\xi\,dx \\ &= \int_\Omega \sigma_i(\varepsilon\phi_i)^{p_i-1}\xi\,dx \\ &= \int_\Omega H_{p_i}(\varepsilon\phi_i)\xi\,dx + \int_\Omega \big[\lambda_i f_i(\zeta_i)+\mu_i g_i(c_i\zeta_i)\big]\xi\,dx \\ &> \int_\Omega \big[\lambda_i f_i(\zeta_i)+\mu_i g_i(\zeta_{i+1})\big]\xi\,dx, \end{aligned}$$

since $H_{p_i}(\varepsilon\phi_i)>0$ and $g_i(c_i\zeta_i)\ge g_i(\zeta_{i+1})$. Thus $\underline{\zeta}$ is a strict supersolution. Taking ε small we can ensure $\underline{\zeta}\le\underline{Z}$ and $\underline{\omega}\not\le\underline{\zeta}$.

Step 4. Application of Lemma 1.3. We now have $\underline{\psi}^0=\underline{0}$, the strict subsolution $\underline{\omega}$, the strict supersolution $\underline{\zeta}$ and the large supersolution $\underline{Z}$ with

$$\underline{0}\le\underline{\zeta}\le\underline{Z},\qquad \underline{\omega}\le\underline{Z},\qquad \underline{\omega}\not\le\underline{\zeta}.$$

Lemma 1.3 yields three solutions in $[\underline{0},\underline{Z}]$, one of which may be $\underline{0}$. In any case there are at least two distinct positive solutions. The minimal positive solution $\underline{u}^{\min}$ in $[\underline{0},\underline{Z}]$ exists by Lemma 1.2. □

3 Systems involving nonlinearities with falling zeroes

We now turn to the case where the nonlinearities f_i possess a unique positive zero and change sign. We assume:

(H4) For each i, $f_i:[0,\infty)\to\mathbb{R}$ is continuous, and $g_i:[0,\infty)\to[0,\infty)$ is continuous, $g_i(s)>0$ for $s>0$, and nondecreasing.

(H5) For each i there exists $r_i > 0$ such that

$$f_i(s)(r_i - s) > 0, \qquad s \neq r_i.$$

Equivalently, $f_i > 0$ on $(0, r_i)$, $f_i(r_i) = 0$, and $f_i < 0$ on (r_i, ∞).

(H6) For each i,

$$\lim_{x\to 0^+} \frac{f_i(x)}{x^{p_i-1}} = 0, \qquad \lim_{x\to 0^+} \frac{g_i(x)}{x^{p_i-1}} = 0.$$

We keep (H3) to control the chain of g_i at infinity.

Theorem 3.1. Assume (H3)–(H5). Then there exists $\Lambda_{**} > 0$ such that if $\lambda_i + \mu_i \geq \Lambda_{**}$ for all i, problem (1.1) has at least one positive weak solution $\underline{u} \in X$.

Proof . The construction of supersolutions is as in Theorem 2.3. We only need to adapt the subsolutions to the falling-zero structure. Fix $\alpha \in (0, \min_{1\leq i\leq n} r_i)$ and define

$$\psi_i(x) := \left(\frac{\alpha}{2}\right)^{\frac{1}{p_i-1}} \left(\frac{p_i-1}{p_i}\right)^{\frac{p_i}{p_i-1}} \phi_i(x)^{\frac{p_i}{p_i-1}}, \qquad i = 1, \dots, n.$$

As before, $-\Delta_{p_i}\psi_i = \frac{\alpha}{2}\big(\sigma_i\phi_i^{p_i} - |\nabla\phi_i|^{p_i}\big)$. On $\Omega\setminus\Omega_\delta$ we have $-\Delta_{p_i}\psi_i \leq \frac{\alpha}{2}\sigma_i$. By (H5), $f_i > 0$ and $g_i > 0$ on $(0, r_i)$, and $\psi_i(x) \in [\frac{\alpha\eta}{2}, \alpha]$ on $\Omega\setminus\Omega_\delta$. Define

$$m_i(\alpha) := \min_{s\in[\frac{\alpha\eta}{2},\alpha]} \big[f_i(s) \wedge g_i(s)\big] > 0.$$

Choose $\Lambda^{\text{sub}}_{**,i}$ so that

$$\frac{\alpha}{2}\sigma_i < (\lambda_i + \mu_i)m_i(\alpha) \quad \text{whenever} \quad \lambda_i + \mu_i \geq \Lambda^{\text{sub}}_{**,i}.$$

Then on $\Omega\setminus\Omega_\delta$,

$$-\Delta_{p_i}\psi_i \leq \frac{\alpha}{2}\sigma_i < (\lambda_i + \mu_i)m_i(\alpha) \leq \lambda_i f_i(\psi_i) + \mu_i g_i(\psi_{i+1}).$$

On Ω_δ, we have $-\Delta_{p_i}\psi_i \leq -\frac{\alpha}{2}m < 0$ (where $m > 0$ is the constant from (2.2)), while $f_i(\psi_i), g_i(\psi_{i+1}) \geq 0$, so the subsolution inequality holds strictly there. Thus for

$$\Lambda^{\text{sub}}_{**} := \max_{1\leq i\leq n} \Lambda^{\text{sub}}_{**,i},$$

$\underline{\psi}$ is a strict subsolution for $\lambda_i + \mu_i \geq \Lambda^{\text{sub}}_{**}$. Supersolutions $\underline{Z}$ are constructed exactly as in Theorem 2.3 (possibly after replacing bounded g_i by $\tilde{g}_i \geq g_i$ with $\tilde{g}_i(x) \to \infty$ and satisfying (H3)). Then, for $\lambda_i + \mu_i \geq \Lambda^{\text{sub}}_{**}$, we can choose C so that $\underline{Z} \geq \underline{\psi}$. Lemma 1.2 yields a positive solution between $\underline{\psi}$ and $\underline{Z}$. □

Theorem 3.2. Assume (H3)–(H6). Then there exists $\widehat{\Lambda}_{**} > 0$ such that if $\lambda_i + \mu_i \geq \widehat{\Lambda}_{**}$ for all i, system (1.1) has at least two distinct positive weak solutions.

