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Prešić type contractions in strongly sequential S-metric spaces

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Abstract

The purpose of this study is to establish several fixed point theorems for certain classes of F-contractions defined on strongly sequential S-metric spaces. In addition, we also derive some fixed point results for the class of F-Prešić-type contractions. To illustrate the applicability of the obtained results, an example along with an application to the solvability of a class of combined fractional integral equations is provided.

Keywords: S-metric space, strong JS-metric space, strongly sequential S-metric space, Wardowski-contraction, Prešić-type contractions, fractional integral equation
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1 Introduction

In order to demonstrate the existence and uniqueness of solutions to various mathematical problems, such as integral and differential equations and their fractional forms, fixed-point theorems play a crucial role. Many real-world problems arising in economics, physics, and engineering, after appropriate modeling, lead to differential or integral equations of arbitrary or even fractional order. The solvability of such equations often relies on fixed-point techniques. Furthermore, the expertise of researchers in numerical analysis is required to approximate the solutions of these equations under specific circumstances.

The generalization of the Banach contraction principle and the concept of metric spaces are two fundamental problems in fixed-point theory that have been extensively investigated by numerous authors. Researchers from various universities and research centers have shown considerable interest in these topics. In particular, many efforts have been devoted to extending the notion of the classical metric space in order to develop more general frameworks. As a result, several generalized metric structures have been introduced, including cone metric spaces [19, 18], controlled metric spaces [29], double-controlled metric spaces [1], JS-metric spaces [22], rectangular metric spaces [6], generalized metric spaces [16], b-metric-like spaces [20], extended b-metric spaces [24], and many others. Furthermore, these generalized structures have been combined with fuzzy and probabilistic concepts, leading to the development of new mathematical models. Research in this direction is still ongoing, and a comprehensive framework has yet to be established (see, for instance, [3]).

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Alongside these developments, substantial work has been carried out on various types of contraction mappings, resulting in well-known contractions such as those introduced by Kannan [25], Chatterjea [7], Reich [33], Ciric [8], Hardy–Rogers [17], Wardowski [39], Jleli–Samet [23], Geraghty [15], Mizoguchi–Takahashi [28], Preši'c [32], as well as their various combinations.

By modifying D-metric [11] and G-metric spaces [30], Sedghi et al. [35] introduced the idea of S-metric spaces. Subsequently, as a generalization of S-metric spaces, the idea of S_b-metric spaces was presented in 2016 [36]. A generalization of S-metric spaces known as S^{JS}-metric spaces was recently introduced by Beg et al. [5]; however, it does not satisfy the symmetry property and the rectangular inequality.

The aim of this paper is to prove some fixed point theorems for F-contractions and F-Prešić-contractions in a strongly sequential S-metric space (s.s.s.m.s.). Let Δ be a nonempty set and let $\alpha_g : \Delta \times \Delta \to [0, \infty]$ be a mapping. For any $\zeta \in \Delta$, we define

$$C(\alpha_g, \Delta, \zeta) = \big\{\{\zeta_n\} \subset \Delta : \lim_{n\to\infty} \alpha_g(\zeta_n, \zeta) = 0\big\}. \tag{1.1}$$

Jleli and Samet [22] introduced a generalization of the concept of metric space, which they termed a JS-metric space.

Definition 1.1. [22] Let $\alpha_g : \Delta \times \Delta \to [0, \infty]$ be a mapping satisfying the following conditions:

1. for every $\zeta, \zeta' \in \Delta$, $\alpha_g(\zeta, \zeta') = 0$ implies $\zeta = \zeta'$;
2. for every $\zeta, \zeta' \in \Delta$, we have $\alpha_g(\zeta, \zeta') = \alpha_g(\zeta', \zeta)$;
3. if $(\zeta, \zeta') \in \Delta \times \Delta$ and $\{\zeta_n\} \in C(\alpha_g, \Delta, \zeta)$, then

$$\alpha_g(\zeta, \zeta') \le p \limsup_{n\to\infty} \alpha_g(\zeta_n, \zeta'),$$

for some $p > 0$.

Then α_g is called a JS-metric and the pair (Δ, α_g) is called a JS-metric space with parameter p.

The concept of strong JS-metric spaces was introduced by Gajić and Ralević as follows.

Definition 1.2. [13] Let $\mathcal{D} : \Delta \times \Delta \to [0, \infty]$ be a mapping satisfying the following conditions:

1. for every $\zeta, \zeta' \in \Delta$, $\mathcal{D}(\zeta, \zeta') = 0$ implies $\zeta = \zeta'$;
2. for every $\zeta, \zeta' \in \Delta$, $\mathcal{D}(\zeta, \zeta') = \mathcal{D}(\zeta', \zeta)$;
3. there exists $p > 0$ such that for every $\zeta, \zeta' \in \Delta$, for all sequences $\{\zeta_n\} \in C(\mathcal{D}, \Delta, \zeta)$ and $\{\zeta'_n\} \in C(\mathcal{D}, \Delta, \zeta')$, we have

$$\mathcal{D}(\zeta, \zeta') \le p \limsup_{n\to\infty} \mathcal{D}(\zeta_n, \zeta'_n).$$

Then, the pair $(\Delta, \mathcal{D})$ is called a *strong JS-metric space.*

Let Δ be a nonempty set and let $\Lambda : \Delta \times \Delta \times \Delta \to [0, \infty]$ be a mapping. For any $\zeta \in \Delta$, we define the set

$$\underline{\mathcal{C}}(\Lambda, \Delta, \zeta) := \Big\{\{\zeta_n\} \subset \Delta : \lim_{n\to\infty} \Lambda(\zeta, \zeta, \zeta_n) = 0\Big\}.$$

Beg et al. [5] introduced the concept of S^{JS}-metric spaces, which generalizes the notion of S-metric spaces.

Definition 1.3. [5] Let $\Lambda : \Delta \times \Delta \times \Delta \to [0, \infty]$ be a mapping satisfying the following conditions:

1. $\Lambda(\zeta, \zeta', \zeta'') = 0$ implies $\zeta = \zeta' = \zeta''$;
2. there exists a constant $K > 0$ such that for any $(\zeta, \zeta', \zeta'') \in \Delta \times \Delta \times \Delta$ and for any sequence $\{\zeta_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta)$, we have

$$\Lambda(\zeta, \zeta', \zeta'') \le K \limsup_{n\to\infty} \Big(\Lambda(\zeta', \zeta', \zeta_n) + \Lambda(\zeta'', \zeta'', \zeta_n)\Big).$$

Then, the pair (Δ, Λ) is called an *S^{JS}-metric space.*

The following theorem is the well-known Banach contraction principle and serves as the foundation for many generalizations in fixed point theory.

Theorem 1.4. [4] Let (Δ, d) be a complete metric space, and let $Ł : \Delta \to \Delta$ be a mapping such that

$$d(Ł\iota, Ł\kappa) \leq \alpha\, d(\iota, \kappa) \quad \text{for all } \iota, \kappa \in \Delta,$$

where $\alpha \in [0, 1)$. Then, there exists a unique point $\sigma \in \Delta$ such that $\sigma = Ł\sigma$. Moreover, for any initial point $\iota_0 \in \Delta$, the sequence defined by $\iota_{n+1} = Ł\iota_n$ converges to σ.

Definition 1.5. [39] Let (Δ, d) be a metric space. A mapping $\mathcal{L} : \Delta \to \Delta$ is called an *F-contraction* if there exists a constant $\tau > 0$ such that for every $\ell, \ell' \in \Delta$,

$$d(\mathcal{L}\ell, \mathcal{L}\ell') > 0 \quad \Rightarrow \quad \tau + F\big(d(\mathcal{L}\ell, \mathcal{L}\ell')\big) \leq F\big(d(\ell, \ell')\big),$$

where $F : (0, +\infty) \to (-\infty, +\infty)$ is a function satisfying the following properties:

(F1) F is strictly increasing;

(F2) For any sequence $\{a_n\}$ of positive real numbers,

$$\lim_{n\to\infty} a_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} F(a_n) = -\infty;$$

(F3) There exists $k \in (0, 1)$ such that

$$\lim_{a\to 0^+} a^k F(a) = 0.$$

Definition 1.6. [12] Let Δ be a non-empty set. A mapping

$$\Lambda : \Delta \times \Delta \times \Delta \longrightarrow [0, +\infty]$$

is called a strongly sequential S-metric (s.s.s.m.) if for all $\zeta, \zeta', \zeta'' \in \Delta$, the following conditions hold:

(a) $\Lambda(\zeta, \zeta', \zeta'') = 0$ iff $\zeta = \zeta' = \zeta''$;
(b) There exists a constant $C > 0$ such that if $\{\zeta_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta)$, $\{\zeta'_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta')$, and $\{\zeta''_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta'')$, then

$$\Lambda(\zeta, \zeta', \zeta'') \leq C\left(\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, \zeta'_n) + \limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, \zeta''_n)\right).$$

Then the pair (Δ, Λ) is called a strongly sequential S-metric space (*s.s.s.m.s.*).

