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Ground state solutions for two classes of fractional Hamiltonian systems

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Abstract

In this paper, we are concerned with the following periodic fractional Hamiltonian system

$$\begin{cases} {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u)(t) + L(t)u(t) = \nabla W(t, u(t)),\ t \in \mathbb{R} \\ u \in H^\alpha(\mathbb{R}). \end{cases}$$

Using variational methods and a version of the concentration compactness principle, we study the existence of ground state solutions for this system under two different classes of superquadratic conditions weaker than the ones known in the literature. To the best of our knowledge, there has been no work focused in this case.

Keywords: Fractional Hamiltonian systems, ground state orbits, periodic potentials, variational methods, concentration compactness principle.
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1 Introduction

In this paper, we are concerned with a class of periodic superquadratic fractional Hamiltonian systems of the following form

$$(\mathcal{FHS}) \qquad \begin{cases} {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u)(t) + L(t)u(t) = \nabla W(t, u(t)),\ t \in \mathbb{R} \\ u \in H^\alpha(\mathbb{R}), \end{cases}$$

where ${}_{-\infty}D^\alpha_t$ and ${}_tD^\alpha_\infty$ are left and right Liouville-Weyl fractional derivatives of order $\frac{1}{2} < \alpha < 1$ on the whole axis respectively, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix valued function, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $\nabla W(t, x) = \frac{\partial W}{\partial x}(t, x)$ is the gradient of $W(t, x)$ with respect to the second variable. Our basic assumption is periodicity of $L(t)$ and $W(t, x)$ in t. As usual, we say that a solution u of $(\mathcal{FHS})$ is a ground state solution if u is nontrivial and minimizes the energy functional of $(\mathcal{FHS})$ among all possible nontrivial solutions.

Fractional differential equations have been receiving great interest recently. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, and so on, see [1, 10, 14, 21, 23]. Therefore, the theory of fractional differential equations

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is an area intensively developed the last decades [1, 10]. Recently, also equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives are an interesting and new field in fractional differential equations theory. Some classical tools have been used to study nonlinear fractional differential equations in the literature. These classical techniques include the topological degree theory [5, 11], the comparison method [17, 43] and some fixed point theorems [4, 42].

During the last four decades, the critical point theory has developed into a wonderful tool for investigating the existence criteria for the solutions of differential equations with variational structures, for example see [19,24] and the references cited therein. On 2013, Torres [34] used the genus properties in critical point theory to prove that system $(\mathcal{FHS})$ possesses a nontrivial solution provided L is positive definite satisfying a coercive condition and W satisfies some suitable conditions among them the so-called Ambrosetti-Rabinowitz superquadratic condition $(\mathcal{AR})$. There exists a constant $\mu > 2$ such that

$$0 < \mu W(t,x) \leq \nabla W(t,x) \cdot x, \ \forall (t,x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}).$$

Here, " $\cdot$ " denotes the standard inner product in $\mathbb{R}^N$ and the associated norm is denoted by $|.|$. Since then, based on critical point theory and variational methods, many mathematicians are interested in the existence and multiplicity of solutions to system $(\mathcal{FHS})$, see [2,3,7,20,30-36,38-41] and the references cited therein. Most of these results were obtained under condition $(\mathcal{AR})$. The $(\mathcal{AR})$ condition is quite natural and important not only to ensure that the energy functional f has a mountain pass geometry, but also to guarantee that the Palais-Smale sequence of f is bounded. However, the $(\mathcal{AR})$ condition is so strong that many functions cannot be involved. For this reason, in recent years, some authors tried to weaken $(\mathcal{AR})$ condition, we refer the readers to [30]-[33]. However, there are many nonnegative superquadratic functions which are not raised in the papers indicated above. In the present paper, motivated by the above papers, we focus on the existence of ground state solutions of $(\mathcal{FHS})$ under some kind of superquadratic conditions weaker than the above mentioned conditions. The remaining of this paper is organized as follows. Section 2 is devoted to some preliminary results. In Sections 3,4, we are interested to the existence of ground state solutions for two different classes of superquadratic potentials. Section 5 is reserved for concrete examples.

2 Preliminaries

In this Section, for the reader's convenience, first we will recall some facts about the fractional calculus on the whole real axis. On the other hand, we will give some preliminaries lemmas for using in the sequel.

2.1 Liouville-Weyl fractional calculus

The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as (see [13,14,23])

$$_{-\infty}I_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-x)^{\alpha-1} u(x) dx \tag{2.1}$$

and

$$_{t}I_{\infty}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (x-t)^{\alpha-1} u(x) dx. \tag{2.2}$$

The Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (see [13,14,23])

$$_{-\infty}D_t^{\alpha}u(t) = \frac{d}{dt}(_{-\infty}I_t^{1-\alpha}u)(t) \tag{2.3}$$

and

$$_{t}D_{\infty}^{\alpha}u(t) = -\frac{d}{dt}(_{t}I_{\infty}^{1-\alpha}u)(t). \tag{2.4}$$

The definitions of (2.3) and (2.4) may be written in an alternative form as follows

$$_{-\infty}D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{u(t)-u(t-x)}{x^{\alpha+1}} dx \tag{2.5}$$

and

$$_{t}D_{\infty}^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{u(t)-u(t+x)}{x^{\alpha+1}} dx. \tag{2.6}$$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}$ of u is defined by

$$\widehat{u}(s) = \int_{-\infty}^{\infty} e^{-ist} u(t) dt.$$

Let u be defined on $\mathbb{R}$. Then the Fourier transform of the Liouville-Weyl integrals and differential operators satisfies (see [13, 14])

$$\widehat{{}_{-\infty}I_t^{\alpha}u}(s) = (is)^{-\alpha}\widehat{u}(s), \tag{2.7}$$

$$\widehat{{}_{t}I_{\infty}^{\alpha}u}(s) = (-is)^{-\alpha}\widehat{u}(s), \tag{2.8}$$

$$\widehat{{}_{-\infty}D_t^{\alpha}u}(s) = (is)^{\alpha}\widehat{u}(s), \tag{2.9}$$

$$\widehat{{}_{t}D_{\infty}^{\alpha}u}(s) = (-is)^{\alpha}\widehat{u}(s). \tag{2.10}$$

Next, we present some properties for Liouville-Weyl fractional integrals and derivatives on the real axis, which were proved in [13]. Denote by $L^p(\mathbb{R}, \mathbb{R}^N)$ $(2 \leq p < \infty)$, the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norms

$$\|u\|_{L^p} = (\int_{\mathbb{R}} |u(t)|^p \, dt)^{\frac{1}{p}},$$

and $L^{\infty}(\mathbb{R}, \mathbb{R}^N)$ the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norm

$$\|u\|_{\infty} = esssup \left\{ |u(t)| \, / t \in \mathbb{R} \right\}.$$

2.2 Fractional derivative spaces

In order to establish the variational structure which enables us to reduce the existence of solutions of $(\mathcal{FHS})$ to find critical points of the corresponding functional, it is necessary to construct the appropriate functional spaces.

