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Research Article

Theoretical Models For Composite Beams, Plates, Sandwich and FGM: a Review of ESL, Layerwise and CUF Approaches

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ABSTRACT

This literature review analyses the latest modelling frameworks for composite structures and functionally gradient materials, emphasizing Equivalent Single-Layer (ESL) theories, layer-by-layer formulations (layerwise) and the Carrera Unified Formulation (CUF) as well as their applications to beams, plates, sandwich structures, and materials with functional gradients (FGM). Our aim is to clarify the modelling trade-offs that determine the theory choice based on the structure's thickness, heterogeneity through thickness, and complexity of multilayer stacks. ESL approaches, ranging from classical theory to first- and higher-order shear models, are distinguished by their low computational cost and ability to conduct large-scale analyses, but often require enriched kinematics or specific corrections to ensure sufficient accuracy in the case of thick structures, marked gradients, or interlaminar effects. Layerwise models, which include discrete and mixed formulations, offer a more precise description of fields across thickness and interfaces, but come with an increased number of degrees of freedom enabling them to analyse sandwich and FGM structures that are susceptible to delamination. The CUF is analysed as a unifying framework that allows for systematic priority of kinematics and controlled adjustment of the trade-off between accuracy and cost, with a goal of convergence towards predictions close to three-dimensional elasticity. Based on the identified works, a comparative synthesis is proposed in terms of accuracy, numerical robustness, and computational cost for the analysis of flexure, vibration, and buckling, as well as for some nonlinear and multiphysical extensions.

Nomenclature:

Symbol	Description
u, v, w :	Displacement components
u0, v0, w0 :	Mid-plane displacements
h :	Total thickness of the structure
f(z) :	Shear or shape function through the thickness
N :	Total number of layers
tk :	Thickness of the (k)-th layer
zk :	Position of the interface of the (k)-th layer
L :	Characteristic length (beam or plate)
σ:	Stress
ε :	Strain
ϕx, ϕy :	Rotations of the transverse normal about the (y)- and (x)-axes, respectively
Ψx, Ψy, Ψz :	Thickness-dependent expansion functions used to approximate displacement fields
ξ:	Normalized thickness coordinate
hc, hf :	Thickness of the core and each face sheet of a sandwich plate
nu, nl :	Number of upper and lower layers, respectively

Abbreviation	Meaning
ESL :	Equivalent Single Layer
CPT :	Classical Plate Theory
FSDT :	First-order Shear Deformation Theory
HSDT :	Higher-order Shear Deformation Theory
LW :	Layerwise
DLWM :	Discrete Layerwise Model
ILWM :	Independent / Integral Layerwise Model
CUF :	Carrera Unified Formulation
FGM :	Functionally Graded Material
FEM :	Finite Element Method
IGA :	Isogeometric Analysis

1. Introduction:

Composite laminated, sandwich, and functional gradient materials (FGM) structures have become a central component of many advanced industrial sectors, such as aeronautics, automotive, energy, civil engineering, and advanced mechanical systems. Their high resistance/mass ratio, high specific rigidity, ability to adapt mechanical and thermal properties, and custom design possibilities are the main reasons for their increasing adoption.

Sandwich structures enable both overall rigidity and energy absorption to be optimized simultaneously, while FGM ensures a gradual transition of properties across the thickness, which reduces stress concentrations and improves durability under severe thermomechanical conditions.

Their use is no longer restricted to aircraft fuselage parts, but they are integrated even in high-performance parts. Ouabdou et al.[1] presented the different uses of composite materials in Table 1.

Table 1. Applications of Composite Materials

Material Type	Key Characteristics	Applications
Carbon Fiber-Reinforced Polymer (CFRP)	Lightweight, high tensile and compressive strength, thermal resistance	Aerospace, automotive, sports equipment
Glass Fiber-Reinforced Polymer (GFRP)	Cost-effective, good impact resistance, and chemical stability	Construction, piping, automotive
Natural Fiber Composites	Renewable, biodegradable, moderate strength	Automotive interiors, construction panels
Hybrid Composites	Tailored properties by combining different fibers	Military, transport, renewable energy

In real industrial situations, these structures are often subject to complex stresses that integrate mechanical, thermal, dynamic, and tribological loads. There have been several studies on the thermo-mechanical, vibratory, and contact analysis of laminated and sandwich components, particularly in critical systems such as braking devices and structures subject to friction, where stress, temperature, and contact pressure distributions play a determining role in performance and reliability. Belhocine et Abdullah [2] [3] Belhocine et Ghazaly [4] [5] Belhocine et Omar [6] and Belhocine et al. [7] highlighted the importance of advanced models capable of accurately representing the effects of transverse shear, interlaminar interactions, and material differences associated with these structures.

In order to model the mechanical behaviour of these structures, different methods have been developed, which stand out for their complexity, precision, and range of validity. Among these methods, Equivalent Single Layer (ESL) methods are widely used by assimilating a multilayer beam to an equivalent monolayer beam. These methods simplify the analysis and reduce the computational cost significantly.

There are three main ESL approaches that can be identified: The Classical Plate Theory (CPT) suggests that a flat section perpendicular to the neutral line remains flat and perpendicular after deformation, regardless of transverse shear effects. However, this assumption is too simplistic for low-slanted beams. The first-order Shear Deformation Theory (FSDT) takes into consideration transverse shear, assuming that plane sections remain flat without remaining perpendicular to the neutral axis after deformation. This theory includes correction coefficients to satisfy the conditions on free surfaces.