Moreover, if Λ also satisfies

(c) $\Lambda(\zeta, \zeta, \zeta') = \Lambda(\zeta', \zeta', \zeta)$ for all $\zeta, \zeta' \in \Delta$,

then we call (Δ, Λ) a *symmetric s.s.s.m.s.*.

In [12], instead of condition a we applied the following condition:

a': $\Lambda(\zeta, \zeta', \zeta'') = 0$ implies $\zeta = \zeta' = \zeta''$ which is stronger than condition a.

Remark 1.7. Taking the constant sequences

$$\zeta'_n = \zeta' \quad \text{and} \quad \zeta''_n = \zeta'', \quad n \in \mathbb{N},$$

it follows that every symmetric s.s.s.m.s. with zero self-distance property is also a *symmetric S^{JS}-metric space*. In fact, we corrected the Remark 2 in [12].

The classes of s.s.s.m.s. are broader than many previously studied classes of abstract metric spaces. In particular, every s.s.s.m.s. includes, as special cases, each of the following: an S-metric space, an S_b-metric space and a dislocated S_b-metric space.

Definition 1.8. [12] Let (Δ, Λ) be a s.s.s.m.s., and let $\{\zeta_n\}$ be a sequence in Δ with $a \in \Delta$. Then:

(i) The sequence $\{\zeta_n\}$ is said to *converge* to $\zeta \in \Delta$ if

$$\{\zeta_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta).$$

(ii) The sequence $\{\zeta_n\}$ is called *Cauchy* if

$$\lim_{n,m\to\infty} \Lambda(\zeta_n, \zeta_n, \zeta_m) = 0.$$

(iii) The space (Δ, Λ) is said to be *complete* if every Cauchy sequence in (Δ, Λ) is convergent.

Example 1.9. [12] Let $\mathcal{L} = \mathbb{R}$ and define

$$\overline{S}(\jmath, \iota, \ell) = |\jmath - \iota| + |\jmath - \ell|, \quad \forall \jmath, \iota, \ell \in \mathcal{L}.$$

It can be observed that $\overline{S}$ defines an s.s.s.m.s. with the constant $C = 1$.

Clearly, the usual triangle inequality expected in an S-metric space does not hold in this example (take $\jmath = 5$, $\iota = 1$, $\ell = 1$, and $a = 0$). Moreover, this space can be regarded as an S_b-metric space or an S_b-metric-like space with $s = 2$ (we corrected Example 1 of [12]).

Proposition 1.10. [12] If $(\Delta, \mathcal{D})$ be a strong JS-metric space, then there exists a mapping

$$\Lambda : \Delta \times \Delta \times \Delta \longrightarrow [0, +\infty)$$

such that (Δ, Λ) forms an s.s.s.m.s.

Remark 1.11. [12]

Note that if (Δ, σ) is a b-metric-like space, then it is also a strong JS-metric space on Δ. By Proposition 1.10, it follows that (Δ, σ) can also generate an s.s.s.m.s. on Δ.

Proposition 1.12. [5, 12] Let (Δ, Λ) be a s.s.s.m.s., and let $\{\zeta_n\}$ be a sequence in Δ such that

$$\lim_{n\to\infty} \Lambda(\zeta_n, \zeta_n, \zeta_n) = 0 \quad \text{and} \quad \{\zeta_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta).$$

Then it follows that

$$\Lambda(\zeta, \zeta, \zeta) = 0.$$

Remark 1.13. [5, 12] Let (Δ, Λ) be an s.s.s.m.s., and let $\{\zeta_n\}$ be a Cauchy sequence in Δ such that $\{\zeta_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta)$. Then we have $\Lambda(\zeta, \zeta, \zeta) = 0$.

Proposition 1.14. [5, 12] In an s.s.s.m.s. (Δ, Λ), any sequence $\{\zeta_n\}$ satisfying

$$\lim_{n\to\infty} \Lambda(\zeta_n, \zeta_n, \zeta_n) = 0$$

can converge to at most one point in Δ.

Remark 1.15. [5, 12] In a complete s.s.s.m.s. (Δ, Λ), any Cauchy sequence can converge to at most one point in Δ.

Now, we introduce a topology on an s.s.s.m.s. Let (Δ, Λ) be an s.s.s.m.s.. For every $\zeta \in \Delta$ and $\eta > 0$, define

$$\mathcal{B}(\zeta, \eta) := \{\xi \in \Delta : \Lambda(\zeta, \zeta, \xi) < \eta\}, \qquad \text{and} \qquad \mathcal{B}[\zeta, \eta] := \{\xi \in \Delta : \Lambda(\zeta, \zeta, \xi) \leq \eta\}.$$

Remark 1.16. [5, 12] It is straightforward to verify that the collection

$$\tau_\Lambda := \{\emptyset\} \cup \{\mathcal{V} \subseteq \Delta \,:\, \mathcal{V} \neq \emptyset, \text{ and for each } \zeta \in \mathcal{V}, \text{ there exists } \eta > 0 \text{ such that } \mathcal{B}(\zeta, \eta) \subseteq \mathcal{V}\}$$

defines a topology on Δ.

The above definition guarantees that each ball contains its center, since, in this paper, we consider all classes of s.s.s.m.s. for which $\Lambda(\zeta, \zeta, \zeta) = 0$.

Proposition 1.17. [5, 12] In an s.s.s.m.s. (Δ, Λ), if a self-mapping $\mathcal{L} : \Delta \to \Delta$ is continuous at $\zeta \in \Delta$, then for any sequence $\{\zeta_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta)$ we have

$$\{\mathcal{L}\zeta_n\} \in \underline{\mathcal{C}}(\Lambda, \Delta, \mathcal{L}\zeta).$$

.

2 Main results

2.1 Weak-Wardowski-Hardy-Rogers type contractions in a s.s.s.m.s.

In this section, we impose additional restrictions on the control function F. Define $\mathbf{F}$ as the class of functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

(F1) F is strictly increasing and continuous,

(F2) For every sequence $\{\mu_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to\infty} \mu_n = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} F(\mu_n) = -\infty.$$

Define

$$\begin{aligned} M(\zeta, \zeta', \zeta'') &= \alpha\Lambda(\zeta, \zeta', \zeta'') + \beta\Lambda(\zeta, \zeta, Ł\zeta) \\ &\quad + \gamma\Lambda(\zeta', \zeta', Ł\zeta') + \rho\frac{\Lambda(\zeta'', \zeta'', Ł\zeta'')}{C} \\ &\quad + \sigma\left(\frac{\Lambda(\zeta, \zeta, Ł\zeta') + \Lambda(\zeta', \zeta', Ł\zeta'') + \Lambda(\zeta'', \zeta'', Ł\zeta)}{3}\right). \end{aligned}$$

Definition 2.1. Let (Δ, Λ) be a complete s.s.s.m.s. with constant $C \geq 1$. A mapping $Ł : \Delta \to \Delta$ is said to be a Λ-Wardowski-Hardy-Rogers type contraction of type A if, for every $\zeta, \zeta', \zeta'' \in \Delta$ with

$$\Lambda(Ł\zeta, Ł\zeta', Ł\zeta'') > 0 \quad \text{and} \quad M(\zeta, \zeta', \zeta'') > 0,$$

the following inequality holds:

$$F\big(\Lambda(Ł\zeta, Ł\zeta', Ł\zeta'')\big) \;\leq\; F(M(\zeta, \zeta', \zeta'')) - \tau. \tag{2.1}$$

Here, the parameters $\alpha, \beta, \gamma, \rho, \sigma \geq 0$ satisfy $\alpha + \beta + \gamma + \rho + \sigma = 1$, $\tau > 0$ and $F \in \mathbf{F}$.

Theorem 2.2. Let (Δ, Λ) be a complete s.s.s.m.s. and let $Ł : \Delta \to \Delta$ be a mapping satisfying:

(i) $Ł$ is a Λ-Wardowski-Hardy-Rogers type contraction of type A,

(ii) there exists $\zeta_0 \in \Delta$ such that

$$\delta(\Lambda, Ł, \zeta_0) := \sup\left\{\Lambda\big(Ł^i\zeta_0, Ł^i\zeta_0, Ł^j\zeta_0\big) : i, j = 1, 2, \dots\right\} < \infty.$$

Then $Ł$ admits at least one fixed point in Δ provided that $Ł$ is continuous or (Δ, Λ) is symmetry. Furthermore, if (Δ, Λ) is symmetric, then this fixed point is unique.