For $\alpha > 0$, define the semi-norm

$$|u|_{I_{-\infty}^{\alpha}} = \|{}_{-\infty}D_t^{\alpha}u\|_{L^2}$$

and the norm

$$\|u\|_{I_{-\infty}^{\alpha}} = (\|u\|_{L^2}^2 + |u|_{I_{-\infty}^{\alpha}}^2)^{\frac{1}{2}},$$

and let

$$I_{-\infty}^{\alpha} = \overline{C_0^{\infty}(\mathbb{R}, \mathbb{R}^N)}^{\|\cdot\|_{I_{-\infty}^{\alpha}}},$$

where $C_0^{\infty}(\mathbb{R}, \mathbb{R}^N)$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with vanishing property at infinity. Now, we can define the fractional Sobolev space $H^{\alpha}(\mathbb{R}, \mathbb{R}^N)$ in terms of the Fourier transform. Choose $0 < \alpha < 1$, define the semi-norm

$$|u|_{\alpha} = \||s|^{\alpha} \widehat{u}\|_{L^2}$$

and the norm

$$\|u\|_{\alpha} = (\|u\|_{L^2}^2 + |u|_{\alpha}^2)^{\frac{1}{2}},$$

and let

$$H^{\alpha}(\mathbb{R}, \mathbb{R}^N) = \overline{C_0^{\infty}(\mathbb{R}, \mathbb{R}^N)}^{\|\cdot\|_{\alpha}}.$$

Moreover, we note that a function $u \in L^2(\mathbb{R}, \mathbb{R}^N)$ belongs to $I_{-\infty}^{\alpha}$ if and only if

$$|s|^{\alpha} \widehat{u} \in L^2(\mathbb{R}, \mathbb{R}^N).$$

Especially, we have

$$|u|_{I_{-\infty}^{\alpha}} = \||s|^{\alpha} \widehat{u}\|_{L^2}.$$

Therefore, $I_{-\infty}^{\alpha}$ and $H^{\alpha}(\mathbb{R}, \mathbb{R}^N)$ are equivalent with equivalent semi-norms and norms. Analogous to $I_{-\infty}^{\alpha}$, we introduce I_{∞}^{α}. Define the semi-norm

$$|u|_{I_{\infty}^{\alpha}} = \|{}_tD_{\infty}^{\alpha}u\|_{L^2}$$

and the norm

$$\|u\|_{I_\infty^\alpha} = (\|u\|_{L^2}^2 + |u|_{I_\infty^\alpha}^2)^{\frac{1}{2}},$$

and let

$$I_\infty^\alpha = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^N)}^{\|.\||_{I_\infty^\alpha}}.$$

Then $I_{-\infty}^\alpha$ and I_∞^α are equivalent with equivalent semi-norms and norms. Let $C(\mathbb{R}, \mathbb{R}^N)$ denote the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$. Then we obtain the following Sobolev lemma.

Lemma 2.1. [20, *Theorem* 2.1]. If $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R}, \mathbb{R}^N) \subset C(\mathbb{R}, \mathbb{R}^N)$, and there exists a constant $C = C_\alpha$ such that

$$\|u\|_\infty = \sup_{t \in \mathbb{R}} |u(t)| \leq C_\alpha \|u\|_\alpha, \forall u \in H^\alpha(\mathbb{R}, \mathbb{R}^N).$$

Remark 2.2. From Lemma 2.1, we know that if $u \in H^\alpha(\mathbb{R}, \mathbb{R}^N)$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [2, \infty]$, because

$$\int_{\mathbb{R}} |u(t)|^p \, dt \leq \|u\|_\infty^{p-2} \|u\|_{L^2}^2.$$

Denote by χ the self-adjoint extension of the operator ${}_tD_\infty^\alpha o_{-\infty}D_t^\alpha + L$ with the domain $\mathcal{D}(\chi) \subset L^2(\mathbb{R})$. Throughout this paper, we consider the following two conditions

(L) $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is $T-$periodic $(T > 0)$ and $L(t)$ is an $N \times N$ symmetric matrix,

$$(\chi) \qquad \sup\Big(\sigma\big({}_tD_\infty^\alpha o_{-\infty}D_t^\alpha + L(t)\big) \cap] - \infty, 0[\Big) < 0 < \inf\Big(\sigma\big({}_tD_\infty^\alpha o_{-\infty}D_t^\alpha + L(t)\big) \cap]0, \infty[\Big).$$

Let $\{\mathcal{E}(\lambda)/ -\infty < \lambda < \infty\}$ and $|\chi|$ be the spectral family and the absolute value of χ, respectively, and $|\chi|^{\frac{1}{2}}$ be the square root of $|\chi|$. Set $U = id - \mathcal{E}(0) - \mathcal{E}(0^-)$. Then U commutes with χ, $|\chi|$ and $|\chi|^{\frac{1}{2}}$, and $\chi = U|\chi|$ is the polar decomposition of χ (see [9]). Let

$$X^\alpha = \mathcal{D}(|\chi|^{\frac{1}{2}}), \ (X^\alpha)^- = \mathcal{E}(0^-)X^\alpha, \ (X^\alpha)^0 = (\mathcal{E}(0) - \mathcal{E}(0^-))X^\alpha, \ (X^\alpha)^+ = (id - \mathcal{E}(0))X^\alpha. \tag{2.11}$$

By (χ), one has $(X^\alpha)^0 = \{0\}$. Hence for any $u \in X^\alpha$, we have $u = u^- + u^+$ and

$$\chi u^- = -|\chi| u, \chi u^+ = |\chi| u, \ \forall u \in X^\alpha \cap \mathcal{D}(\chi), \tag{2.12}$$

where $u^- = \mathcal{E}(0^-)u \in (X^\alpha)^-, \ u^+ = (id - \mathcal{E}(0))u \in (X^\alpha)^+$. We can define an inner product

$$\prec u, v \succ = \prec |\chi|^{\frac{1}{2}} u, |\chi|^{\frac{1}{2}} v \succ_{L^2}, \ \forall u, v \in X^\alpha \tag{2.13}$$

and the corresponding norm

$$\|u\| = \prec u, u \succ^{\frac{1}{2}}, \ \forall u \in X^\alpha. \tag{2.14}$$

Remark 2.3. The fractional norm thus defined is invariant under translation, in fact by (2.5), for $u \in \big(X^\alpha\big)^+$, denote by $v(t) = u(t+T)$, then we have

$$\begin{aligned}
\|v\|^2 &= \prec v, v \succ = \prec \chi v, v \succ_{L^2} \\
&= \int_{\mathbb{R}} \Big[{}_tD_\infty^\alpha o_{-\infty}D_t^\alpha v(t) \cdot v(t) + L(t)v(t) \cdot v(t)\Big] dt \\
&= \int_{\mathbb{R}} \Big[\,|{}_{-\infty}D_t^\alpha v(t)|^2 + L(t)v(t) \cdot v(t)\Big] dt \\
&= \int_{\mathbb{R}} \Big[\Big(\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(t+T) - u(t+T-x)}{x^{\alpha+1}} dx\Big)^2 + L(t)u(t+T) \cdot u(t+T)\Big] dt.
\end{aligned}$$

By the change of variables $s = t + T$ and as L is periodic, one gets

$$\begin{aligned}
\|v\|^2 &= \int_{\mathbb{R}} \Big[\Big(\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(s) - u(s-x)}{x^{\alpha+1}} dx\Big)^2 + L(s)u(s) \cdot u(s)\Big] ds \\
&= \int_{\mathbb{R}} \Big[\,|{}_{-\infty}D_s^\alpha u(s)|^2 + L(s)u(s) \cdot u(s)\Big] ds = \|u\|^2.
\end{aligned}$$

Similarly, for $u \in \left(X^\alpha\right)^-$ and $v(t) = u(t+T)$, by the same calculation as before, we obtain $\|v\| = \|u\|$. Hence, for all $u \in X^\alpha$ and $v(t) = u(t+T)$, we have $\|u\| = \|v\|$, which means that the fractional norm is invariant under translation.

Since $X^\alpha = H^\alpha(\mathbb{R})$ with equivalent norms under condition (L), X^α is continuously embedded in $L^s(\mathbb{R})$ for all $2 \leq s \leq \infty$ and compactly embedded in $L^s(\mathbb{R})$ for all $2 < s \leq \infty$. Hence, for all $2 \leq s \leq \infty$, there exists a constant $\eta_s > 0$ such that

$$\|u\|_{L^s} \leq \eta_s \|u\|, \ \forall u \in X^\alpha. \tag{2.15}$$

In addition, one has the decomposition $X^\alpha = (X^\alpha)^- \oplus (X^\alpha)^+$ orthogonal with respect to both inner product $\prec .,. \succ$ and $\prec .,. \succ_{L^2}$. In view of (2.12) and (2.14), we have for all $u \in X^\alpha$

$$\int_{\mathbb{R}} \left[\, |_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t)\cdot u(t)\right] dt = \left\|u^+\right\|^2 - \left\|u^-\right\|^2. \tag{2.16}$$

Now, in order to prove our main results, the following critical point theorem will be needed.

Definition 2.4. Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. A functional $f \in C^1(X, \mathbb{R})$ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in X, one has $f(u) \leq \liminf_{n\to\infty} f(u_n)$, and f' is said to be weakly sequentially continuous if $\lim_{n\to\infty} f'(u_n)v = f'(u)v$ for each $v \in X$.