In order to obtain a more precise modelling, especially for thick beams, the Higher-order Shear Deformation Theory (HSDT) introduces displacement fields with quadratic distribution, which allows a better representation of transverse deformations.

However, the ESL approximation is limited in the case of highly heterogeneous anisotropic composite materials, where global modelling cannot accurately represent the behaviour of the different layers, that's why it is interesting to use methods called Layerwise, which use ESL models for each layer individually, while ensuring continuity of stress and displacements at the interfaces, among these, we can distinguish between layer-by-layer methods, whose complexity increases with the number of layers, and zigzag methods, which are more efficient because they do not depend directly on the number of layers.

3D elasticity methods provide a complete and very accurate representation of mechanical behaviour, although they are mathematically complex and have high computational costs, limiting them to validation studies or specific cases.

In this situation, it's important to consider the trade-off between precision, computational cost and the nature of the composite laminate when choosing the most appropriate modelling method. A summary of the different deformation theories applied to composite laminates is presented in Figure 1.

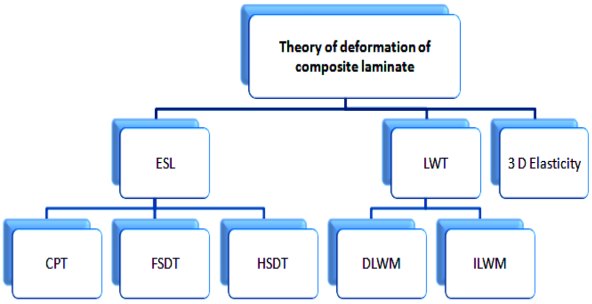

Fig. 1. Schematic classification of plate and beam modelling approaches.

This paper provides a thorough study of recent advancements in Equivalent Single-Layer (ESL), layerwise, and Carrera Unified Formulation (CUF). The review looks at the theoretical foundations and kinematic assumptions employed in analysing plates and beams, with a particular focus on laminated composites, sandwich structures, and functionally graded materials (FGMs)

2. Equivalent Single Layer (ESL) Methods Review:

The study of thin composite beams and plates is done using CPT, FSDT, or simplified HSDT methods.

The CPT method is more adapted to thin plate since it neglects the transverse shear and normal stress, following Z, i.e. $\gamma_{xz}=0$; $\gamma_{yz}=0$ and $\sigma_z=0$. The displacement is given by the equations (1)

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) - Z\frac{\partial w_0}{\partial x} \\ v(x,y,z,t) &= v_0(x,y,t) - Z\frac{\partial w_0}{\partial x} \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \qquad (1)$$

Unlike CPT, the First-Order Shear Deformation Theory (FSDT) takes into consideration transverse shear stresses. According to this theory, the displacement field and transverse stresses remain constant over the thickness which is not true in reality. This implies the introduction of a correction factor. The displacement is given by the equations (2)

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + Z.\phi_x \\ v(x,y,z,t) &= v_0(x,y,t) + Z.\phi_y \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \qquad (2)$$

Theories of elasticity show that the field of displacement is not actually linear but quadratic. This has led to the development of HSDT theory for thick beams where transverse deformations cannot be neglected. The displacement equations are giving by Reddy and Robbins [8] in equation (3)

$$\begin{aligned} u(x,y,z,t) &= u_0 + z.\phi_x + z^2.\Psi_x + z^3.\theta_x \\ v(x,y,z,t) &= v_0 + z.\phi_y + z^2.\Psi_y + z^3.\theta_y \\ w(x,y,z,t) &= w_0 + z.\phi_z + z^2.\Psi_z \end{aligned} \qquad (3)$$

where u_0; v_0; w_0; ϕ_x; ϕ_y; ϕ_z; Ψ_x; Ψ_y; Ψ_z; θ_x; θ_y; θ_z; are unknown function of position and time. Since the displacement vectors are quadratic, they cancel on the top and bottom of a laminate. The HSDT method can be used for thick plates without a correction factor.

Wang et al. [9] give the different configurations of beam deformation for ESL theories in Figure 2

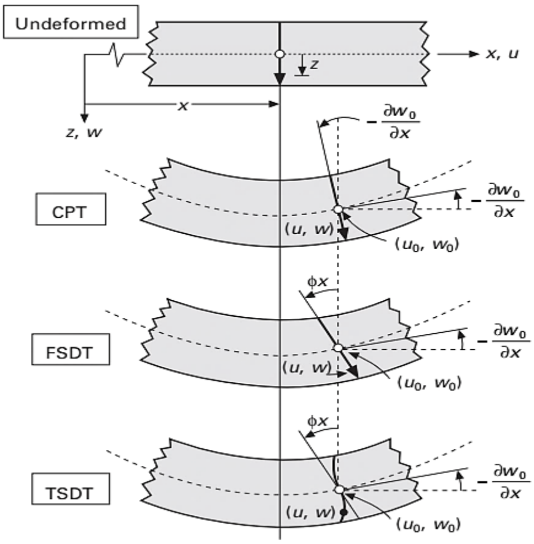

Fig. 2. Schematic representation of through-thickness displacement field configurations for classical plate theory (CPT), first-order shear deformation theory (FSDT), and higher-order shear deformation theories (HSDT). [9]

2.1. Laminate Composite

2.1.1. Laminate Beams and Plates:

The researchers examined show a gradual evolution of beam and plates theories, from simplified kinematic formulations to unified, multiphysical, and numerically robust models.