Proof . As in Theorem 2.4, we use a three-solution argument. The zero vector $\underline{0}$ is a solution and thus a subsolution. For $\lambda_i + \mu_i$ large, Theorem 3.1 provides a strict positive subsolution $\underline{\omega}$ of the form constructed above and a large supersolution $\underline{Z}$. To obtain a small strict supersolution, use (H6). By continuity near zero and the limit conditions, for each i there exists $\varepsilon_0 > 0$ such that

$$\sigma_i s^{p_i-1} > \lambda_i f_i(s) + \mu_i g_i(c_i s) \quad \text{for } s \in (0, \varepsilon_0),$$

where the c_i are as before. Let $\varepsilon \in (0, \min\{\varepsilon_0, \alpha/2\})$ and $\zeta_i := \varepsilon\phi_i$. Then

$$\begin{aligned}
\int_\Omega |\nabla\zeta_i|^{p_i-2}\nabla\zeta_i\cdot\nabla\xi\,dx &= \varepsilon^{p_i-1}\int_\Omega |\nabla\phi_i|^{p_i-2}\nabla\phi_i\cdot\nabla\xi\,dx \\
&= \int_\Omega \sigma_i(\varepsilon\phi_i)^{p_i-1}\xi\,dx \\
&> \int_\Omega \big[\lambda_i f_i(\zeta_i) + \mu_i g_i(c_i\zeta_i)\big]\xi\,dx \\
&\geq \int_\Omega \big[\lambda_i f_i(\zeta_i) + \mu_i g_i(\zeta_{i+1})\big]\xi\,dx
\end{aligned}$$

for $0 \le \xi \in C_0^\infty(\Omega)$, so $\underline{\zeta}$ is a strict supersolution. Choosing ε small we ensure $\underline{\zeta} \le \underline{Z}$ and $\underline{\omega} \not\le \underline{\zeta}$. Lemma 1.3 then yields three solutions in $[\underline{0}, \underline{Z}]$, at least two of which are positive. This holds for $\lambda_i + \mu_i$ larger than some $\widehat{\Lambda}_{**}$. $\square$

4 Examples

We conclude with some explicit examples of nonlinearities satisfying our hypotheses.

Example 4.1 (Power-type combined effects). Let

$$f_i(x) = a_i x^{\alpha_i} - c_i, \qquad g_i(x) = b_i x^{\beta_i} - C_i, \qquad x \ge 0,$$

where $a_i, b_i \ge 0$, $c_i, C_i \ge 0$, $\alpha_i \ge 0$, and

$$\alpha_i < p_i - 1, \qquad \prod_{i=1}^n \beta_i < \prod_{i=1}^n (p_i - 1).$$

Then f_i and g_i satisfy (H1)–(H3) and Theorem 2.3 applies.

Example 4.2 (Piecewise power-type nonlinearities). Define

$$f_i(x) = \begin{cases} x^{\alpha_1^{(i)}}, & x \le 1, \\ \dfrac{\alpha_1^{(i)}}{\alpha_2^{(i)}} x^{\alpha_2^{(i)}} + \left(1 - \dfrac{\alpha_1^{(i)}}{\alpha_2^{(i)}}\right), & x > 1, \end{cases} \qquad g_i(x) = \begin{cases} x^{\beta_1^{(i)}}, & x \le 1, \\ \dfrac{\beta_1^{(i)}}{\beta_2^{(i)}} x^{\beta_2^{(i)}} + \left(1 - \dfrac{\beta_1^{(i)}}{\beta_2^{(i)}}\right), & x > 1, \end{cases}$$

where $\alpha_1^{(i)}, \alpha_2^{(i)}, \beta_1^{(i)}, \beta_2^{(i)} \ge 0$ satisfy

$$\alpha_1^{(i)} > p_i - 1 \text{ if } p_i \in \mathbb{N}, \qquad \alpha_1^{(i)} > [p_i] \text{ if } p_i \notin \mathbb{N},$$

similarly for $\beta_1^{(i)}$, and

$$\alpha_2^{(i)} < p_i - 1, \qquad \prod_{i=1}^n \beta_2^{(i)} < \prod_{i=1}^n (p_i - 1).$$

Then (H1)–(H3) and the smoothness conditions near zero in Theorem 2.4 are satisfied, and there exist at least two positive solutions for large parameters.

Example 4.3 (Falling zeros). Let

$$f_i(x) = x^{\gamma_i}(1 - x), \qquad g_i(x) \text{ as in the previous examples,}$$

with $\gamma_i > p_i - 1$, so that f_i has a falling zero at $x = 1$ and $f_i(x) < 0$ for $x > 1$. Then f_i and g_i satisfy (H3)–(H6) and Theorems 3.1 and 3.2 apply: for large parameters the system (1.1) has at least one, and under (H6) at least two positive solutions.

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