Proof . We define

$$\delta(\Lambda, Ł^{p+1}, \zeta_0) := \sup\left\{\Lambda\left(Ł^{p+i}\zeta_0, Ł^{p+i}\zeta_0, Ł^{p+j}\zeta_0\right) : i, j = 1, 2, \cdots\right\},$$

for all $p \geq 1$. Clearly, $\delta(\Lambda, Ł^{p+1}, \zeta_0) \leq \delta(\Lambda, Ł, \zeta_0) < \infty$ for all $p \geq 1$. Then for all $p \geq 1$ and for all $i, j = 1, 2, \cdots$,

$$\begin{aligned} &F(\Lambda(Ł^{p+i}\zeta_0, Ł^{p+i}\zeta_0, Ł^{p+j}\zeta_0)) \\ \leq &F\Big(\alpha\Lambda(Ł^{p-1+i}\zeta_0, Ł^{p-1+i}\zeta_0, Ł^{p-1+j}\zeta_0) + (\beta+\gamma)\Lambda(Ł^{p-1+i}\zeta_0, Ł^{p-1+i}\zeta_0, Ł^{p+i}\zeta_0) + \rho\frac{\Lambda(Ł^{p-1+j}\zeta_0, Ł^{p-1+j}\zeta_0, Ł^{p+j}\zeta_0)}{C} \\ &+ \sigma(\frac{\Lambda(Ł^{p-1+i}\zeta_0, Ł^{p-1+i}\zeta_0, Ł^{p+i}\zeta_0) + \Lambda(Ł^{p-1+i}\zeta_0, Ł^{p-1+i}\zeta_0, Ł^{p+j}\zeta_0) + \Lambda(Ł^{p-1+j}\zeta_0, Ł^{p-1+j}\zeta_0, Ł^{p+i}\zeta_0)}{3})\Big) - \tau \\ &\leq F(\eta\delta(\Lambda, Ł^p, \zeta_0)) - \tau, \end{aligned}$$

where $\eta = \alpha + \beta + \gamma + \rho + \sigma$. Taking the sup on i, j in the left hand side of the above inequality, for all $p \geq 1$,

$$\begin{aligned} F\left(\delta(\Lambda, Ł^{p+1}, \zeta_0)\right) &= F\left(\sup_{i,j\geq 1} \Lambda(Ł^{p+i}\zeta_0, Ł^{p+i}\zeta_0, Ł^{p+j}\zeta_0)\right) \\ &\leq F(\delta(\Lambda, Ł^{p}, \zeta_0)) - \tau \\ &\leq F(\delta(\Lambda, Ł^{p-1}, \zeta_0)) - \tau - \tau \\ &\;\vdots \\ &\leq F(\delta(\Lambda, Ł, \zeta_0)) - p\tau. \end{aligned}$$

since $\eta = 1$. Let $\zeta_i = Ł\zeta_{i-1} = Ł^i\zeta_0$ for all $i \in \mathbb{N}$. For all $m > n \geq 1$ we have,

$$\begin{aligned} F\left(\Lambda(\zeta_n, \zeta_n, \zeta_m)\right) &= F(\Lambda(Ł^n\zeta_0, Ł^n\zeta_0, Ł^m\zeta_0)) \\ &\leq F\left(\delta(\Lambda, Ł^n, \zeta_0)\right) \\ &\leq F\left(\delta(\Lambda, Ł, \zeta_0)\right) - (n-1)\tau \longrightarrow -\infty \quad (\text{as } n \to \infty). \end{aligned}$$

We infer from $(F2)$ that,

$$\Lambda(\zeta_n, \zeta_n, \zeta_m) \to 0 \quad (\text{as } n \to \infty).$$

Therefore, $\{\zeta_n\}$ is a Cauchy sequence in Δ. From the completeness of Δ, $\{\zeta_n\}$ is convergent. Let $\lim_{n\to\infty} \zeta_n = \zeta \in \Delta$. So, we have $\lim_{n\to\infty} \Lambda(\zeta, \zeta, \zeta_n) = 0$.

On the other hand, using condition (b) of Definition 1.6 and the continuity assumption of Ł, we have

$$\Lambda(\zeta, \zeta, Ł\zeta) \leq C\Big(\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, \zeta_{n+1})\Big) = 0.$$

Hence, we have $Ł\zeta = \zeta$, *i.e.*, $\zeta \in \Delta$ is a fixed point of Ł. If we do not have the continuity assumption of Ł we should use the contractive condition 2.1 and Definition 1.6 as follows:

If $\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, Ł\zeta) = 0$, then as we know that

$$\Lambda(\zeta, \zeta, Ł\zeta) \leq C\Big(\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, Ł\zeta)\Big),$$

we conclude that $\Lambda(\zeta, \zeta, Ł\zeta) = 0$ which implies that $\zeta = Ł\zeta$.

So, we suppose that $\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, Ł\zeta) > 0$. We know that

$$\begin{aligned} &F(\Lambda(\zeta_{n+1}, \zeta_{n+1}, Ł\zeta)) = F(\Lambda(Ł\zeta_n, Ł\zeta_n, Ł\zeta)) \\ &\leq F\Big(\alpha\Lambda(\zeta_n, \zeta_n, \zeta) + (\beta+\gamma)\Lambda(\zeta_n, \zeta_n, Ł\zeta_n) + \rho\frac{\Lambda(\zeta, \zeta, Ł\zeta)}{C} + \sigma(\frac{\Lambda(\zeta_n, \zeta_n, Ł\zeta_n) + \Lambda(\zeta_n, \zeta_n, Ł\zeta) + \Lambda(\zeta, \zeta, Ł\zeta_n)}{3})\Big) - \tau. \end{aligned}$$

Taking the limsup and using Definition 1.6 in the above we obtain that:

$$\begin{aligned} \limsup_{n\to\infty} F(\Lambda(\zeta_{n+1}, \zeta_{n+1}, Ł\zeta)) &= F(\limsup_{n\to\infty} \Lambda(Ł\zeta_n, Ł\zeta_n, Ł\zeta)) \\ &\leq F\Big(\limsup_{n\to\infty} \rho\frac{\Lambda(\zeta, \zeta, Ł\zeta)}{C} + \sigma(\frac{\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, Ł\zeta) + \limsup_{n\to\infty} \Lambda(\zeta, \zeta, Ł\zeta_n)}{3})\Big) \\ &\quad - \tau. \end{aligned}$$

Consequently, from the assumption $\lim_{n\to\infty} \Lambda(\zeta, \zeta, \zeta_n) = 0$, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} \Lambda(\zeta_{n+1}, \zeta_{n+1}, Ł\zeta) &< \rho\frac{\Lambda(\zeta, \zeta, Ł\zeta)}{C} + \sigma(\frac{\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, Ł\zeta)}{3}) \\ &\leq \rho\limsup_{n\to\infty} \Lambda(Ł\zeta_n, Ł\zeta_n, Ł\zeta) + \sigma(\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, Ł\zeta)). \end{aligned}$$

which is impossible, so $\Lambda(\zeta,\zeta,Ł\zeta) \le C\Big(\limsup_{n\to\infty}\Lambda(\zeta_n,\zeta_n,Ł\zeta)\Big) = 0$. Hence $Ł\zeta=\zeta$. Now, if ζ and ζ' be two fixed points of Ł in Δ with $\Lambda(\zeta,\zeta,\zeta)=0$ and $\Lambda(\zeta',\zeta',\zeta')=0$, then

$$\begin{aligned} F(\Lambda(\zeta,\zeta,\zeta')) &= F(\Lambda(Ł\zeta,Ł\zeta,Ł\zeta')) \\ &\le F\Big(\alpha\Lambda(\zeta,\zeta,\zeta')+(\beta+\gamma)\Lambda(\zeta,\zeta,Ł\zeta)+\rho\Lambda(\zeta',\zeta',Ł\zeta'))+\sigma(\frac{\Lambda(\zeta,\zeta,Ł\zeta)+\Lambda(\zeta,\zeta,Ł\zeta')+\Lambda(\zeta',\zeta',Ł\zeta)}{3})\Big)-\tau \\ &\le F((\alpha+\sigma)\Lambda(\zeta,\zeta,\zeta'))-\tau \\ &< F((\alpha+\sigma)\Lambda(\zeta,\zeta,\zeta')), \end{aligned}$$

which is impossible, so this implies that $\Lambda(\zeta,\zeta,\zeta')=0$, hence $\zeta=\zeta'$. □

Define

$$M(\zeta,\zeta') = \alpha\,\Lambda(\zeta,\zeta,\zeta')+\beta\,\Lambda(\zeta,\zeta,Ł\zeta)+\gamma\,\frac{\Lambda(\zeta',\zeta',Ł\zeta')}{C}+\sigma\,\frac{\Lambda(\zeta,\zeta,Ł\zeta')+\Lambda(\zeta',\zeta',Ł\zeta)}{2}.$$

Definition 2.3. Let (Δ,Λ) be a complete s.s.s.m.s. with $C\ge 1$. A mapping $Ł:\Delta\to\Delta$ is called a Λ-Wardowski–Hardy–Rogers type contraction of type B if

$$F\big(\Lambda(Ł\zeta,Ł\zeta,Ł\zeta')\big) \le F\big(M(\zeta,\zeta')\big)-\tau$$

for all $\zeta,\zeta'\in\Delta$ with $\Lambda(Ł\zeta,Ł\zeta,Ł\zeta')>0$ and $M(\zeta,\zeta')>0$, for some $\alpha,\beta,\gamma,\sigma\ge 0$ with $\alpha+\beta+\gamma+\sigma=1$, where $F\in\mathbf{F}$ and $\tau>0$.