Sayyad et al. (2015) [11] and Sayyad et al. (2016) [12] have consistently contributed to the development of accurate kinematic models for thermo-mechanical, vibratory, and stability of laminated composite plates by introducing theories with a limited number of variables capable of accurately describing the effects of transverse shear.

In continuation of this approach, Ghugal and Gajbhiye (2016)[13] extend the analysis to thick isotropic plates, highlighting the interest of high-order theories in properly capturing the flexural response when classical assumptions become ineffective. Shimpi et al. (2017)[14] illustrate the transition from plates to one-dimensional structures by proposing a single variable formulation for rectangular beams, with the aim of reconciling mathematical simplicity and mechanical precision.

In parallel, Nguyen et al. (2020)[15] highlight a unification approach by developing a complex model for laminated composite beams, emphasizing kinematic coherence between different shear theories. The study by Benbouras et al. (2022)[16] supports these theoretical developments experimentally by analysing the non-linear behaviour of a carbon/epoxy composite subjected to a three-point bending, which provides elements for physical validation for analytical models.

Then, Ntaflos et al. (2024)[17] illustrate the extension towards functional structures by proposing a unified theory for piezoelectric beams that integrates geometric non-linearities, thus paving the way for smart structures. Finally, Benbakhti et al. (2024)[18] demonstrate the practical potential of these advanced approaches by using a real structural application in the field of civil engineering highlighting the gradual transition from advanced theories in the academic field to concrete engineering issues.

Table 2. Review of ESL applied to Laminate Beams and plates

Ref.	Studied structure	Equation	f(z)	Thin / Thick	Dominant method	Methodological justification
Sayyad et al. (2015) **Sayyad et al. (2016)**	**Laminated composite plates**	$u(x,y,z)=-z.\frac{\partial w_b}{\partial x}+f(z).\frac{\partial w_s}{\partial x}$ $v(x,y,z)=-z.\frac{\partial w_b}{\partial y}+f(z).\frac{\partial w_s}{\partial y}$ $w(x,y,z)=w_b(x,y,t)+w_s(x,y,t)$	$f(z)=-z[\frac{5}{3}(\frac{z}{h})^2-\frac{1}{4}]$ **Sayyad et al.**	**Thin to moderately thick**	**HSDT (4 variables)**	**Transverse shear effects included without normal deformation.** **Balanced accuracy–cost formulation for bending, vibration, and buckling**
Ghugal & Gajbhiye (2016)	**Isotropic plates**	$U(x,y,z)=u_0-z.\frac{\partial w}{\partial x}+f_1(z).\phi_x(x,y)+f_2(z).\Psi_x(x,y)$ $V(x,y,z)=v_0-z.\frac{\partial w}{\partial y}+f_1(z).\phi_y(x,y)+f_2(z).\Psi_y(x,y)$ $W(x,y,z)=w(x,y)+g_1(z).\phi_z(x,y)+g_2(z).\Psi_z(x,y)$	• $f_1(z)=z\left[1-\frac{4z^2}{3h^2}\right]$ $f_2(z)=z\left[1-\frac{16z^4}{5h^4}\right]$ • $g_1(z)=\left[1-4\frac{z^2}{h^2}\right]$ $g_2(z)=\left[1-16\frac{z^4}{h^4}\right]$	**Thick**	**Fifth-order HSDT**	**Dominant transverse shear effects in thick plates**
Shimpi et al. (2017)	**Rectangular beam**	$u=-z\frac{\partial w_b}{\partial x}+f(z).\frac{\partial w_s}{\partial x}$ $w=w_b(x,t)+w_s(x,t)$	$f(z)=h[\frac{1}{4}\left(\frac{z}{h}\right)-\frac{5}{3}(\frac{z}{h})^3]$	**Thin**	**Single-variable SDT**	**Negligible shear deformation; minimal computational cost**
Nguyen et al. (2020)	**Laminated composite beams**	$u(x,y,z)=u_0-z.\frac{\partial w_0}{\partial x}+C_f f(z).\frac{\partial}{\partial x}\Delta w_0$ $v(x,y,z)=v_0-z.\frac{\partial w_0}{\partial y}+C_f f(z).\frac{\partial}{\partial y}\Delta w_0$ $w(x,y,z)=w_0+C_f w_0$	• $f(z)=0$ ***Reissner and Mindlin*** • $f(z)=\frac{-4z^3}{3h^2}$ ***Reddy*** [19] • $f(z)=\frac{-5z^3}{3h^2}+\frac{z}{4}$ **Shimpi** [14] • $f(z)=\frac{-z}{8}-\frac{2z^3}{h^2}+\frac{2z^5}{h^4}$ **Nguyen-Xuan** [20] • $f(z)=-9z+\frac{10z^3}{h^2}+\frac{6z}{5h^4}^{5}+\frac{8z^7}{7h^6}$ **Nguyen et al.** [21] • $f(z)=-\frac{17z^3}{10h^2}+\frac{22z^5}{25h^4}$ **Nguyen et al.** [22] • $f(z)=tan^{-1}\left(\sin\left(\frac{\pi z}{h}\right)\right)-z$ ***Thai*** **et al.** [23]	**Thin to moderately thick**	**Unified beam model (SDT–HSDT)**	**Single framework adaptable to different kinematic assumptions**
Benbouras et al. (2022)	**Carbon/epoxy composite**	$u(x,y,z)=u_0(x,y)-z.\frac{\partial w_0}{\partial x}$ $v(x,y,z)=v_0(x,y)-z.\frac{\partial w_0}{\partial y}$ $w(x,y,z)=w_0(x,y)$		**Thick specimen**	**Analytical–experimental approach**	**Experimental validation beyond purely kinematic modeling**
Ntaflos et al. (2024)	**Piezoelectric beams**	$u(x,y,z,t)=u_0(x,t)-z.\frac{\partial w_0(x,t)}{\partial x}+f(z).\varphi(x,t)$ $v(x,y,z,t)=0$ $w(x,y,z,t)=w_0(x,t)$ $\phi(x,y,z,t)=g(z)\phi_0(x,t)$	• $f(z)=0$ **Euler Bernouilli** • $f(z)=z$ **Timoshenko** • $f(z)=\frac{zh^2}{8}-\frac{z^3}{6}$ **Ambartsumian** • $f(z)=\frac{5z}{4}-\frac{5z^3}{3h^2}$ **Kruszewski** [24] • $f(z)=z-\frac{4}{3}\left(\frac{z^3}{h^2}\right)$ **Reddy** [25] • $f(z)=\frac{h}{\pi}\sin(\frac{\pi z}{h})$ **Touratier** [26] • $f(z)=-h\sinh\left(\frac{z}{h}\right)+z\cosh\left(\frac{1}{2}\right)$ **Soldatos** [27] • $f(z)=z\exp[-2\left(\frac{z}{h}\right)^2]$ **Karama** [28] • $g(z)=-\frac{z}{h}$ **Goldschmidtboeing** [29] • $g(z)=\frac{2z}{h}$ **Komeili** [30] • $g(z)=\frac{h}{\pi}\cos\left(\frac{\pi z}{h}\right)$ **Fernandes et Pouget** [31] • $g(z)=1-(\frac{2z}{h})^2$ **Wang et al.**[32] • $g(z)=\cos\left(\frac{\pi z}{h}\right)$ **Baroudi et al.** [33]	**Thin to moderately thick**	**Unified electromechanical SDT/HSDT**	**Coupled geometric and electromechanical effects**
Benbakhti et al. (2024)	**Real composite structural element**	$u(x,y,z)=u_0(x,y)-z.\frac{\partial w_b}{\partial x}-f(z).\frac{\partial w_s}{\partial x}$	$f(z)=sinh^{-1}\left(\frac{3z}{h}\right)-z\frac{6}{h\sqrt{3}}$	**Thick**	**Analytical structural model**	**Engineering-oriented application rather than theory refinement**