Lemma 2.5. [15, 16] Let $(X, \|.\|)$ be a Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $f \in C^1(X, \mathbb{R})$ be of the form

$$f(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 \right) - g(u), \ u = u^- + u^+ \in X^- \oplus X^+.$$

Assume that the following conditions are satisfied

(1) $g \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous,

(2) g' is weakly sequentially continuous,

(3) there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that

$$\alpha = \inf f(S_\rho^+) > \sup f(\partial\Lambda),$$

where

$$S_\rho^+ = \left\{u \in X^+ / \|u\| = \rho\right\}, \ \Lambda = \left\{v + se / v \in X^-, s \geq 0, \|v + se\| \leq r\right\}.$$

Then there exist a constant $c \in [\alpha, \sup f(\Lambda)]$ and a sequence $(u_n) \subset X$ satisfying

$$f(u_n) \to c, \ \|f'(u_n)\| \left(1 + \|u_n\|\right) \to 0.$$

Definition 2.6. Let (u_n) be a bounded sequence in a Banach space. We say that (u_n) is vanishing if, for each $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |u_n|^2 \, dt = 0$$

and (u_n) is nonvanishing if there exist $\sigma > 0$, $R > 0$ and $(y_n) \subset \mathbb{R}$ such that

$$\liminf_{n\to\infty} \int_{y_n-R}^{y_n+R} |u_n|^2 \, dt \geq \sigma, \ \forall n \in \mathbb{N}.$$

In the vanishing case, we have the following result, which is a special case of Lion's concentration compactness principle.

Lemma 2.7. [18, Lemma 21] Let (u_n) be a bounded sequence, if for any $R > 0$

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |u_n|^2 \, dt = 0,$$

then $u_n \to 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$.

3 First class of superquadratic potentials

Let $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ be a symmetric matrix-valued function and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function, differentiable with respect to the second variable with continuous derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$. To state the main result of this Section, we still need the following conditions

(W_1) $W(t,x)$ is $T-$periodic in t and $W(t,x) \geq 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$;

(W_2) $W(t,0) = 0$ and $\nabla W(t,x) = o(|x|)$ uniformly for $t \in \mathbb{R}$;

(W_3) $\lim_{|x|\to\infty} \frac{W(t,x)}{|x|^2} = +\infty$, almost everywhere $t \in \mathbb{R}$;

(W_4) $\widetilde{W}(t,x) \geq 0$, $\forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$, and there exists $g \in C(\mathbb{R}_+^*, \mathbb{R}_+^*)$ with $\lim_{s\to\infty} g(s) = +\infty$ such that

$$\frac{|\nabla W(t,x)|}{|x|} \geq \frac{1}{4\eta_2^2} \Rightarrow |\nabla W(t,x)| \leq \frac{|x|}{g(|x|)}\widetilde{W}(t,x),$$

where η_2 is the Sobolev embedding constant given in (2.15).

Theorem 3.1. Assume that (L), (χ) and $(W_1) - (W_4)$ are satisfied. Then the fractional Hamiltonian system $(\mathcal{FHS})$ possesses a ground state solution.

Proof of Theorem 3.1. We are going to establish the corresponding variational framework to obtain the existence of ground state solution of $(\mathcal{FHS})$. For this end, define the energy functional f associated to system $(\mathcal{FHS})$

$$f(u) = \frac{1}{2}\int_{\mathbb{R}} \left[\, |{}_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t)\cdot u(t)\right] dt - \int_{\mathbb{R}} W(t,u(t))dt$$

defined on the Hilbert space X^α introduced in Section 2. By (2.16), f can be rewritten as

$$f(u) = \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 \Big) - \int_{\mathbb{R}} W(t,u(t))dt$$

for $u = u^- + u^+ \in X^\alpha = \big(X^\alpha\big)^- \oplus \big(X^\alpha\big)^+$. It is well known that f is continuously differentiable on X^α and

$$\begin{aligned} f'(u)v &= \int_{\mathbb{R}} \Big[{}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha)v(t) + L(t)u(t)\cdot v(t)\Big] dt - \int_{\mathbb{R}} \nabla W(t,u(t))\cdot v(t)dt \\ &= \prec u, v \succ - \int_{\mathbb{R}} \nabla W(t,u(t))\cdot v(t)dt \end{aligned}$$

for all $u, v \in X^\alpha$. Moreover, the nontrivial critical points of f on X^α are solutions of $(\mathcal{FHS})$. In the following, we will proceed by successive lemmas.

Lemma 3.2. Assume that (L), (χ), (W_1) and (W_2) are satisfied. Then the functional

$$g(u) = \int_{\mathbb{R}} W(t,u(t))dt,\ u \in X^\alpha$$

is nonnegative, weakly sequentially lower semi-continuous and g' is weakly sequentially continuous.

Proof . Let $u_n \rightharpoonup u$ in X^α. Taking a subsequence if necessary, we have $u_n \to u$ almost everywhere on $\mathbb{R}$. By (W_2), Fatou's lemma, we have

$$g(u) \leq \liminf_{n\to\infty} g(u_n),$$

which means that g is weakly sequentially lower semi-continuous. $g^{'}$ is weakly sequentially continuous on X^α is due to [37]. □

Lemma 3.3. Assume that (L), (χ), (W_1) and (W_2) are satisfied. Then there exists a constant $\rho > 0$ such that $\gamma = \inf(S_\rho^+) > 0$, where

$$S_\rho^+ = \Big\{u \in \big(X^\alpha\big)^+ / \left\|u\right\| = \rho\Big\}.$$

Proof . By (W_2), we have

$$\forall \epsilon > 0,\ \exists r > 0/\forall t \in \mathbb{R},\ |x| \le r,\ |\nabla W(t,x)| \le \epsilon\, |x|\,,$$

which by the Mean Value Theorem implies

$$0 \le W(t,x) = \int_0^1 \nabla W(t,sx)\cdot x ds \le \frac{\epsilon}{2}\, |x|^2\,. \tag{3.1}$$

Let $u \in \left(X^{\alpha}\right)^+$ be such that $\|u\| \le \frac{r}{\eta_\infty}$, then by (2.15), one has $\|u\|_{L^\infty} \le r$. Hence

$$\begin{aligned} f(u) &= \frac{1}{2}\, \|u\|^2 - \int_{\mathbb{R}} W(t,u(t))dt \\ &\ge \frac{1}{2}\, \|u\|^2 - \int_{\mathbb{R}} \frac{\epsilon}{2}\, |u(t)|^2\, dt \\ &\ge \frac{1}{2}(1-\epsilon\eta_2)\, \|u\|^2\,. \end{aligned}$$

Take $\epsilon = \frac{\eta_2}{2}$, we obtain

$$f(u) \ge \frac{1}{4}\, \|u\|^2\,,\ \forall\, \|u\| \le \frac{r}{\eta_\infty},\ u \in \left(X^{\alpha}\right)^+.$$

It suffices to take $\rho = \frac{r}{\eta_\infty}$ and $\gamma = \frac{\rho^2}{4}$. The proof of Lemma 3.3 is completed. □

Lemma 3.4. Assume that (L), (χ), (W_1) and (W_3) are satisfied. Let $e \in \left(X^{\alpha}\right)^+$ with $\|e\| = 1$. Then there is $r_0 > \rho$ such that $\sup f(\partial\Lambda) \le 0$ for $r \ge r_0$, where

$$\Lambda = \left\{ w + se/w \in \left(X^{\alpha}\right)^-, s \ge 0, \|w + se\| \le r \right\}.$$

Proof . We have

$$\partial\Lambda = \left\{ w \in \left(X^{\alpha}\right)^- / \|w\| \le r \right\} \cup \left\{ w + se/w \in \left(X^{\alpha}\right)^-, s \ge 0, \|w + se\| = r \right\}.$$