The proof of the following result will proceed in a manner analogous to that of Theorem 2.4.

Theorem 2.4. Let (Δ,Λ) be a complete s.s.s.m.s., and let $Ł:\Delta\to\Delta$ be a mapping such that:

(i) Ł is a Λ-Wardowski–Hardy–Rogers type contraction of type B;

(ii) there exists $\zeta_0\in\Delta$ such that

$$\delta(\Lambda,Ł,\zeta_0) := \sup\big\{\Lambda\big(Ł^i\zeta_0,\,Ł^i\zeta_0,\,Ł^j\zeta_0\big) : i,j=1,2,\ldots\big\}<\infty.$$

Then Ł has at least one fixed point in Δ, provided that Ł is continuous or (Δ,Λ) is symmetric. Moreover, if (Δ,Λ) is symmetric, then the fixed point is unique.

From this point onward, we assume that $\Lambda(Ł\zeta,Ł\zeta,Ł\zeta')$ and the expression on the right-hand side are positive. By setting $\sigma=0$ in Theorem 2.4, we obtain a result for Reich-type contractions, which is a special case of the above theorem and yields the following statement:

Corollary 2.5. Let (Δ,Λ) be a complete s.s.s.m.s. with $C\ge 1$ and $Ł:\Delta\to\Delta$ be a mapping such that:

(i) Ł is a Λ-Wardowski-Reich type contraction, that is,

$$F(\Lambda(Ł\zeta,Ł\zeta,Ł\zeta')) \le F\Big(\alpha\Lambda(\zeta,\zeta,\zeta')+\beta\Lambda(\zeta,\zeta,Ł\zeta)+\gamma\frac{\Lambda(\zeta',\zeta',Ł\zeta')}{C}\Big)-\tau,$$

for all $\zeta,\zeta'\in\Delta$ and for some $\alpha,\beta,\gamma\ge 0$ with $\alpha+\beta+\gamma=1$ where $F\in\mathbf{F}$ and $\tau>0$.

(ii) there is $\zeta_0\in\Delta$ so that

$$\delta(\Lambda,Ł,\zeta_0) := \sup\big\{\Lambda\left(Ł^i\zeta_0,Ł^i\zeta_0,Ł^j\zeta_0\right) : i,j=1,2,\cdots\big\}<\infty.$$

Then Ł has a unique fixed point in Δ.

For a detailed discussion of the definitions of Kannan, Chatterjea, Reich, and Hardy–Rogers contractions, the reader is referred to [7, 17, 25, 26, 33].

Remark 2.6. Taking $\alpha = 0$ in Corollary 2.5, the Kannan fixed point theorem is extended.

Taking $\beta = \gamma = \sigma = 0$ and $\alpha = 1$ as a special case of Theorem 2.4, we have the following result.

Corollary 2.7. Let (Δ, Λ) be a complete s.s.s.m.s. and $Ł : \Delta \to \Delta$ be a mapping such that:

(i) $Ł$ is a Λ-Wardowski-contraction, that is,

$$\Lambda(Ł\zeta, Ł\zeta, Ł\zeta') > 0 \text{ and } \Lambda(\zeta, \zeta, \zeta') > 0 \Rightarrow F(\Lambda(Ł\zeta, Ł\zeta, Ł\zeta')) \leq F(\Lambda(\zeta, \zeta, \zeta')) - \tau,$$

for all $\zeta, \zeta' \in \Delta$, where $F \in \mathbf{F}$,

(ii) there is $\zeta_0 \in \Delta$ so that

$$\delta(\Lambda, Ł, \zeta_0) := \sup\left\{\Lambda\left(Ł^i\zeta_0, Ł^i\zeta_0, Ł^j\zeta_0\right) : i, j = 1, 2, \cdots\right\} < \infty.$$

Then $Ł$ has one unique fixed point in Δ.

Taking $\alpha = \beta = \gamma = 0$ and $\sigma = 1$ in Theorem 2.4, the Chatterjea fixed point theorem is generalized.

Corollary 2.8. Let (Δ, Λ) be a complete s.s.s.m.s. with $C \geq 1$, and let $Ł : \Delta \to \Delta$ be a mapping such that:

(i) $Ł$ is a Λ-Wardowski–Chatterjea type contraction; that is, for all $\zeta, \zeta' \in \Delta$, if

$$\Lambda(Ł\zeta, Ł\zeta, Ł\zeta') > 0 \quad \text{and} \quad \frac{\Lambda(\zeta, \zeta, Ł\zeta') + \Lambda(\zeta', \zeta', Ł\zeta)}{2} > 0,$$

then

$$F\Big(\Lambda(Ł\zeta, Ł\zeta, Ł\zeta')\Big) \leq F\Big(\frac{\Lambda(\zeta, \zeta, Ł\zeta') + \Lambda(\zeta', \zeta', Ł\zeta)}{2}\Big) - \tau,$$

where $F \in \mathbf{F}$.

(ii) There exists $\zeta_0 \in \Delta$ such that

$$\delta(\Lambda, Ł, \zeta_0) := \sup\Big\{\Lambda(Ł^i\zeta_0, Ł^i\zeta_0, Ł^j\zeta_0) : i, j = 1, 2, \dots\Big\} < \infty.$$

Then $Ł$ has a unique fixed point in Δ provided that $Ł$ is continuous or (Δ, Λ) is symmetric.

Setting $F(t) = \ln t$, in Theorem 2.4, we have

Corollary 2.9. Let (Δ, Λ) be a complete s.s.s.m.s. with $C \geq 1$, and let $Ł : \Delta \to \Delta$ be a mapping. Suppose there exist constants $a, b, c, d \in [0, 1)$ with $a + b + c + d < 1$ such that:

(i) $Ł$ is a Λ-Hardy–Rogers type contraction; that is, for all $\zeta, \zeta' \in \Delta$,

$$\Lambda(Ł\zeta, Ł\zeta, Ł\zeta') \leq a\,\Lambda(\zeta, \zeta, \zeta') + b\,\Lambda(\zeta, \zeta, Ł\zeta) + c\,\frac{\Lambda(\zeta', \zeta', Ł\zeta')}{C} + d\,\frac{\Lambda(\zeta, \zeta, Ł\zeta') + \Lambda(\zeta', \zeta', Ł\zeta)}{2}.$$

(ii) There exists $\zeta_0 \in \Delta$ such that

$$\delta(\Lambda, Ł, \zeta_0) := \sup\Big\{\Lambda(Ł^i\zeta_0, Ł^i\zeta_0, Ł^j\zeta_0) : i, j = 1, 2, \dots\Big\} < \infty.$$

Then $Ł$ has at least one fixed point in Δ provided that $Ł$ is continuous or (Δ, Λ) is symmetric. Moreover, if (Δ, Λ) is symmetric, then the fixed point is unique.

Taking $b = c = d = 0$ in the above corollary, we have:

Corollary 2.10. Let (Δ, Λ) be a complete s.s.s.m.s. and let $Ł : \Delta \to \Delta$ be a mapping satisfying:

(i) $Ł$ is a Λ-Banach type contraction; that is, there exists $a \in [0, 1)$ such that for all $\zeta, \zeta' \in \Delta$,

$$\Lambda(Ł\zeta, Ł\zeta, Ł\zeta') \leq a\,\Lambda(\zeta, \zeta, \zeta').$$

(ii) There exists $\zeta_0 \in \Delta$ such that

$$\delta(\Lambda, Ł, \zeta_0) := \sup\left\{\Lambda(Ł^i\zeta_0, Ł^i\zeta_0, Ł^j\zeta_0) : i, j = 1, 2, \dots\right\} < \infty.$$

Then Ł admits a unique fixed point in Δ.