$$v(x,y,z) = v_0(x,y) - z.\frac{\partial w_b}{\partial y} - f(z).\frac{\partial w_s}{\partial y}$$

$$w(x,y,z) = w_b(x,y) + w_s(x,y)$$

2.1.2. Sandwich Structures:

The works of Naik and Sayyad (2018) [34] propose a unified analysis of laminated and sandwich plates, suggesting a high-order theory with only five independent kinematic unknowns capable of coherently reproducing the mechanical behavior of multilayer structures, while maintaining a relatively compact formulation. Thai et al. (2014) [35] have developed a method for functionally gradient material sandwich plates, which explicitly integrates the continuous variation of mechanical properties through thickness. This allows for a more accurate representation of transitions between layers and better anticipation of static and dynamic responses. Li et al. (2019) [36] then developed a functional gradient sandwich beam model to move from plates to one-dimensional structures, highlighting the effect of coupling between material gradation and shear deformation on the accuracy of the analysis. Four independent functions are used to describe the beam's displacement field: axial displacement, transverse displacement, and variables associated with rotation/shear. This work is distinguished by the integration of additional mixed variables (in connection with shear forces) into the formulation of finite elements to improve accuracy without increasing the number of basic kinematic unknowns.

Sayyad and Naik (2019) [37] reinforce the conceptual coherence between these different formulations by proposing a generalized displacement model applicable to laminated and sandwich plates with an emphasis on continuity and precise evaluation of transverse shear stresses. Yaghoobi and Taheri (2020) [38] then examine the introduction of material defects in porous core-reinforced sandwich plates, demonstrating that porosity and nano-structural reinforcement have a significant impact on buckling capacity and statistical dispersion of mechanical responses. As an extension of this work, Zenkour and Alghanmi (2022) [39] combine the effects of functional gradation and porosity in sandwich plates with elastic foundations highlighting the importance of quasi-3D to accurately capture the relationship between porous hearts and functionally graded skins. Finally, Belkhodja et al. (2023) [40] have advanced HSDT models for hard or soft core FGM sandwich plates, which integrate thermo-mechanical effects. It is an important starting point in the advanced modeling of sandwich structures, bringing together hardware complexity, diversity of core configurations and multiphysical solicitations within a unified theoretical framework.