We have $f(w) \le 0$ for all $w \in \left(X^{\alpha}\right)^-$. We claim that $f(w+se) \to -\infty$ as $\|w+se\| \to \infty$. Arguing indirectly, assume that there exists a sequence $(w_n + s_n e) \subset \left(X^{\alpha}\right)^- \oplus \mathbb{R}e$ with $\|w_n + s_n e\| \to \infty$ such that $f(w_n + s_n e) \ge -M$ for all $n \in \mathbb{N}$, where M is a constant. Set $\frac{w_n + s_n e}{\|w_n + s_n e\|} = v_n^- + \tau_n e$, then $\|v_n^- + \tau_n e\| = 1$. Passing to a subsequence if necessary, we may assume that $\tau_n \to \bar{\tau}$, $v_n^- \rightharpoonup v^-$ and $v_n^- \to v^-$ almost everywhere on $\mathbb{R}$. Hence

$$\frac{-M}{\|w_n + s_n e\|^2} \le \frac{f(w_n + s_n e)}{\|w_n + s_n e\|^2} = \frac{\tau_n^2}{2} - \frac{1}{2}\left\|v_n^-\right\|^2 - \int_{\mathbb{R}} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} dt. \tag{3.2}$$

If $\bar{\tau} = 0$, then it follows from (3.2) that

$$0 \le \frac{1}{2}\left\|v_n^-\right\|^2 + \int_{\mathbb{R}} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} dt \le \frac{\tau_n^2}{2} + \frac{M}{\|w_n + s_n e\|^2}$$

which yields $\|v_n^-\| \to 0$ and so $1 = \|v_n^- + \tau_n e\| \to 0$, a contradiction. If $\bar{\tau} \neq 0$, then it follows from (3.2), (W_3) and Fatou's lemma

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \left[\frac{\tau_n^2}{2} - \frac{1}{2}\left\|v_n^-\right\|^2 - \int_{\mathbb{R}} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} dt \right] \\ &\le \frac{\bar{\tau}^2}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} dt \\ &\le \frac{\bar{\tau}^2}{2} - \int_{\mathbb{R}} \liminf_{n\to\infty} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} dt \\ &= -\infty, \end{aligned}$$

a contradiction again. Hence $f(w+se) \to -\infty$ as $\|w+se\| \to \infty$, and then there exists $r_0 > \rho$ such that $\sup f(\partial\Lambda) \le 0$ for $r \ge r_0$. □

Lemma 3.5. Assume that (L), (χ), (W_1) and (W_4) are satisfied. Then any sequence (u_n) satisfying

$$f(u_n) \to c > 0,\ \|f'(u_n)\| (1 + \|u_n\|) \to 0 \text{ as } n \to \infty \tag{3.3}$$

is bounded in X^α.

Proof . To prove the boundedness of (u_n), arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and $\|v_n\|_{L^s} \leq \eta_s \|v_n\| = \eta_s$ for $2 \leq s \leq \infty$. Observe that

$$c + o(1) = f(u_n) - \frac{1}{2} f'(u_n)u_n = \int_{\mathbb{R}} \widetilde{W}(t, u_n)dt. \tag{3.4}$$

Since $\lim_{s\to\infty} g(s) = +\infty$, there exists $R > 0$ such that

$$g(s) \geq 8(c + 1)\eta_\infty^2,\ \forall s \geq R. \tag{3.5}$$

Let

$$A_n = \left\{ t \in \mathbb{R} / \frac{|\nabla W(t, u_n)|}{|u_n|} \leq \frac{1}{4\eta_2^2} \right\} \text{ and } B_n = \{t \in \mathbb{R}/ |u_n| \geq R\}. \tag{3.6}$$

Hence, it follows from (3.6) that

$$\begin{aligned} \int_{A_n} \frac{|\nabla W(t, u_n)|}{\|u_n\|} |v_n| \, dt &= \int_{A_n} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n|^2 \, dt \\ &\leq \frac{1}{4\eta_2^2} \int_{A_n} |v_n|^2 \, dt \leq \frac{1}{4}. \end{aligned} \tag{3.7}$$

From (W_4), (3.4)-(3.6), one has

$$\begin{aligned} \int_{(\mathbb{R}\setminus A_n)\cap B_n} \frac{|\nabla W(t, u_n)|}{\|u_n\|} |v_n| \, dt &= \int_{(\mathbb{R}\setminus A_n)\cap B_n} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n|^2 \, dt \\ &\leq \|v_n\|_{L^\infty}^2 \int_{(\mathbb{R}\setminus A_n)\cap B_n} \frac{|\nabla W(t, u_n)|}{|u_n|} dt \\ &\leq \eta_\infty^2 \int_{(\mathbb{R}\setminus A_n)\cap B_n} \frac{\widetilde{W}(t, u_n)}{g(|u_n|)} dt \\ &\leq \eta_\infty^2 \frac{1}{8(c + 1)\eta_\infty^2} \int_{(\mathbb{R}\setminus A_n)\cap B_n} \widetilde{W}(t, u_n)dt \\ &\leq \frac{1}{8(c + 1)})(c + o(1)) \leq \frac{1}{8} + o(1) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \int_{(\mathbb{R}\setminus A_n)\cap(\mathbb{R}\setminus B_n)} \frac{|\nabla W(t, u_n)|}{\|u_n\|} |v_n| \, dt &\leq \frac{\|v_n\|_{L^\infty}}{\|u_n\|} \int_{(\mathbb{R}\setminus A_n)\cap(\mathbb{R}\setminus B_n)} |\nabla W(t, u_n)| \, dt \\ &\leq \frac{\|v_n\|_{L^\infty}}{\|u_n\|} \int_{(\mathbb{R}\setminus A_n)\cap(\mathbb{R}\setminus B_n)} \frac{|u_n|}{g(|u_n|)} \widetilde{W}(t, u_n)dt \\ &\leq \frac{\eta_\infty}{\|u_n\|} \sup_{|x|\leq R} \frac{|x|}{g(|x|)} \int_{(\mathbb{R}\setminus A_n)\cap(\mathbb{R}\setminus B_n)} \widetilde{W}(t, u_n)dt \\ &\leq \frac{c_1}{\|u_n\|} \int_{(\mathbb{R}\setminus A_n)\cap(\mathbb{R}\setminus B_n)} \widetilde{W}(t, u_n)dt \\ &= \frac{c_1(c + o(1))}{\|u_n\|} = o(1). \end{aligned} \tag{3.9}$$

Now, combining (3.3) with (3.7)-(3.9) yields

$$\begin{aligned}1+o(1) &= \frac{\|u_n\|^2 - f'(u_n)u_n}{\|u_n\|^2}\\ &= \frac{1}{\|u_n\|^2}\int_{\mathbb{R}} \nabla W(t,u_n)\cdot u_n dt\\ &= \frac{1}{\|u_n\|}\int_{\mathbb{R}} \nabla W(t,u_n)\cdot v_n dt\\ &\leq \frac{1}{\|u_n\|}\int_{\mathbb{R}} |\nabla W(t,u_n)|\,|v_n|\,dt\\ &= \int_{A_n}\frac{|\nabla W(t,u_n)|}{\|u_n\|}|v_n|\,dt + \int_{(\mathbb{R}\backslash A_n)\cap B_n}\frac{|\nabla W(t,u_n)|}{\|u_n\|}|v_n|\,dt + \int_{(\mathbb{R}\backslash A_n)\cap(\mathbb{R}\backslash B_n)}\frac{|\nabla W(t,u_n)|}{\|u_n\|}|v_n|\,dt\\ &\leq \frac{3}{8}+o(1).\end{aligned} \tag{3.10}$$

This contradiction implies that (u_n) is bounded in X^α. $\square$

Now, by Lemmas 2.5 and 3.2-3.4, there exist a constant $c \geq \alpha$ and a sequence $(u_n) \subset X^\alpha$ satisfying

$$f(u_n) \to c \; and \; \|f'(u_n)\|\,(1+\|u_n\|) \to 0. \tag{3.11}$$

In view of Lemma 3.5, (u_n) is bounded. So, there exists a constant $R > 0$ such that

$$\|u_n\| \leq R, \;\forall n \in \mathbb{N}. \tag{3.12}$$

By virtue of (W_1) and (W_2), for all $\epsilon > 0$, there exists $r_\epsilon \in]0, R\eta_2[$ such that

$$|\nabla W(t,x)| \leq \epsilon\,|x|\,,\; \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\; |x| \leq r_\epsilon.$$