Example 2.11. Consider $\Delta = \mathbb{R}$ and $\Lambda(\zeta, \zeta', \zeta'') = 0$ for all $\zeta = \zeta' = \zeta''$ and $\Lambda(\zeta, \zeta', \zeta'') = 2\zeta^2 + {\zeta'}^2 + {\zeta''}^2 + |\zeta - \zeta'|^p + |\zeta - \zeta''|^p$, otherwise. Note that $\Lambda(\zeta, \zeta', \zeta'') = 0$ implies that $\zeta = \zeta' = \zeta''$, then the first condition of the Definition 1.6 is satisfied. Now, we show that Λ satisfies the condition (b) of Definition 1.6. let $\{\zeta_n\}$, $\{\zeta'_n\}$ and $\{\zeta''_n\}$ are convergent sequences such that $\lim_{n\to\infty} \Lambda(\zeta, \zeta, \zeta_n) = 0$, $\lim_{n\to\infty} \Lambda(\zeta', \zeta', \zeta'_n) = 0$ and $\lim_{n\to\infty} \Lambda(\zeta'', \zeta'', \zeta''_n) = 0$. So, $\lim_{n\to\infty} \zeta_n = \zeta$, $\lim_{n\to\infty} \zeta'_n = \zeta'$ and $\lim_{n\to\infty} \zeta''_n = \zeta''$. We have

$$\begin{aligned}
\Lambda(\zeta, \zeta', \zeta'') &= 2\zeta^2 + {\zeta'}^2 + {\zeta''}^2 + |\zeta - \zeta'|^p + |\zeta - \zeta''|^p \\
&\leq 2^{2p-1} \limsup_{n\to\infty}(3\zeta^2 + \zeta_n^2 + |\zeta - \zeta_n|^p + 3{\zeta'}^2 + {\zeta'_n}^2 + |\zeta' - {\zeta'}_n|^p + 3{\zeta''}^2 + {\zeta''_n}^2 + |\zeta'' - \zeta''_n|^p \\
&\quad + 3\zeta_n^2 + {\zeta'_n}^2 + 3\zeta_n^2 + {\zeta''_n}^2 + |\zeta_n - \zeta'_n|^p + |\zeta_n - \zeta''_n|^p) \\
&\leq 2^{2p-1}(\limsup_{n\to\infty} \Lambda(\zeta, \zeta, \zeta_n) + \Lambda(\zeta', \zeta', {\zeta'}_n) + \Lambda(\zeta'', \zeta'', \zeta''_n) + \Lambda(\zeta_n, \zeta_n, \zeta'_n) + \Lambda(\zeta_n, \zeta_n, \zeta''_n)).
\end{aligned}$$

Since $\lim_{n\to\infty} \Lambda(\zeta, \zeta, \zeta_n) = 0$, $\lim_{n\to\infty} \Lambda(\zeta', \zeta', \zeta'_n) = 0$ and $\lim_{n\to\infty} \Lambda(\zeta'', \zeta'', \zeta''_n) = 0$, then

$$\Lambda(\zeta, \zeta', \zeta'') \leq 2^{2p-1}(\limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, \zeta'_n) + \limsup_{n\to\infty} \Lambda(\zeta_n, \zeta_n, \zeta''_n)),$$

where $C = 2^{2p-1}$ and $p \geq 1$. Then Λ is a s.s.s.m. on Δ. Define $Ł : \Delta \to \Delta$ by $Ł\zeta = \frac{1}{6}\sinh^{-1}\zeta$ for all $\zeta \in \Delta$. Then Ł satisfies all the conditions of Corollary 2.7, for $\tau = \frac{1}{2}$ and $F(t) = \ln t$. Clearly, Ł has a unique fixed point $0 \in \Delta$, because

$$\begin{aligned}
\tau + F(\Lambda(Ł\zeta, Ł\zeta, Ł\zeta')) &= \frac{1}{2} + \ln(\frac{3}{36}(\sinh^{-1}\zeta)^2 + \frac{1}{36}(\sinh^{-1}\zeta')^2 + |\frac{1}{6}\sinh^{-1}\zeta - \frac{1}{6}\sinh^{-1}\zeta'|^p) \\
&\leq \ln(\frac{1}{3}(3\zeta^2 + {\zeta'}^2 + |\zeta - \zeta'|^p)) \\
&\leq \ln(3\zeta^2 + {\zeta'}^2 + |\zeta - \zeta'|^p) \\
&\leq F(\Lambda(\zeta, \zeta, \zeta')).
\end{aligned}$$

2.2 Wardowski-Prešić type contractions

In this section, we present results for Prešić contractive mappings in a s.s.s.m.s. Many generalizations and extensions of the Banach Contraction Principle (BCP) have been developed (see [31]). In [32], Prešić established the following result:

Theorem 2.12. [32] Let (Δ, d) be a complete metric space and let $Ł : \Delta^k \to \Delta$, where k is a positive integer. Assume that

$$d(Ł(\zeta_1, \dots, \zeta_k), Ł(\zeta_2, \dots, \zeta_{k+1})) \leq \sum_{i=1}^{k} \alpha_i\, d(\zeta_i, \zeta_{i+1}), \tag{2.2}$$

for all $\zeta_1, \dots, \zeta_{k+1} \in \Delta$, where $\alpha_i \geq 0$ and $\sum_{i=1}^k \alpha_i \in [0, 1)$. Then Ł has a unique fixed point $\zeta^* \in \Delta$, that is,

$$Ł(\zeta^*, \dots, \zeta^*) = \zeta^*.$$

Moreover, for any initial points $\zeta_1, \dots, \zeta_{k+1} \in \Delta$, the sequence $\{\zeta_n\}$ defined by

$$\zeta_{n+k} = Ł(\zeta_n, \zeta_{n+1}, \dots, \zeta_{n+k-1})$$

converges to ζ^*.

It is evident that for $k = 1$, Theorem 2.12 coincides with the Banach Contraction Principle. Ćirić and Prešić generalized this result in [9] as follows:

Theorem 2.13. [9] Let (Δ, d) be a complete metric space and k a positive integer. Consider a mapping $Ł : \Delta^k \to \Delta$ satisfying

$$d(Ł(\zeta_1, \ldots, \zeta_k), Ł(\zeta_2, \ldots, \zeta_{k+1})) \leq \alpha \max\{d(\zeta_i, \zeta_{i+1}) : 1 \leq i \leq k\}, \tag{2.3}$$

for all $\zeta_1, \ldots, \zeta_{k+1} \in \Delta$, where $\alpha \in [0, 1)$. Then Ł admits a fixed point $\zeta^* \in \Delta$, and for any initial points $\zeta_1, \ldots, \zeta_{k+1} \in \Delta$, the sequence $\{\zeta_n\}$ defined by

$$\zeta_{n+k} = Ł(\zeta_n, \zeta_{n+1}, \ldots, \zeta_{n+k-1})$$

converges to ζ^*. Moreover, the fixed point is unique if

$$d(Ł(\rho, \ldots, \rho), Ł(\varrho, \ldots, \varrho)) < d(\rho, \varrho)$$

for all distinct $\rho, \varrho \in \Delta$.

However, as noted in [37], the above theorem directly follows from the results in [38]. The main result of this section is now within reach. To begin, we first prove the following lemma.

Lemma 2.14. Suppose that $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ are n s.s.s.m. on $\Delta_1, \Delta_2, \ldots, \Delta_n$, respectively, and let $\varpi : [0, \infty)^n \longrightarrow [0, \infty)$ so that $\varpi(\sigma_1, \ldots, \sigma_n) = 0$ if and only if $\sigma_i = 0$ for all $i = 1, 2, \ldots, n$ and

$$\begin{aligned}
&\varpi\Big[C_1\Big(\limsup_{m\to\infty} \Lambda_1((\zeta_m)_{11}, (\zeta_m)_{11}, (\zeta_m)_{21}) + \limsup_{m\to\infty} \Lambda_1((\zeta_m)_{11}, (\zeta_m)_{11}, (\zeta_m)_{31})\Big),\\
&\quad C_2\Big(\limsup_{m\to\infty} \Lambda_2((\zeta_m)_{12}, (\zeta_m)_{12}, (\zeta_m)_{22}) + \limsup_{m\to\infty} \Lambda_2((\zeta_m)_{12}, (\zeta_m)_{12}, (\zeta_m)_{32})\Big),\\
&\vdots\\
&\quad C_m\Big(\limsup_{m\to\infty} \Lambda_n((\zeta_m)_{1n}, (\zeta_m)_{1n}, (\zeta_m)_{2n}) + \limsup_{m\to\infty} \Lambda_n((\zeta_m)_{1n}, (\zeta_m)_{1n}, (\zeta_m)_{3n})\Big)\Big]\\
&\leq C\Big(\limsup_{m\to\infty} \varpi\Big[\Lambda_1((\zeta_m)_{11}, (\zeta_m)_{11}, (\zeta_m)_{21}), \Lambda_2((\zeta_m)_{12}, (\zeta_m)_{12}, (\zeta_m)_{22}), \cdots, \Lambda_n((\zeta_m)_{1n}, (\zeta_m)_{1n}, (\zeta_m)_{2n})\Big]\\
&\quad + \limsup_{m\to\infty} \varpi\Big[\Lambda_1((\zeta_m)_{11}, (\zeta_m)_{11}, (\zeta_m)_{31}), \Lambda_2((\zeta_m)_{12}, (\zeta_m)_{12}, (\zeta_m)_{32}), \cdots, \Lambda_n((\zeta_m)_{1n}, (\zeta_m)_{1n}, (\zeta_m)_{3n})\Big]\Big),
\end{aligned}$$

for all $\zeta_{ij} \in [0, \infty)$ and $(\zeta_m)_{ij} \in \underline{\mathcal{C}}(\Lambda, \Delta, \zeta_{ij})$, where $C = \varpi(C_1, C_2, \ldots C_n) \geq C_j$, for all $1 \leq j \leq n$. Then

$$\widetilde{\Lambda}((\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), (\zeta_{21}, \zeta_{22}, ..., \zeta_{2n}), (\zeta_{31}, \zeta_{32}, ..., \zeta_{3n})) = \varpi(\Lambda_1(\zeta_{11}, \zeta_{21}, \zeta_{31}), \Lambda_2(\zeta_{12}, \zeta_{22}, \zeta_{32}), \ldots, \Lambda_n(\zeta_{1n}, \zeta_{2n}, \zeta_{3n})),$$

is a s.s.s.m. in $[\Delta_1 \times \Delta_2 \times \ldots \times \Delta_n]^3$.