Table 3. Review of ESL applied to Sandwich structures

Ref.	Studied structure	Equation	f(z)	Thin / Thick	Dominant method	Number of unknowns	Methodological justification
Naik & Sayyad (2018)	Laminated and sandwich plates	$u(x,z)=u_0(x)-z.\frac{dw_0}{dx}+\left[z-\frac{4z^3}{3h^2}\right]\phi_x(x)+\left[z-\frac{16z^5}{5h^4}\right].\Psi_x(x)$ $w(x,z)=w_0(x)+\left[1-\frac{4z^2}{h^2}\right]\phi_z(x)+\left[1-\frac{16z^4}{h^4}\right].\Psi_z(x)$	ϕ_x and ψ_x are rotations of the normal to the middle plane about y axis ϕ_z and ψ_z represent higher-order transverse cross-sectional deformation	Moderately thick	Fifth-order HSDT	5	Higher-order shear deformation required to accurately capture transverse shear effects in sandwich configurations
Thai et al. (2014)	Functionally graded isotropic and sandwich plates	$u(x,y,z)=u_0(x,y)-z.\frac{dw}{dx}+f(z)\beta_x(x,y)$ $v(x,y,z)=v_0(x,y)-z.\frac{dw}{dy}+f(z)\beta_y(x,y)$ $w(x,y,z)=w(x,y)$	• $f(z)=tan^{-1}\left(\sin\left(\frac{\pi}{h}z\right)\right)$ • $f(z)=sinh^{-1}\left(\sin\left(\frac{\pi}{h}z\right)\right)$	Moderately thick to thick	Generalized SDT + Isogeometric Analysis (IGA)	5-6	Generalized kinematics combined with IGA for accurate shear representation in FG and sandwich plates
Li et al. (2019)	Functionally graded sandwich beams	$u_x(x,y)=u(x)-y.\frac{dw(x)}{dx}+f(y)\left[\frac{dw(x)}{dx}-\theta(x)\right]$ $u_x(x,y)=w(x)$	• f(y)=0 Classical beam theory • f(y)=y First-order beam theory (Timoshenko model) • $f(y)=y\left(1-\frac{4y^2}{3h^2}\right)$ Reddy [19] • $f(y)=\frac{h}{\pi}\sin\left(\frac{\pi y}{h}\right)$ Touratier [26] • $f(y)=h\sinh\left(\frac{y}{h}\right)-y\cosh\left(\frac{1}{2}\right)$ Soldatos [27] • $f(y)=y\exp[-2\left(\frac{y}{h}\right)^2]$ Karama [28] • $f(y)=y.3.\exp[\frac{-2(^y/_h)^2}{\ln(3)}]$ Aydogdu [41]	Moderately thick to thick	Mixed higher-order shear deformable beam element (HSDT-based FEM)	≥5	Mixed formulation improves accuracy of transverse shear stress prediction in FG sandwich beams
Yaghoobi & Taheri (2020)	Sandwich plates with porous core reinforced by GPL	$u(x,y,z)=u_0(x,y)-z.\frac{\partial w_b}{\partial x}-f(z).\frac{\partial w_s}{\partial x}$ $v(x,y,z)=v_0(x,y)-z.\frac{\partial w_b}{\partial y}-f(z).\frac{\partial w_s}{\partial y}$ $w(x,y,z)=w_b(x,y)+w_s(x,y)$	$f(z)=\frac{4z^3}{3h^2}$	Thick	Analytical HSDT-based buckling model	≥5	Porous core and GPL reinforcement demand refined shear modeling for buckling capacity prediction
Zenkour & Alghanmi (2022)	FG porous sandwich plates on elastic foundations	$u_1(x,y,z)=u(x,y)-z.\frac{\partial w}{\partial x}-f(z).\phi_1(x,y)$ $v(x,y,z)=v(x,y)-z.\frac{\partial w}{\partial y}-f(z).\phi_2(x,y)$ $w(x,y,z)=w(x,y)+f'(z).\phi_3(x,y)$	$f(z)=z\left(1-\frac{4}{3}\frac{z^2}{h^2}\right)$	Thick	Refined quasi-3D theory	≥6	Inclusion of transverse normal deformation essential for porous sandwich plates and foundation effects
Belkhodja et al. (2023)	FGM sandwich plates with hard/soft core (symmetric & asymmetric)	$u(x,y,z)=u_0(x,y)-z.\frac{\partial w_b}{\partial x}+f(z).\theta_x$ $v(x,y,z)=v_0(x,y)-z.\frac{\partial w_b}{\partial y}+f(z).\theta_y$ $w(x,y,z)=w_0(x,y)$	$f(z)=\frac{h}{2}\sin\left(\frac{2z}{h}\right)-\frac{4}{3}\frac{z^3}{h^2}\cos(1)$	Moderately thick to thick	Quasi-3D / advanced HSDT (2D–3D)	≥6	Thermo-mechanical loading and core heterogeneity require quasi-3D kinematics

2.1.3. Functional Gradient Materials (FGM)

A. Functionally Graded Materials: Classic Configurations

Vu et al. (2018) **[42]** then elaborate a refined formulation of the TSDT type, associated with a meshless method to minimize unknowns while maintaining good prediction capability for FGM plates. Sayyad and Ghugal (2018) **[43]** illustrate how the concepts of high-order theories apply to FGM beams using an exponential gradation law by offering analytical solutions for evaluating bending, buckling, and vibrations, we use these examples as a methodological basis for the plate models. To continue with two-dimensional plates, Li et al. (2020) **[44]** proposed a new shear strain theory specifically for FGM plates focusing on improving the description of transverse shear, without resorting to correction factors. Belkhodja et al. (2020) **[45]** enriched this approach by introducing an HSDT formulation based on exponential and trigonometric functions, offering greater flexibility in the modelling of displacement fields, and significantly improving bending accuracy, vibration, and buckling. In the work of Vinh et al. (2021) **[46]**, we continue this trend towards models with a limited number of variables by proposing a single-variable theory for the vibratory analysis of FGM plates, clearly focusing on the compromise between simplicity of the model and precision of the results. More recently, Yadav et al. (2023) **[47]** addressed, at the same time the effects of shear and normal deformation using a sinusoidal theory, highlighting the importance of transverse stretching in the mechanical response of FGM plates. According to Hoang and Thanh (2024) **[48]**, these developments continue to include FGM plates with variable thickness in two directions, thus showing that the combination of material gradation and geometric variation imposes higher kinematic requirements, which confirms the need for refined theories to properly capture the dynamic behaviour of these advanced structures.