Let $C_\epsilon = \max\left\{\frac{|\nabla W(t,x)|}{|x|^2}/t \in \mathbb{R},\; r_\epsilon \leq |x| \leq R\eta_2\right\}$. Then we have

$$|\nabla W(t,x)| \leq \epsilon\,|x| + C_\epsilon\,|x|^2\,,\; \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\; |x| \leq R\eta_2. \tag{3.13}$$

Since X^α is continuously embedded in $L^s(\mathbb{R})$ for $s = 2, 3$, there exists a constant $c_2 > 0$ such that $\|u_n\|^2_{L^2} + \|u_n\|^3_{L^3} \leq c_2$. If

$$\delta = \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^*}\int_{I(y,2)} \left|u_n^+\right|^2 dt = 0$$

where $I(y,2)$ is the interval of $\mathbb{R}$ centered at y with radius 2, then by Lemma 2.7, $u_n^+ \to 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$. By (2.15) and (3.12), one has

$$\|u_n\|_{L^\infty} \leq \eta_\infty R,\; \forall n \in \mathbb{N}. \tag{3.14}$$

Combining (3.13), (3.14) and Hölder's inequality yields for a positive constant c_3

$$\begin{aligned}2c+o(1) = 2f(u_n) &= \left\|u_n^+\right\|^2 - \left\|u_n^-\right\|^2 - 2\int_{\mathbb{R}} W(t,u_n)dt\\ &\leq \left\|u_n^+\right\|^2 = f'(u_n)u_n^+ + \int_{\mathbb{R}} \nabla W(t,u_n)\cdot u_n^+ dt\\ &\leq f'(u_n)u_n^+ + \int_{\mathbb{R}}\left[\epsilon\,|u_n| + C_\epsilon\,|u_n|^2\right]\left|u_n^+\right| dt\\ &\leq f'(u_n)u_n^+ + \epsilon\,\|u_n\|_{L^2}\left\|u_n^+\right\|_{L^2} + C_\epsilon\,\|u_n\|^2_{L^3}\left\|u_n^+\right\|_{L^3}\\ &\leq \epsilon c_3 + o(1).\end{aligned}$$

This is a contradiction since $\epsilon > 0$ is arbitrary. Thus $\delta > 0$. Going to a subsequence if necessary, we can assume the existence of $(k_n) \subset \mathbb{Z}$ such that

$$\int_{I(k_n,2)} \left|u_n^+\right|^2 dt \geq \frac{\delta}{2},\; \forall n \in \mathbb{N}.$$

Let us define $w_n(t) = u_n(t + k_n T)$ so that

$$\int_{I(0,2)} \left|w_n^+\right|^2 dt \geq \frac{\delta}{2}, \ \forall n \in \mathbb{N}. \tag{3.15}$$

Since $W(t,x)$ is $T-$ periodic in t, we have $\|w_n\| = \|u_n\|$ and

$$f(w_n) \to c \in [\gamma, \sup f(\Lambda)] \ and \ \|f'(w_n)\| (1 + \|w_n\|) \to 0. \tag{3.16}$$

Passing to a subsequence, we get $w_n \rightharpoonup w_0$ in X^α, $w_n \to w_0$ in $L^s_{loc}(\mathbb{R})$ for $2 \leq s < \infty$ and $w_n \to w_0$ almost everywhere on $\mathbb{R}$. Since f' is weakly sequentially continuous, one has

$$f'(w_n)v \to f'(w_0)v, \ \forall v \in X^\alpha,$$

and by (3.16), $f'(w_n) \to 0$ in $\left(X^\alpha\right)'$. Then we have $f'(w_0) = 0$. By (3.15) and the fact $w_n \to w_0$ in $L^s_{loc}(\mathbb{R})$, we have

$$\int_{I(0,2)} \left|w_0^+\right|^2 dt \geq \lim_{n\to\infty} \int_{I(0,2)} \left|w_n^+\right|^2 dt \geq \frac{\delta}{2}.$$

Hence $w_0^+ \neq 0$ and $w_0 \in \mathcal{M}$. Therefore $\mathcal{M}$ is not empty. Now, let

$$c = \inf_{u \in \mathcal{M}} f(u).$$

For any critical point $u \in \mathcal{M} \setminus \{0\}$, assumption (W_4) implies

$$f(u) = f(u) - \frac{1}{2} f'(u)u = \int_{\mathbb{R}} \big(\frac{1}{2}\nabla W(t,u)\cdot u - W(t,u)\big)dt \geq 0.$$

Therefore $c \geq 0$. We shall prove that $c > 0$ and there is $u_0 \in \mathcal{M} \setminus \{0\}$ such that $f(u_0) = c$. Let $(u_n) \subset \mathcal{M} \setminus \{0\}$ be such that $f(u_n) \to c$. Then the proof of Lemma 3.5 shows that (u_n) is bounded. So, there exists a constant $R_0 > 0$ such that $\|u_n\| \leq R_0$ for all $n \in \mathbb{N}$. By (2.15), we get

$$\|u_n\|_{L^\infty} \leq \eta_\infty R_0 = R, \ \forall n \in \mathbb{N}. \tag{3.17}$$

As in (3.13), for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\nabla W(t,x)| \leq \epsilon |x| + C_\epsilon |x|^2, \ \forall t \in \mathbb{R}, \ |x| \leq R.$$

So

$$\begin{aligned} \left\|u_n^+\right\|^2 &\leq \int_{\mathbb{R}} \nabla W(t,u_n)\cdot u_n^+ dt \\ &\leq \int_{\mathbb{R}} \left[\epsilon |u_n| + C_\epsilon |u_n|^2\right] \left|u_n^+\right| dt \\ &\leq \epsilon \|u_n\|_{L^2} \left\|u_n^+\right\|_{L^2} + C_\epsilon \|u_n\|^2_{L^3} \left\|u_n^+\right\|_{L^3} \\ &\leq \epsilon\eta_3^2 \|u_n\|^2 + C_\epsilon \eta_3^2 \|u_n\|^2 \|u_n\|_{L^3}. \end{aligned} \tag{3.18}$$

Similarly, we have

$$\left\|u_n^-\right\|^2 \leq \epsilon\eta_2^2 \|u_n\|^2 + C_\epsilon \eta_3^2 \|u_n\|^2 \|u_n\|_{L^3}. \tag{3.19}$$

From (3.18) and (3.19), we get

$$\|u_n\|^2 \leq 2\epsilon\eta_2^2 \|u_n\|^2 + 2C_\epsilon \eta_3^2 \|u_n\|^2 \|u_n\|_{L^3},$$

which implies

$$\|u_n\|_{L^3} \geq d, \ \forall n \in \mathbb{N} \tag{3.20}$$

for a positive constant d. If

$$\delta = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^*} \int_{I(y,2)} |u_n|^2 dt = 0,$$

then, by Lemma 2.7, $u_n \to 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$. This is a contradiction with (3.20). Thus $\delta > 0$. Taking a subsequence if necessary, we can assume as above the existence of $(k_n) \subset \mathbb{Z}$ such that

$$\int_{I(k_n,2)} |u_n|^2\, dt \geq \frac{\delta}{2},\ \forall n \in \mathbb{N}.$$

Let us define $w_n(t) = u_n(t + k_n T)$ so that

$$\int_{I(0,2)} |w_n|^2\, dt \geq \frac{\delta}{2},\ \forall n \in \mathbb{N}. \tag{3.21}$$

Since $W(t,x)$ is $T-$periodic in t, we have $\|w_n\| = \|u_n\|$ and

$$f(w_n) \to c \ and \ \|f'(w_n)\|\,(1 + \|w_n\|) \to 0. \tag{3.22}$$

Passing to a subsequence, we get $w_n \rightharpoonup w_0$ in E, $w_n \to w_0$ in $L^s_{loc}(\mathbb{R})$ for $2 \leq s < \infty$ and $w_n \to w_0$ almost everywhere on $\mathbb{R}$. Since f' is weakly sequentially continuous, then we have $f'(w_0) = 0$. Since $w_n \to w_0$ in $L^2_{loc}(\mathbb{R})$, then (3.21) implies

$$\int_{I(0,2)} |w_0|^2\, dt = \lim_{n\to\infty} \int_{I(0,2)} |w_n|^2\, dt \geq \frac{\delta}{2}.$$

Hence $w_0 \neq 0$ and $c = f(w_0) > 0$. The proof of Theorem 3.1 is completed.