Proof . We now prove that $\widetilde{\Lambda}$ fulfills condition (b) of Definition 1.6. Let $\zeta_{ij} \in \Delta_j$ for all $1 \leq i \leq 3$ and $1 \leq j \leq n$. So,

$$\begin{aligned}
&\widetilde{\Lambda}((\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), (\zeta_{21}, \zeta_{22}, ..., \zeta_{2n}), (\zeta_{31}, \zeta_{32}, ..., \zeta_{3n}))\\
&= \varpi(\Lambda_1(\zeta_{11}, \zeta_{21}, \zeta_{31}), \Lambda_2(\zeta_{12}, \zeta_{22}, \zeta_{32}), \ldots, \Lambda_n(\zeta_{1n}, \zeta_{2n}, \zeta_{3n}))\\
&\leq \varpi\Big[C_1\Big(\limsup_{m\to\infty} \Lambda_1((\zeta_m)_{11}, (\zeta_m)_{11}, (\zeta_m)_{21}) + \limsup_{m\to\infty} \Lambda_1((\zeta_m)_{11}, (\zeta_m)_{11}, (\zeta_m)_{31})\Big),\\
&\quad C_2\Big(\limsup_{m\to\infty} \Lambda_2((\zeta_m)_{12}, (\zeta_m)_{12}, (\zeta_m)_{22}) + \limsup_{m\to\infty} \Lambda_2((\zeta_m)_{12}, (\zeta_m)_{12}, (\zeta_m)_{32})\Big),\\
&\vdots\\
&\quad C_n\Big(\limsup_{m\to\infty} \Lambda_n((\zeta_m)_{1n}, (\zeta_m)_{1n}, (\zeta_m)_{2n}) + \limsup_{m\to\infty} \Lambda_n((\zeta_m)_{1n}, (\zeta_m)_{1n}, (\zeta_m)_{3n})\Big)\Big]\\
&\leq C\Big(\limsup_{m\to\infty} \varpi\Big[\Lambda_1((\zeta_m)_{11}, (\zeta_m)_{11}, (\zeta_m)_{21}), \Lambda_2((\zeta_m)_{12}, (\zeta_m)_{12}, (\zeta_m)_{22}), \cdots, \Lambda_n((\zeta_m)_{1n}, (\zeta_m)_{1n}, (\zeta_m)_{2n})\Big]\\
&\quad + \limsup_{m\to\infty} \varpi\Big[\Lambda_1((\zeta_m)_{11}, (\zeta_m)_{11}, (\zeta_m)_{31}), \Lambda_2((\zeta_m)_{12}, (\zeta_m)_{12}, (\zeta_m)_{3n}), \cdots, \Lambda_n((\zeta_m)_{1n}, (\zeta_m)_{1n}, (\zeta_m)_{3n})\Big]\Big)\\
&= C\Big(\limsup_{m\to\infty} \widetilde{\Lambda}\big(((\zeta_m)_{11}, (\zeta_m)_{12}, ..., (\zeta_m)_{1n}), ((\zeta_m)_{11}, (\zeta_m)_{12}, ..., (\zeta_m)_{1n}), ((\zeta_m)_{21}, (\zeta_m)_{22}, ..., (\zeta_m)_{2n})\big)\\
&\quad + \limsup_{m\to\infty} \widetilde{\Lambda}\big(((\zeta_m)_{11}, (\zeta_m)_{12}, ..., (\zeta_m)_{1n}), ((\zeta_m)_{11}, (\zeta_m)_{12}, ..., (\zeta_m)_{1n}), ((\zeta_m)_{31}, (\zeta_m)_{32}, ..., (\zeta_m)_{3n})\big).
\end{aligned}$$

□

In the subsequent results, we suppose that on both sides of the contractive inequality, the function F operates over positive values.

Theorem 2.15. Let (Δ, Λ) be a s.s.s.m.s., and let $Ł : \Delta^n \to \Delta$ be a mapping satisfying the following:

(i) $Ł$ is a Λ-Wardowski-Ćirić-Prešić-type contraction, meaning that

$$F\Big(\Lambda\big(Ł(\zeta_{11}, \zeta_{12}, \dots, \zeta_{1n}), Ł(\zeta_{11}, \zeta_{12}, \dots, \zeta_{1n}), Ł(\zeta_{21}, \zeta_{22}, \dots, \zeta_{2n})\big)\Big) \leq F\Big(\max_{1 \leq j \leq n}\{\Lambda(\zeta_{1j}, \zeta_{1j}, \zeta_{2j})\}\Big) - \tau, \tag{2.4}$$

for all $\zeta_{ij} \in \Delta$, $1 \leq i \leq 2$, $1 \leq j \leq n$, with $F \in \mathbf{F}$ and $\tau > 0$.

(ii) There exist $\zeta_1, \dots, \zeta_n \in Ł$ such that

$$\delta(\Lambda, Ł, (\zeta_1, \dots, \zeta_n) = \widetilde{\zeta}) := \sup\Big\{\Lambda\big(\widetilde{Ł}^i\widetilde{\zeta}, \widetilde{Ł}^i\widetilde{\zeta}, \widetilde{Ł}^j\widetilde{\zeta}\big) : i, j = 1, 2, \dots\Big\} < \infty,$$

where

$$\widetilde{Ł}\widetilde{\zeta} = (\underbrace{Ł(\zeta_1, \dots, \zeta_n), \dots, Ł(\zeta_1, \dots, \zeta_n)}_{n \text{ times}}) \quad \text{and} \quad \widetilde{\zeta} = (\zeta_1, \dots, \zeta_n).$$

Then $Ł$ admits a unique fixed point in Δ.

Proof . Let $\widetilde{Ł} : \Delta^n \to \Delta^n$ be defined as

$$\widetilde{Ł}(\sigma_1, \dots, \sigma_n) = \big(Ł(\sigma_1, \dots, \sigma_n), \dots, Ł(\sigma_1, \dots, \sigma_n)\big).$$

Also, define the s.s.s.m. $\widetilde{\Lambda} : \Delta^n \times \Delta^n \times \Delta^n \to \mathbb{R}$ by

$$\widetilde{\Lambda}\big((\zeta_{11}, \dots, \zeta_{1n}), (\zeta_{21}, \dots, \zeta_{2n}), (\zeta_{31}, \dots, \zeta_{3n})\big) := \max_{1 \leq j \leq n} \Lambda(\zeta_{1j}, \zeta_{2j}, \zeta_{3j}).$$

According to (2.4) we have

$$\begin{aligned}
&F\Big(\widetilde{\Lambda}(\widetilde{Ł}(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), \widetilde{Ł}(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), \widetilde{Ł}(\zeta_{21}, \zeta_{22}, ..., \zeta_{2n})\Big)\\
=&F\Big(\widetilde{\Lambda}\Big((Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), \dots, Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n})),\\
&(Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), \dots, Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}))\\
&(Ł(\zeta_{21}, \zeta_{22}, ..., \zeta_{2n}), Ł(\zeta_{21}, \zeta_{22}, ..., \zeta_{2n}), \dots, Ł(\zeta_{21}, \zeta_{22}, ..., \zeta_{2n}))\Big)\\
=&F\Big(\max\{\Lambda(Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{21}, \zeta_{22}, ..., \zeta_{2n})),\\
&\Lambda(Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{21}, \zeta_{22}, ..., \zeta_{2n})),\\
&\dots\\
&\Lambda(Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{21}, \zeta_{22}, ..., \zeta_{2n}))\}\Big)\\
=&F\Big(\Lambda(Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), Ł(\zeta_{21}, \zeta_{22}, ..., \zeta_{2n}))\Big)\\
&\leq F\Big(\max_{1 \leq j \leq n}\{\Lambda(\zeta_{1j}, \zeta_{1j}, \zeta_{2j})\}\Big) - \tau\\
=&F\Big(\widetilde{\Lambda}((\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), (\zeta_{11}, \zeta_{12}, ..., \zeta_{1n}), (\zeta_{21}, \zeta_{22}, ..., \zeta_{2n}))\Big) - \tau.
\end{aligned}$$

Thus, $\widetilde{Ł}$ satisfies the assumptions of Corollary 2.7. Hence, by Corollary 2.7, $\widetilde{Ł}$ admits a fixed point, which implies the existence of $\sigma_1, \dots, \sigma_n \in \Delta$ such that

$$Ł(\sigma_1, \dots, \sigma_n) = \sigma_1 = \dots = \sigma_n.$$

Therefore, $Ł$ possesses a fixed point of Prešić-type. □

Similarly, we can have the following result.