B. Enhanced FGM (Nano-Enhanced, CNT, GPL, Nano Effects)

According to Nguyen-Quoc et al. (2018) [49], the modelling of FG-CNTRC is based on the combination of a high-order shear theory and a micromechanical description of the carbon nanotube reinforcement, highlighting the noticeable improvement in stiffness and static response compared to conventional FGM plates. Arefi and Soltan Arani (2018) [50] examine the issue of FGM nanobeams in a one-dimensional framework but more strongly connected on the multiphysical level. they simultaneously integrate thermal, electrical and magnetic effects and highlight the crucial importance of the nano scale and coupled fields in the evaluation of bending behaviour. With the aim of improving geometric and material accuracy, Wang et al. (2021) [10] propose a new HSDT specially designed for two-dimensional gradation FGM nanoplate by highlighting that changes in two directions amplify scale effects and reinforce the importance of an enriched kinematics. Nguyen et al. (2020) [51] mainly focus on the development of a three-variable high shear theory, integrated in an isogeometric framework, to study free vibrations and buckling, and instabilities of LPG reinforced porous FGM plates. Their contribution clearly highlights the competing role of porosity, which tends to reduce overall stiffness, and LPG reinforcement, which improves mechanical performance and highlights the effectiveness of a formulation with a limited number of unknowns in capturing these effects. In comparison, Wang et al. (2024) [52] adopt a more detailed plate theory, which focuses not only on vibration and buckling but also on static flexion, focusing particularly on the precise description of transverse shear and on the robustness of the model with respect to a variety of material parameters. Belarbi et al. (2023) [53] refocused their analysis on the FG-CNT beams using a hyperbolic shear theory that allows accurately capturing the impact of the CNT reinforcement on thermo-mechanical behaviour while maintaining a fairly compact formulation.

Considering these works together, we can observe a consistent change from classical FG-CNT plates to more complex nano-reinforced FGM structures, which integrate porosity, multidirectional gradation, and multiphysical effects. This confirms that determining the choice of shear deformation theory becomes crucial as hardware complexity and analysis scale increase.

C. Porous FGM

The research of Shahsavari et al. (2018) [54] aims to analyse porous FGM plates by proposing a quasi-hyperbolic 3D theory capable of considering both the effects of porosity and the effect of elastic foundations on the vibratory response**.** Phung-Van et al. (2018) [55] extend this issue to porous FGM nanoplates by using scale effects and an isogeometric formulation to make it more refined at the nano-scale, which allows a more precise description of static and dynamic behaviours is possible when dimension characteristics become similar to the microstructure.

Slimani et al. (2024) [56] continue to focus on improving kinematic models and have developed a quasi-Refined 3D HSDT type for static bending analysis of porous FGM plates, highlighting the importance of clearly considering transverse stretching and improving accuracy compared to classical HSDT models.

2.1.4. Curved Structures:

The study by Guo et al. [57] addresses the numerical topic of curved laminated composite beams using a domain decomposition method with an emphasis on managing general elastic boundary conditions and static and dynamic analysis, without proposing a new beam kinematics. in contrast, the research of Belarbi et al. [58] and Avhad & Sayyad [59] specifically target functionally graded sandwich beams, whose main objective is to reduce the numerical cost by elaborating high-order theories applied to beams, with a limited number of unknowns, to apply them respectively to buckling and static deformation. Ansari & Kumar [60], Arefi et al. [61], and Tran et al. [62] extend the framework to curved surface structures of shell type, which exhibit complex geometries such as truncated cones and double-ended curved nanoshells and additional physical effects such as CNT reinforcement, thickness stretching, porosity, non-locality, and interaction with an elastic foundation.