4 Second class of superquadratic potentials

Let L and W be defined as in Section 3 and note that the ground state solution for $(\mathcal{FHS})$ in Theorem 3.1 is in fact a nontrivial solution u which satisfies $f(u) = \inf_{\mathcal{M}} f$, where

$$\mathcal{M} = \left\{ v \in X^\alpha \setminus \{0\} / f'(v) = 0 \right\},$$

$X^\alpha = \left(X^\alpha\right)^- \oplus \left(X^\alpha\right)^+$ is the working space on which the energy functional f associated with $(\mathcal{FHS})$ is defined. On 2005, Pankov [22] was first introduced the following set

$$\mathcal{N}^- = \left\{ u \in X^\alpha \setminus \left(X^\alpha\right)^- / f'(u)u = f'(u)v, \forall v \in \left(X^\alpha\right)^- \right\},$$

which is a subset of the Nehari manifold

$$\mathcal{N} = \left\{ u \in X^\alpha \setminus \{0\} / f'(u)v = 0, \forall v \in \left(X^\alpha\right)^- \right\}.$$

In general, $\mathcal{M}$ is a very small subset of $\mathcal{N}^-$. By definition, if $u \in X^\alpha \setminus \{0\}$ satisfies $f(u) = \inf_{\mathcal{N}^-} f$ and $f'(u) = 0$, it is said a ground state solution of $(\mathcal{FHS})$. Using variational methods and critical point theory, many authors study the existence of ground state solutions which minimizes the energy on the Nehari-Pankov manifold $\mathcal{N}^-$ for Schrödinger equation, see [6, 27]-[29] and the references therein. However, to the best of our knowledge there is no similar results on the existence of a ground state solution for fractional Hamiltonian system $(\mathcal{FHS})$. We point out that it is much more difficult to find a solution u of $(\mathcal{FHS})$ which satisfies $f(u) = \inf_{\mathcal{N}^-} f$ than one satisfying $f(u) = \inf_{\mathcal{M}} f$. Motivated by the above papers, in the following, we will study the existence of ground state solution for $(\mathcal{FHS})$ which minimizes the associated energy on the Nehari-Pankov manifold $\mathcal{N}^-$. More precisely, we obtain the following result.

Theorem 4.1. Assume that (L), (χ), $(W_1) - (W_3)$ and the following condition are satisfied

$$(W_5) \qquad \frac{1-\theta^2}{2}\nabla W(t,x)\cdot x - \theta \nabla W(t,x)\cdot y + W(t,\theta x + y) - W(t,x) \geq 0,\ \forall \theta \geq 0,\ x, y \in \mathbb{R}^N.$$

Then the fractional Hamiltonian system $(\mathcal{FHS})$ possesses a nontrivial solution satisfying $f(u) = \inf_{\mathcal{N}^-} f$.

Proof of Theorem 4.1. To prove Theorem 4.1, we need the following several lemmas:

Lemma 4.2. Assume that (L), (χ), (W_1) and (W_5) are satisfied. Then for all $\theta \geq 0$, $u \in X^\alpha$ and $w \in \big(X^\alpha\big)^-$

$$f(u) \geq f(\theta u + w) + \frac{1}{2}\|w\|^2 + \frac{1-\theta^2}{2} f'(u)u - \theta f'(u)w.$$

Proof . By (W_5), we have for all $\theta \geq 0$, $u \in X^\alpha$ and $w \in \big(X^\alpha\big)^-$

$$\begin{aligned}
f(u) - f(\theta u + w) &= \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\Big) - \int_{\mathbb{R}} W(t,u(t))dt - \frac{1}{2}\Big(\left\|\theta u^+\right\|^2 - \left\|\theta u^- + w\right\|^2\Big) + \int_{\mathbb{R}} W(t,\theta u + w)dt \\
&= \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\Big) - \frac{1}{2}\Big(\theta^2\left\|u^+\right\|^2 - \theta^2\left\|u^-\right\|^2 - \|w\|^2 - \theta \prec u^-, w \succ\Big) \\
&+ \int_{\mathbb{R}} \big[W(t,\theta u + w) - W(t,u)\big]dt \\
&= \frac{1}{2}\|w\|^2 + \frac{1-\theta^2}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 - \int_{\mathbb{R}} \nabla W(t,u)\cdot u dt\Big) - \theta\Big[\prec u, w \succ - \int_{\mathbb{R}} \nabla W(t,u)\cdot w dt\Big] \\
&+ \int_{\mathbb{R}} \Big[\frac{1-\theta^2}{2}\nabla W(t,u)\cdot u - \theta\nabla W(t,u)\cdot w + W(t,\theta u + w) - W(t,u)\Big]dt \\
&= \frac{1}{2}\|w\|^2 + \frac{1-\theta^2}{2} f'(u)u - \theta f'(u)w \\
&+ \int_{\mathbb{R}} \Big[\frac{1-\theta^2}{2}\nabla W(t,u)\cdot u - \theta\nabla W(t,u)\cdot w + W(t,\theta u + w) - W(t,u)\Big]dt \\
&\geq \frac{1}{2}\|w\|^2 + \frac{1-\theta^2}{2} f'(u)u - \theta f'(u)w.
\end{aligned}$$

The proof of Lemma 4.2 is completed. □

Corollary 4.3. Assume that (L), (χ), (W_1) and (W_5) are satisfied. Then for any $u \in \mathcal{N}^-$, one has

$$f(u) \geq f(\theta u + w), \ \forall \theta \geq 0, \ w \in \big(X^\alpha\big)^-.$$

Proof . It is evident since $u \in \mathcal{N}^-$ implies $f'(u)u = f'(u)w = 0$. □

Corollary 4.4. Assume that (L), (χ), (W_1) and (W_5) are satisfied. Then, for all $u \in X^\alpha$ and $\theta \geq 0$

$$f(u) \geq \frac{\theta^2}{2}\|u\|^2 + \frac{1-\theta^2}{2} f'(u)u + \theta^2 f'(u)u^- - \int_{\mathbb{R}} W(t,\theta u^+)dt. \tag{4.3}$$

Proof . Take $w = -\theta u^-$ in Lemma 4.2, one has

$$\begin{aligned}
f(u) &\geq f(\theta u^+) + \frac{\theta^2}{2}\left\|u^-\right\|^2 + \frac{1-\theta^2}{2} f'(u)u + \theta^2 f'(u)u^- \\
&= \frac{\theta^2}{2}\left\|u^+\right\|^2 - \int_{\mathbb{R}} W(t,\theta u^+)dt + \frac{\theta^2}{2}\left\|u^-\right\|^2 + \frac{1-\theta^2}{2} f'(u)u + \theta^2 f'(u)u^- \\
&= \frac{\theta^2}{2}\|u\|^2 + \frac{1-\theta^2}{2} f'(u)u + \theta^2 f'(u)u^- - \int_{\mathbb{R}} W(t,\theta u^+)dt,
\end{aligned}$$

which ends the proof of Corollary 4.4. □

Lemma 4.5. Assume that (L), (χ), (W_1), (W_2) and (W_5) are satisfied. Then

(i) there exists $\rho > 0$ such that

$$m = \inf_{\mathcal{N}^-} f \geq \gamma = \inf\Big\{f(u)/u \in \big(X^\alpha\big)^+, \ \|u\| = \rho\Big\} > 0,$$

(ii)
$$\left\|u^+\right\| \geq \max\Big\{\left\|u^-\right\|, \sqrt{2m}\Big\}, \ \textit{for all } u \in \mathcal{N}^-.$$

Proof . (i) Take $w = -\theta u^-$ in Corollary 4.3, one has

$$f(u) \geq f(\theta u^+),\ \forall \theta \geq 0,\ u \in \mathcal{N}^-.$$

Let ρ be defined in Lemma 3.3, we have

$$\gamma = \inf\Big\{f(u)/u \in \big(X^\alpha\big)^+,\ \|u\| = \rho\Big\} > 0.$$

Then, we have for all $u \in \mathcal{N}^-$ (take $\theta = \frac{\rho}{\|u^+\|}$)

$$f(u) \geq f(\rho \frac{u^+}{\|u^+\|}) \geq \gamma = \inf\Big\{f(v)/v \in \big(X^\alpha\big)^+,\ \|v\| = \rho\Big\}.$$

(ii) For $u \in \mathcal{N}^-$, we have

$$m \leq \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\Big) - \int_{\mathbb{R}} W(t, u(t))dt \leq \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\Big),$$

which implies $2m + \left\|u^-\right\|^2 \leq \left\|u^+\right\|^2$ and hence $\|u^+\| \geq \max\big\{\sqrt{2m}, \|u^-\|\big\}$. The proof of Lemma 4.5 is completed. □

Lemma 4.6. Assume that (L), (χ), $(W_1) - (W_3)$ and (W_5) are satisfied. Then any sequence $(u_n) \subset X^\alpha$ satisfying

$$f(u_n) \to c > 0,\ f'(u_n)u_n^\pm \to 0 \tag{4.4}$$

is bounded.