Theorem 2.16. Let (Δ, Λ) be a s.s.s.m.s. and let $Ł : \Delta^n \to \Delta$ be a mapping such that

(i) $Ł$ is a Λ-Wardowski–Hardy–Rogers–Ćirić–Prešić-type contraction, i.e. for all $\zeta_{ij} \in \Delta$ $(i = 1, 2,\ 1 \le j \le n)$ we have

$$
\begin{aligned}
&F\Big(\Lambda\big(Ł(\zeta_{11}, \zeta_{12}, \dots, \zeta_{1n}), Ł(\zeta_{11}, \zeta_{12}, \dots, \zeta_{1n}), Ł(\zeta_{21}, \zeta_{22}, \dots, \zeta_{2n})\big)\Big) \\
&\le F\Big(\alpha \max_{1\le j\le n}\{\Lambda(\zeta_{1j}, \zeta_{1j}, \zeta_{2j})\} + \beta \max_{1\le j\le n}\{\Lambda(\zeta_{1j}, \zeta_{1j}, Ł(\zeta_{11}, \zeta_{12}, \dots, \zeta_{1n}))\} + \gamma \frac{\max_{1\le j\le n}\{\Lambda(\zeta_{2j}, \zeta_{2j}, Ł(\zeta_{21}, \zeta_{22}, \dots, \zeta_{2n}))\}}{C} \\
&\quad + \sigma\Big(\frac{\max_{1\le j\le n}\{\Lambda(\zeta_{1j}, \zeta_{1j}, Ł(\zeta_{21}, \zeta_{22}, \dots, \zeta_{2n}))\} + \max_{1\le j\le n}\{\Lambda(\zeta_{2j}, \zeta_{2j}, Ł(\zeta_{11}, \zeta_{12}, \dots, \zeta_{1n}))\}}{2}\Big)\Big) - \tau,
\end{aligned} \tag{2.5}
$$

where $F \in \mathbf{F}$, $\tau > 0$, $\alpha, \beta, \gamma, \sigma \ge 0$ and

$$\alpha + \beta + \gamma + \sigma = 1,$$

(ii) There exist $\zeta_1, \dots, \zeta_n \in Ł$ such that

$$\delta(\Lambda, Ł, (\zeta_1, \dots, \zeta_n) = \widetilde{\zeta}) := \sup\Big\{\Lambda\big(\widetilde{Ł}^i\widetilde{\zeta}, \widetilde{Ł}^i\widetilde{\zeta}, \widetilde{Ł}^j\widetilde{\zeta}\big) : i, j = 1, 2, \dots\Big\} < \infty,$$

where

$$\widetilde{Ł}\widetilde{\zeta} = (\underbrace{Ł(\zeta_1, \dots, \zeta_n), \dots, Ł(\zeta_1, \dots, \zeta_n)}_{n \text{ times}}) \quad \text{and} \quad \widetilde{\zeta} = (\zeta_1, \dots, \zeta_n).$$

Then $Ł$ has at least one fixed point in Δ provided that $Ł$ is continuous or (Δ, Λ) is symmetric. Furthermore, if (Δ, Λ) is symmetric and for every fixed point σ of $Ł$ in Δ one has $\Lambda(\sigma, \sigma, \sigma) = 0$, then σ is unique.

3 Application

The following fractional integrals are recalled:

Let $\omega, s \in \mathbb{R}_+$. For any function $f \in L^p[a, b]$, $1 \le p < \infty$, $1 \le a < b < \infty$ and for $s \in [a, b]$, [14] defines the Hadamard fractional integral of order ω as

$$\mathcal{J}_a^\omega f(s) = \frac{1}{\Gamma(\omega)} \int_a^s \left(\ln\left(\frac{s}{\xi}\right)\right)^{\omega-1} f(\xi)\frac{d\xi}{\xi}.$$

Therefore, for $a = 1$, we have

$$\mathcal{J}^\omega f(s) = \frac{1}{\Gamma(\omega)} \int_1^s \left(\ln\left(\frac{s}{\xi}\right)\right)^{\omega-1} f(\xi)\frac{d\xi}{\xi},\ \omega > 0,\ s \ge 1.$$

Katugampola fractional integral ${}^\sigma\mathcal{I}_a^\gamma$ of order γ has been defined in [27] as,

$${}^\sigma\mathcal{I}_a^\omega f(s) = \frac{\sigma^{1-\omega}}{\Gamma(\omega)} \int_a^s \frac{\xi^{\sigma-1}}{(s^\sigma - \xi^\sigma)^{1-\omega}} f(\xi)d\xi,\ \ \omega > 0,\ \sigma > 0.$$

For $a = 1$, we have

$${}^\sigma\mathcal{I}^\omega f(s) = \frac{\sigma^{1-\omega}}{\Gamma(\omega)} \int_1^s \frac{\xi^{\sigma-1}}{(s^\sigma - \xi^\sigma)^{1-\omega}} f(\xi)d\xi.$$

This section examines the fractional integral equation that follows:

$$\Upsilon(s) = \mathcal{G}\left(s, \mathcal{K}(s, \Upsilon(s)), \mathcal{J}^\omega\Upsilon(s), {}^\sigma\mathcal{I}^\omega\Upsilon(s)\right), \tag{3.1}$$

where $\omega, \sigma > 0$ and $s \in [1, T]$. In fact, we want to prove the existence of a solution for this equation using fixed point results. This equation has been considered in [10] in which the authors prove the existence of a solution for it using

the measure of noncompactness tool.
Assume that $\mathbf{E} = C([1,T])$, the set of continuous functions defined on $[1,T]$. For $\jmath \in \mathbf{E}$, define

$$||\jmath||_\infty = \sup_{t\in[1,T]} |\jmath(t)|.$$

Here, $(\mathbf{E}, ||.||_\infty)$ is a Banach space. It is easy to prove that $\mathbf{E}$ is a complete s.s.s.m.s. if we choose Λ as follows:

$$\Lambda(\zeta,\zeta',\zeta'') = ||\zeta-\zeta'||_\infty + ||\zeta'-\zeta''||_\infty.$$

Consider the following assumptions:

(A) $\mathcal{G} : [1,T]\times\mathbb{R}^3 \to \mathbb{R},\ \mathcal{K} : [1,T]\times\mathbb{R}\to\mathbb{R}$ are continuous and there exist constants $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \geq 0$ such that

$$\begin{aligned}&F\Big[\left|\mathcal{G}(s,\mathcal{K}(s,\varUpsilon(s)),I_1\varUpsilon(s),I_2\varUpsilon(s)) - \mathcal{G}(s,\bar{\mathcal{K}}(s,\varUpsilon(s)),\bar{I}_1\varUpsilon(s),\bar{I}_2\varUpsilon(s))\right|\Big]\\ &\leq F\big(\alpha_1\left|\mathcal{K}(s,\varUpsilon(s))-\bar{\mathcal{K}}(s,\varUpsilon(s))\right| + \alpha_2\left|I_1\varUpsilon(s)-\bar{I}_1\varUpsilon(s)\right| + \alpha_3\left|I_2\varUpsilon(s)-\bar{I}_2\varUpsilon(s)\right|\big) - \tau,\end{aligned}$$

for all $s\in[1,T]$ and

$$|\mathcal{K}(s,J_1(s)) - \mathcal{K}(s,J_2(s))| \leq \alpha_4\left|J_1(s)-J_2(s)\right|, J_1,J_2:\mathbb{R}\to\mathbb{R}.$$

(B)

$$\alpha_1\alpha_4 + \frac{\alpha_2}{\Gamma(\omega+1)}(\ln T)^\omega + \frac{\alpha_3}{\Gamma(\omega+1)}\frac{1}{\sigma\omega}(T^\sigma-1)^\omega \leq 1,$$

(C) there exists $\varUpsilon_0 \in \mathbb{E}$ such that

$$\sup\{\sup_{s\in[1,T]}(|\mathcal{G}^i\left(s,\mathcal{K}(s,\varUpsilon_0(s)),\mathcal{J}^\omega\varUpsilon_0(s),{}^\sigma\mathcal{I}^\omega\varUpsilon_0(s)\right) - \mathcal{G}^j\left(s,\mathcal{K}(s,\varUpsilon_0(s)),\mathcal{J}^\omega\varUpsilon_0(s),{}^\sigma\mathcal{I}^\omega\varUpsilon_0(s)\right)|),\ i,j=1,2,...\}<\infty.$$

Theorem 3.1. If conditions (A)-(C) are satisfied, then equation (3.1) possesses a solution in $\mathbf{E} = C([1,T])$.