Table 4. Review of ESL applied to FGM and curved structures

Functionally graded materials: Classic configurations

Ref.	Studied structure	Equation	f(z)	Thin / Thick	Dominant method	Number of unknowns	Methodological justification
Thai et al. (2014)	FG isotropic and sandwich plates	$u_1(x,y,z) = u(x,y) - z.\frac{\partial w}{\partial x} - f(z).\phi_1(x,y)$ $v(x,y,z) = v(x,y) - z.\frac{\partial w}{\partial y} - f(z).\phi_2(x,y)$ $w(x,y,z) = w(x,y) + f'(z).\phi_3(x,y)$	$f(z) = h.\sinh\left(\frac{z}{h}\right) - z\cosh(\frac{1}{2})$	Moderately thick → thick	Generalized SDT + IGA	5–6	Generalized shear kinematics required for FG and sandwich plates; IGA ensures smooth higher-order fields
Vu et al. (2018)	Through-thickness FG plates	$u(x,y,z) = u_0(x,y) - z.\frac{\partial w_b}{\partial x} - f(z).\frac{\partial w_s}{\partial x}$ $v(x,y,z) = v_0(x,y) - z.\frac{\partial w_b}{\partial y} - f(z).\frac{\partial w_s}{\partial y}$ $w(x,y,z) = w_b(x,y) + w_s(x,y)$	$f(z) = z - \frac{5}{4}z + \frac{5z^3}{3h^2}$ Vu et al.	Thin → moderately thick	Refined TSDT + meshfree	4	Simplified higher-order kinematics with reduced computational cost
Sayyad & Ghugal (2018)	Exponential FGM beams	$u(x,z,t) = u_0(x,t) - z.\frac{\partial w_b(x,t)}{\partial x} - f(z).\frac{\partial w_s(x,t)}{\partial x}$ $w(x,t) = w_b(x,t) + w_s(x,t)$	w_b and w_s are the bending and shear components of transverse displacement of a point on the neutral axis of the beam. $f(z) = z.[1 - e^{2.(z/h)^2}]$	Thin → moderately thick	HSDT beam theory (analytical)	4–5	Shear deformation significant in graded beams
Li et al. (2020)	FGM plates	$u(x,y,z) = u_0(x,y) - z.\frac{\partial w_b}{\partial x} + f(z).\theta_x$ $v(x,y,z) = v_0(x,y) - z.\frac{\partial w_b}{\partial y} + f(z).\theta_y$ $w(x,y,z) = w_0(x,y)$	$f(z) = \frac{\pi m h^3 - e^{-\frac{mh}{\pi}z[\pi m h^3 \cos\left(\frac{\pi}{h}\right)z - \pi^3 h \sin\left(\frac{\pi}{h}\right)z]}}{\pi^4 + m^2 h^4}$	Thin → moderately thick	Novel refined SDT/HSDT	4–5	Improved shear strain distribution without full quasi-3D complexity
Belkhodja et al. (2020)	FGM plates	$u(x,y,z) = u_0(x,y) - z.\frac{\partial w_b}{\partial x} + f(z).\theta_x$ $v(x,y,z) = v_0(x,y) - z.\frac{\partial w_b}{\partial y} + f(z).\theta_y$ $w(x,y,z) = w_0(x,y)$	$f(z) = \frac{h}{2}\sin\left(\frac{2z}{h}\right) - \frac{4}{3}\frac{z^3}{h^2}\cos(1)$	Thin → moderately thick	Exponential–trigonometric HSDT	5	Enhanced shear deformation representation for graded material
Vinh et al. (2021)	Rectangular FGM plates	$u(x,y,z) = -z.\frac{\partial w_b}{\partial x} + f(z).\frac{\partial w_s}{\partial x}$ $v(x,y,z) = -z.\frac{\partial w_b}{\partial y} + f(z).\frac{\partial w_s}{\partial y}$ $w(x,y,z) = w_b(x,y,t) + w_s(x,y,t)$	$f(z) = \left(\frac{127z^4}{200h^3} - \frac{77z^2}{125h} + \frac{39h}{500}\right)\sin\left(\frac{\pi z}{h}\right)$	Thin	Single-variable SDT	1	Shear effects minimized; lowest computational cost
Yadav et al. (2023)	FGM plates	$U(x,y,z) = u_0 - z.w_{,x} + f(z).\phi(x,y)$ $V(x,y,z) = v_0 - z.w_{,y} + f(z).\Psi(x,y)$ $W(x,y,z) = w(x,y) + g(z)\xi(x,y)$	$f(z) = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)$ $g(z) = \frac{h}{\pi}\cos\left(\frac{\pi z}{h}\right)$	Moderately thick	Sinusoidal SDT with normal deformation	5–6	Inclusion of transverse normal strain improves accuracy
Hoang & Thanh (2024)	FGM plates with 2D variable thickness	$u_x(x,z) = u_0(x) - z.\frac{\partial w_0}{\partial x} + f(z).\varphi_x$ $u_z(x,z) = w_0(x)$	$f(z) = \frac{1}{4\pi}\mathrm{Sec}[\frac{1}{12}]^2(2\pi z\cos\left[\frac{1}{6}\right] + h\cos\left[\frac{z}{3h)}\right]\sin\left[\frac{2\pi z}{h}\right])$	Variable (thin → thick)	Trigonometric SDT	4–5	Variable thickness requires adaptable shear representation

Enhanced FGM (nano-enhanced, CNT, GPL, nano effects)

Nguyen-Quoc et al. (2018)	FG-CNTRC plates	$u(x,y,z) = u_0 + \left(z - \frac{4z^3}{3t^2}\right)\beta_x - \frac{4z^3}{3h^2}\phi_x$ $v(x,y,z) = v_0 + \left(z - \frac{4z^3}{3t^2}\right)\beta_y - \frac{4z^3}{3h^2}\phi_y$ $w(x,y,z) = w_0$	β_x and β_y are the rotations around y-axis and x-axis, respectively	Thin → moderately thick	C^0-HSDT	5	Higher-order shear deformation compatible with C^0 continuity
Arefi & Soltan Arani (2018)	Magneto-electro-thermo-	$U_1(x,z,t) = u(x,t) + z.\theta(x,t) - C.z^3(\frac{\partial w(x,t)}{\partial x} + \theta(x,t))$ $U_3(x,z,t) = w(x,t)$	$C = 4/3h^2$ θ is the total bending rotation of the cross section	Nano (thin)	Higher-order shear deformation theory (multi-physics)	≥5	Strong multi-field coupling dominates kinematic