Proof . To prove the boundedness of (u_n), arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. By (2.15), $\|v_n\|_{L^2} \leq \eta_2$. If

$$\delta = \limsup_{n\to\infty} \sup_{s\in\mathbb{R}} \int_{s-T}^{s+T} \left|v_n^+\right|^2 dt = 0,$$

then by Lemma 2.7, $v_n^+ \to 0$ in $L^s(\mathbb{R})$ for $s \in]2, \infty[$. Fix $R > [2(1+c)]^{\frac{1}{2}}$. By virtue of (W_1) and (W_2), for $\epsilon = \frac{1}{4(R\eta_2)^2}$, there exists $r \in]0, R\eta_\infty[$ such that

$$W(t,x) \leq \frac{1}{4(R\eta_2)^2} |x|^2,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ |x| \leq r. \tag{4.5}$$

Let

$$\beta = \max\left\{\frac{W(t,x)}{|x|^3}/t \in \mathbb{R},\ r \leq |x| \leq R\eta_\infty\right\}. \tag{4.6}$$

Then $0 \leq \beta < \infty$. Hence, it follows from (4.5) and (4.6) that

$$W(t,x) \leq \frac{1}{4(R\eta_2)^2} |x|^2 + \beta |x|^3,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ |x| \leq R\eta_\infty. \tag{4.7}$$

Combining (4.7) with the fact $\|v_n^+\|_{L^\infty} \leq \eta_\infty$, one has

$$\limsup_{n\to\infty} \int_{\mathbb{R}} W(t, Rv_n^+)dt \leq \frac{1}{4\eta_2^2} \limsup_{n\to\infty} \int_{\mathbb{R}} \left|v_n^+\right|^2 dt + R^3\beta \limsup_{n\to\infty} \int_{\mathbb{R}} \left|v_n^+\right|^3 dt \leq \frac{1}{4}. \tag{4.8}$$

Set $\theta_n = \frac{R}{\|u_n\|}$. Hence, by virtue of (4.3), (4.4) and (4.8), one has

$$\begin{aligned}
&c + o(1) = f(u_n) \\
&\geq \frac{\theta_n^2}{2} \|u_n\|^2 - \int_{\mathbb{R}} W(t, \theta_n u_n^+)dt + \frac{1-\theta_n^2}{2} f'(u_n)u_n + \theta_n^2 f'(u_n)u_n^- \\
&\geq \frac{R^2}{2} - \int_{\mathbb{R}} W(t, \frac{R}{\|u_n\|}u_n^+)dt + \frac{1}{2}\Big(1 - \frac{R^2}{\|u_n\|^2}\Big) f'(u_n)u_n + \frac{R^2}{\|u_n\|^2} f'(u_n)u_n^- \\
&\geq \frac{R^2}{2} - \int_{\mathbb{R}} W(t, \frac{R}{\|u_n\|}u_n^+)dt + o(1) \\
&\geq \frac{R^2}{2} - \frac{1}{4} + o(1) > c + 1 - \frac{1}{4} + o(1) = c + \frac{3}{4} + o(1).
\end{aligned}$$

This contradiction shows that $\delta > 0$. We may assume that there exists $(k_n) \subset \mathbb{Z}$ such that

$$\int_{(k_n-2)T}^{(k_n+2)T} \left|v_n^+\right|^2 dt > \frac{\delta}{2}, \ \forall n \in \mathbb{N}.$$

Let $w_n(t) = v_n(t + k_nT)$. Then

$$\int_{-2T}^{2T} \left|w_n^+\right|^2 dt > \frac{\delta}{2}, \ \forall n \in \mathbb{N}. \tag{4.9}$$

Now, we define $\widetilde{u}_n(t) = u_n(t + k_nT)$. Then $\frac{\widetilde{u}_n}{\|u_n\|} = w_n$ and $\|w_n\| = 1$. Up to a subsequence if necessary, we can assume $w_n \rightharpoonup w$ in X^α, $w_n \to w$ in $L^2_{loc}(\mathbb{R})$ and $w_n \to w$ almost everywhere on $\mathbb{R}$. Obviously, (4.9) implies that $w \neq 0$. Hence, it follows from (4.4), (W_3) and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{f(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\left(\left\|w_n^+\right\|^2 - \left\|w_n^-\right\|^2\right) - \int_{\mathbb{R}} \frac{W(t, u_n)}{\|u_n\|^2} dt\right. \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\left(\left\|w_n^+\right\|^2 - \left\|w_n^-\right\|^2\right) - \int_{\mathbb{R}} \frac{W(t, u_n)}{|\widetilde{u}_n|^2} |w_n|^2 dt\right. \\
&\leq \frac{1}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}} \frac{W(t, u_n)}{|\widetilde{u}_n|^2} |w_n|^2 dt \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Thus (u_n) is bounded. The proof of Lemma 4.6 is completed. □

Lemma 4.7. Assume that (L), (χ), $(W_1) - (W_3)$ and (W_5) are satisfied. Then there exist a constant $c \in [\gamma, \sup f(\Lambda)]$ and a sequence $(u_n) \subset X^\alpha$ satisfying

$$f(u_n) \to c \textit{ and } \|f'(u_n)\| (1 + \|u_n\|) \to 0. \tag{4.10}$$

Proof . By Lemmas 3.2, 4.5(i), the functional f satisfies all the conditions of Lemma 2.5. Hence, by Lemma 2.5, there exist a constant $c \in [\gamma, \sup f(\Lambda)]$ and a sequence $(u_n) \subset X^\alpha$ satisfying (4.10). The proof of Lemma 4.7 is completed. □

Lemma 4.8. Assume that (L), (χ), $(W_1) - (W_3)$ and (W_5) are satisfied. Then there exist a constant $c \in [\gamma, m]$ and a sequence $(u_n) \subset X^\alpha$ satisfying

$$f(u_n) \to c \textit{ and } \|f'(u_n)\| (1 + \|u_n\|) \to 0. \tag{4.11}$$

Proof . Choose $(v_n) \subset \mathcal{N}^-$ such that

$$m \leq f(v_n) < m + \frac{1}{n}, \ \forall n \in \mathbb{N}. \tag{4.12}$$

By Lemma 4.5, $\left\|v_+^+\right\| \geq \sqrt{2m}$. Set $e_n = \frac{v_n^+}{\left\|v_n^+\right\|}$. Then $e_n \in \left(X^\alpha\right)^+$ and $\|e_n\| = 1$. In view of Lemma 3.4, there exists $r_n > \max\{\rho, \|v_n\|\}$ such that $\sup f(\partial\Lambda_n) \leq 0$, where

$$\Lambda_n = \left\{w + se_n / w \in \left(X^\alpha\right)^-, \ s \geq 0, \ \|w + se_n\| \leq r_n\right\}, \ n \in \mathbb{N}. \tag{4.13}$$

Hence, applying Lemma 4.6 to the above set Λ_n, there exist a constant $c_n \in [\gamma, \sup f(\Lambda_n)]$ and a sequence $(u_{n,k}) \subset X^\alpha$ satisfying

$$f(u_{n,k}) \to c_n \textit{ and } \|f'(u_{n,k})\| (1 + \|u_{n,k}\|) \to 0, \ \forall n \in \mathbb{N}. \tag{4.14}$$