Proof . Define the operator $\mathcal{Q} : \mathbf{E}\to\mathbf{E}$ as follows:

$$(\mathcal{Q}\varUpsilon)(s) = \mathcal{G}\left(s,\mathcal{K}(s,\varUpsilon(s)),\mathcal{J}^\omega\varUpsilon(s),{}^\sigma\mathcal{I}^\omega\varUpsilon(s)\right).$$

We prove that $\mathcal{Q}$ is a Λ-Wardowski-contraction. So, we have

$$\begin{aligned}F\Big(\left|(\mathcal{Q}\varUpsilon)(s) - (\mathcal{Q}\bar{\varUpsilon})(s)\right|\Big) &= F\Big(\left|\mathcal{G}\left(s,\mathcal{K}(s,\varUpsilon(s)),\mathcal{J}^\omega\varUpsilon(s),{}^\sigma\mathcal{I}^\omega\varUpsilon(s)\right) - \mathcal{G}\left(s,\mathcal{K}(s,\bar{\varUpsilon}(s)),\mathcal{J}^\omega\bar{\varUpsilon}(s),{}^\sigma\mathcal{I}^\omega\bar{\varUpsilon}(s)\right)\right|\Big)\\ &\leq F\big(\alpha_1\left|\mathcal{K}(s,\varUpsilon(s))-\mathcal{K}(s,\bar{\varUpsilon}(s))\right| + \alpha_2\left|\mathcal{J}^\omega\varUpsilon(s)-\mathcal{J}^\omega\bar{\varUpsilon}(s)\right| + \alpha_3\left|{}^\sigma\mathcal{I}^\omega\varUpsilon(s)-{}^\sigma\mathcal{I}^\omega\bar{\varUpsilon}(s)\right|\big) - \tau.\end{aligned}$$

Also,

$$\begin{aligned}\left|\mathcal{J}^\omega\varUpsilon(s)-\mathcal{J}^\omega\bar{\varUpsilon}(s)\right| &= \left|\frac{1}{\Gamma(\omega)}\int_1^s\left(\ln\left(\frac{s}{\xi}\right)\right)^{\omega-1}(\varUpsilon(\xi)-\bar{\varUpsilon}(\xi))\frac{d\xi}{\xi}\right|\\ &\leq \frac{1}{\Gamma(\omega)}\int_1^s\left(\ln\left(\frac{s}{\xi}\right)\right)^{\omega-1}\left|\varUpsilon(\xi)-\bar{\varUpsilon}(\xi)\right|\frac{d\xi}{\xi}\\ &\leq \frac{||\varUpsilon-\bar{\varUpsilon}||_\infty}{\Gamma(\omega)}\int_1^s\left(\ln\left(\frac{s}{\xi}\right)\right)^{\omega-1}\frac{d\xi}{\xi}\\ &\leq \frac{||\varUpsilon-\bar{\varUpsilon}||_\infty}{\Gamma(\omega+1)}(\ln T)^\omega,\end{aligned}$$

and

$$
\begin{aligned}
\left|{}^{\sigma}\mathcal{I}^{\omega}\varUpsilon(s)-{}^{\sigma}\mathcal{I}^{\omega}\bar{\varUpsilon}(s)\right| &= \left|\frac{\sigma^{1-\omega}}{\Gamma(\omega)}\int_1^s \frac{\xi^{\sigma-1}}{(s^\sigma-\xi^\sigma)^{1-\omega}}\varUpsilon(\xi)d\xi-\frac{\sigma^{1-\omega}}{\Gamma(\omega)}\int_1^s \frac{\xi^{\sigma-1}}{(s^\sigma-\xi^\sigma)^{1-\omega}}\bar{\varUpsilon}(\xi)d\xi\right| \\
&\leq \frac{\sigma^{1-\omega}}{\Gamma(\omega)}\left|\frac{1}{\sigma}\int_1^s \sigma\frac{\xi^{\sigma-1}}{(s^\sigma-\xi^\sigma)^{1-\omega}}(\varUpsilon(\xi)-\bar{\varUpsilon}(\xi))d\xi\right| \\
&\leq \frac{||\varUpsilon-\bar{\varUpsilon}||_\infty\sigma^{1-\omega}}{\Gamma(\omega)}\left|\frac{1}{\sigma}\int_1^s \sigma\xi^{\sigma-1}(s^\sigma-\xi^\sigma)^{\omega-1}d\xi\right| \\
&= \frac{||\varUpsilon-\bar{\varUpsilon}||_\infty\sigma^{1-\omega}}{\Gamma(\omega)}\frac{1}{\sigma}\left|\int_1^s \sigma\xi^{\sigma-1}(s^\sigma-\xi^\sigma)^{\omega-1}d(\xi)\right| \\
&= \frac{||\varUpsilon-\bar{\varUpsilon}||_\infty}{\Gamma(\omega)}\frac{1}{\sigma}\Big[\frac{-(s^\sigma-\xi^\sigma)^\omega}{\omega}\Big]_1^s \\
&\leq \frac{||\varUpsilon-\bar{\varUpsilon}||_\infty}{\Gamma(\omega+1)}\frac{1}{\sigma\omega}(T^\sigma-1)^\omega.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
F\Big[\left|(\mathcal{Q}\varUpsilon)(s)-(\mathcal{Q}\bar{\varUpsilon})(s)\right|\Big] &\leq F\big(\alpha_1\alpha_4||\varUpsilon-\bar{\varUpsilon}||_\infty+\frac{\alpha_2||\varUpsilon-\bar{\varUpsilon}||_\infty}{\Gamma(\omega+1)}(\ln T)^\omega+\frac{\alpha_3||\varUpsilon-\bar{\varUpsilon}||_\infty}{\Gamma(\omega+1)}\frac{1}{\sigma\omega}(T^\sigma-1)^\omega\big)-\tau \\
&= F\Big(||\varUpsilon-\bar{\varUpsilon}||_\infty\big(\alpha_1\alpha_4+\frac{\alpha_2}{\Gamma(\omega+1)}(\ln T)^\omega+\frac{\alpha_3}{\Gamma(\omega+1)}\frac{1}{\sigma\omega}(T^\sigma-1)^\omega\big)\Big)-\tau \\
&\leq F\big(||\varUpsilon-\bar{\varUpsilon}||_\infty\big)-\tau.
\end{aligned}
$$

So by taking the supremum from the left side, we have

$$
F\big(\left|\left|(\mathcal{Q}\varUpsilon)-(\mathcal{Q}\bar{\varUpsilon})\right|\right|_\infty\big)\leq F\big(||\varUpsilon-\bar{\varUpsilon}||_\infty\big)-\tau.
$$

Thus,

$$
\begin{aligned}
F\big(\Lambda\left(\mathcal{Q}\varUpsilon,\mathcal{Q}\varUpsilon,\mathcal{Q}\bar{\varUpsilon}\right)\big) &= F\big(||\mathcal{Q}\varUpsilon-\mathcal{Q}\bar{\varUpsilon}||_\infty\big) \\
&\leq F\big(||\varUpsilon-\bar{\varUpsilon}||_\infty\big)-\tau \\
&\leq F\big(\Lambda\left(\varUpsilon,\varUpsilon,\bar{\varUpsilon}\right)\big)-\tau.
\end{aligned}
$$

Therefore, it is clear that the mapping $\mathcal{Q}$ satisfies all the requirements of Corollary 2.7. Consequently, equation (3.1) admits a solution in $\mathbf{E}=C(I)$. □

4 Conclusions

In the framework of a s.s.s.m.s, we established several fixed point results for generalized F-contraction and F-Prešić-contraction type mappings. We concluded the study by addressing the solvability of a fractional integral equation. Our results complement and extend the related findings reported in [9, 12, 13]. An important question naturally arises: can weak-Wardowski contractions be employed in place of Wardowski contractions in this framework? Weak-Wardowski contractions have been investigated in [21].

In [12], replacing condition a in Definition 1.6, we considered the following condition:

$$
a':\Lambda(\zeta,\zeta',\zeta'')=0 \text{ implies } \zeta=\zeta'=\zeta'',
$$

which is weaker than condition a. Under this assumption, the resulting framework can be viewed as a combination of sequential S-metric spaces and partial S-metric spaces [2, 34]. Consequently, the notions of convergence, continuity, and the structure of balls are modified. For instance, the set of convergent sequences is defined as follows. For any $\zeta\in\Delta$, we define

$$
\underline{\mathcal{C}}(\Lambda,\Delta,\zeta):=\Big\{\{\zeta_n\}\subset\Delta:\lim_{n\to\infty}\Lambda(\zeta,\zeta,\zeta_n)=\Lambda(\zeta,\zeta,\zeta)\Big\}.
$$

Therefore, this structure can be further investigated in future studies.

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