By virtue of Corollary 4.3, we can get

$$f(v_n) \geq f(sv_n + w), \ \forall s \geq 0, \ w \in \left(X^\alpha\right)^-. \tag{4.15}$$

Since $v_n \in \Lambda_n$, it follows from (4.13) and (4.15) that $f(v_n) = \sup f(\Lambda_n)$. Hence, by (4.12) and (4.14), one has

$$f(u_{n,k}) \to c_n < m + \frac{1}{n} \textit{ and } \|f'(u_{n,k})\| (1 + \|u_{n,k}\|) \to 0, \ \forall n \in \mathbb{N}. \tag{4.16}$$

Now, we can choose a sequence $(k_n) \subset \mathbb{N}$ such that

$$f(u_{n,k_n}) < m + \frac{1}{n} \textit{ and } \|f'(u_{n,k_n})\| (1 + \|u_{n,k_n}\|) < \frac{1}{n}, \ \forall n \in \mathbb{N}. \tag{4.17}$$

Let $u_n = u_{n,k_n}$ for $n \in \mathbb{N}$, then going to a subsequence if necessary, we obtain

$$f(u_n) \to c \in [\alpha, m] \textit{ and } \|f'(u_n)\| (1 + \|u_n\|) \to 0.$$

The proof of Lemma 4.8 is completed. □

Next, in view of Lemma 4.8, there exist a constant $c \in [\gamma, m]$ and a sequence $(u_n) \subset X^\alpha$ satisfying

$$f(u_n) \to c \textit{ and } \|f'(u_n)\| (1 + \|u_n\|) \to 0. \tag{4.18}$$

By Lemma 4.6, (u_n) is bounded. By proceeding as in the proof of Theorem 3.1, there exist a constant $\delta > 0$ and a sequence $(w_n) \subset X^\alpha$ satisfying

$$\int_{I(0,2)} \left|w_n^+\right|^2 dt \geq \frac{\delta}{2}, \ \forall n \in \mathbb{N} \tag{4.19}$$

and

$$f(w_n) \to c \textit{ and } \|f'(w_n)\| (1 + \|w_n\|) \to 0. \tag{4.20}$$

By Lemma 4.6, (w_n) is bounded. Hence, passing to a subsequence, we may assume that $w_n \rightharpoonup w_0$ in X^α, $w_n \to w_0$ in $L^2_{loc}(\mathbb{R})$ and $w_n \to w_0$ almost everywhere on $\mathbb{R}$. As in the proof of Theorem 3.1, we get $f'(w_0) = 0$ and

$$\int_{I(0,2)} \left|w_0^+\right|^2 dt \geq \frac{\delta}{2},$$

which implies that $w_0^+ \neq 0$ and $w_0 \in \mathcal{N}^-$. Hence $f(w_0) \geq m$. On the other hand, by using (4.20), (W_5) and Fatou's lemma, we have

$$\begin{aligned} m \geq c &= \lim_{n\to\infty} \left[f(w_n) - \frac{1}{2} f'(w_n) w_n\right] \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} \left[\frac{1}{2} \nabla W(t, w_n) \cdot w_n - W(t, w_n)\right] dt \\ &\geq \int_{\mathbb{R}} \lim_{n\to\infty} \left[\frac{1}{2} \nabla W(t, w_n) \cdot w_n - W(t, w_n)\right] dt \\ &\geq \int_{\mathbb{R}} \left[\frac{1}{2} \nabla W(t, w_0) \cdot w_0 - W(t, w_0)\right] dt \\ &= f(w_0) - \frac{1}{2} f'(w_0) w_0 = f(w_0). \end{aligned}$$

This shows that $m \geq f(w_0)$ and so $f(w_0) = m = \inf_{\mathcal{N}^-} f$. The proof of Theorem 4.1 is completed.

5 Examples

In this Section, we give some examples for our results.

Example 5.1. Let

$$W(t, x) = \cos^2 t \, |x|^2 \ln(1 + |x|), \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

It is easy to check assumptions $(W_1)-(W_3)$. It remains to prove (W_4). We have

$$\lim_{|x|\to 0}\frac{|\nabla W(t,x)|}{|x|}\le\lim_{|x|\to 0}\Big[2\ln(1+|x|)+\frac{|x|}{1+|x|}\Big]=0.$$

So, there exists a positive constant r_0 such that

$$|x|\le r_0\to\frac{|\nabla W(t,x)|}{|x|}<\frac{1}{4\eta_2^2},\ \forall x\in\mathbb{R}.$$

By contra-position, we get

$$\forall x\in\mathbb{R}^N,\ \frac{|\nabla W(t,x)|}{|x|}\ge\frac{1}{4\eta_2^2}\to|x|\ge r_0.$$

On the other hand, we have

$$\widetilde{W}(t,x)=\frac{1}{2}cos^2t\frac{|x|^3}{1+|x|}$$

and then

$$\frac{|\nabla W(t,x)|}{\widetilde{W}(t,x)\,|x|}=2\frac{2(1+|x|)ln(1+|x|)+|x|}{|x|^3}.$$

Set

$$g_0(s)=\frac{s^2}{4ln(1+s)},\ s\ge r_0,$$

we get

$$\lim_{|x|\to\infty}\frac{|\nabla W(t,x)|}{\widetilde{W}(t,x)\,|x|}g_0(|x|)=1$$

hence $(t,x)\mapsto\frac{|\nabla W(t,x)|}{\widetilde{W}(t,x)|x|}g_0(|x|)$ is bounded on $\mathbb{R}\times\{x\in\mathbb{R}^N/\,|x|\ge r_0\}$ and achieves its maximum. Let

$$M_0=\max_{t\in\mathbb{R},|x|\ge r_0}\frac{|\nabla W(t,x)|}{\widetilde{W}(t,x)\,|x|}g_0(|x|)\text{ and }g(s)=\frac{g_0(s)}{M_0}.$$

Then, we have

$$|\nabla W(t,x)|\le\frac{|x|}{g(|x|)}\widetilde{W}(t,x).$$

Therefore $W(t,x)$ satisfies the condition (W_4).

Example 5.2. Let

$$W(t,x)=\theta(t)\Big[\,|x|^{\frac{13}{4}}-|x|^{\frac{11}{4}}+|x|^{\frac{9}{4}}\,\Big]$$

where $\theta\in C(\mathbb{R},\mathbb{R}^+)$ is periodic. It is easy to see that W satisfies $(W_1)-(W_3)$. Let us prove condition (W_4). We have

$$\nabla W(t,x)=\frac{\theta(t)}{4}\Big[13\,|x|-11\,|x|^{\frac{1}{2}}+1\Big]x$$

and

$$\widetilde{W}(t,x)=\frac{\theta(t)}{8}\Big[5\,|x|-3\,|x|^{\frac{1}{2}}+1\Big]\,|x|^{\frac{9}{4}}$$

so

$$\frac{|\nabla W(t,x)|}{\widetilde{W}(t,x)}=2\frac{13\,|x|-11\,|x|^{\frac{1}{2}}+1}{5\,|x|-3\,|x|^{\frac{1}{2}}+1}\,|x|^{-1}.$$

Set $M_0=2\sup_{t\in\mathbb{R}^+}\frac{13t^2-11t+1}{5t^2-3t+1}$, then we get

$$\frac{|\nabla W(t,x)|}{\widetilde{W}(t,x)}\le\frac{|x|}{g(|x|)}$$

where $g(s) = \frac{s^2}{M_0}, t \in \mathbb{R}^+$. Therefore

$$|\nabla W(t,x)| \leq \frac{|x|}{g(|x|)} \widetilde{W}(t,x) \text{ for all } x \neq 0$$

with $g \in C(\mathbb{R}^*_+, \mathbb{R}^*_+)$. Therefore W satisfies (W_4).

Example 5.3. Let

$$W(t,x) = a(t)\Big[\,|x|^p + (p-2)\,|x|^{p-\epsilon}\sin^2\big(\frac{|x|^\epsilon}{\epsilon}\big)\Big],$$

where $p > 2$, $a \in C(\mathbb{R}, \mathbb{R}^*_+)$ is periodic and $0 < \epsilon < p - 2$. It is easy to check that W satisfies $(W_1) - (W_3)$ and (W_5).

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