	elastic FGM nanobeams						complexity at nanoscale
Wang et al. (2021)	2D-FG nanoplates	$u(x,y,z) = u_0(x,y) - z.\frac{\partial w}{\partial x} + f(z).\beta_x(x,y)$ $v(x,y,z) = v_0(x,y) - z.\frac{\partial w}{\partial y} + f(z).\beta_y(x,y)$ $w(x,y,z) = w_0(x,y)$	βx and βy are rotations of the plate. $f(z) = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\left(e^{\frac{1}{2}\cos\left(\frac{\pi z}{h}\right)} + 1\right) - z\left(\frac{1}{2} - \frac{4z^2}{3h^2}\right)$	Nano (thin)	Higher-order SDT (2D gradation)	≥5	Two-directional material gradation increases kinematic richness
Nguyen et al. (2020)	FG porous plates reinforced with GPL	$u(x,y,z) = u_0 - z.\frac{\partial w_0}{\partial x} + C_f f(z).\frac{\partial}{\partial x}\Delta w_0$ $v(x,y,z) = v_0 - z.\frac{\partial w_0}{\partial y} + C_f f(z).\frac{\partial}{\partial y}\Delta w_0$ $w(x,y,z) = w_0 + C_f w_0$	$f(z) = -9z + \frac{10z^3}{h^2} + \frac{6z}{5h^4}^5 + \frac{8z^7}{7h^6}$	Moderately thick	Three-variable HSDT + IGA	3	Reduced-variable formulation balancing accuracy and efficiency
Wang et al. (2024)	FG porous plates reinforced with GPL	$u(x,y,z) = u_0(x,y) - z.\frac{\partial w_b}{\partial x} - f(z).\frac{\partial w_s}{\partial x}$ $v(x,y,z) = v_0(x,y) - z.\frac{\partial w_b}{\partial y} - f(z).\frac{\partial w_s}{\partial y}$ $w(x,y,z) = w_b(x,y) + w_s(x,y)$	$f(z) = -\frac{z}{4} + \frac{5z^3}{3h^2}$	Moderately thick → thick	Simple refined SDT/HSDT	3–4	Reduced-order theory optimized for GPL-reinforced porous plates
Belarbi et al. (2023)	FG-CNT reinforced beams	$u_x(x,z) = u_0(x) - z.\frac{\partial w_0}{\partial x} + f(z).\varphi_x$ $u_z(x,z) = w_0(x)$	$f(z) = \frac{h.\tanh\left(\frac{2z}{h}\right) + 2z.(\tanh(1)^2 - 1)}{2\tanh(1)^2}$	Thin → moderately thick	Hyperbolic HSDT	5	CNT reinforcement amplifies shear deformation effects
Porous FGM							
Shahsavari et al. (2018)	FG porous plates on elastic foundations	$u(x,y,z,t) = u_0(x,y,t) - z.\frac{\partial w_b}{\partial x} - f(z).\frac{\partial w_s}{\partial x}$ $v(x,y,z,t) = v_0(x,y,t) - z.\frac{\partial w_b}{\partial y} - f(z).\frac{\partial w_s}{\partial y}$ $w(x,y,z,t) = w_b(x,y,t) + w_s(x,y,t) + w_{st}(x,y,z,t)$	$f(z) = -\left[r_1\left(\frac{z}{h}\right) + r_2\sinh\left(\frac{z}{h}\right)\right]h$ Where $r_1 = \frac{\cosh(\theta)}{24\sinh(\theta) - 11\cosh(\theta)} - 1$ $r_2 = \frac{-1}{24\sinh(\theta) - 11\cosh(\theta)}$ $\theta = \frac{1}{2}$ $g(z) = 1 - \frac{df(z)}{dz}$	Thick	Quasi-3D hyperbolic theory	≥6	Porosity and foundation effects require transverse normal deformation
Phung-Van et al. (2019)	Porous FG nanoplates	$u(x,y,z) = u_0(x,y) - z.\frac{\partial w}{\partial x} + f(z).\beta_x(x,y)$ $v(x,y,z) = v_0(x,y) - z.\frac{\partial w}{\partial y} + f(z).\beta_y(x,y)$ $w(x,y,z) = w_0(x,y)$	βx and βy are rotations of the plate. $f(z) = z - \frac{4}{3h^2}z^3$	Nano (thin)	Quasi-3D + IGA	≥6	Size effects and porosity necessitate 3D-like kinematics
Slimani et al. (2024)	Porous FGM plates	$u(x,y,z) = u_0(x,y) - z.\frac{\partial w_b}{\partial x} + f(z).\frac{\partial w_s}{\partial x}$ $v(x,y,z) = v_0(x,y) - z.\frac{\partial w_b}{\partial y} + f(z).\frac{\partial w_s}{\partial y}$ $w(x,y,z) = w_b(x,y) + g(z).w_s(x,y)$	$f(z) = \left(\frac{5}{4}\right).z - \left(\frac{5}{3}\right)\left(\frac{z^3}{h^2}\right)$; $g(z) = r.\frac{df}{dz}$ and $r = \left(\frac{2}{15}\right)$	Thick	Refined quasi-3D HSDT	≥6	Full shear and normal deformation modeling required
Curved structures							
Guo et al. (2019)	Curved laminated composite beam	$u_i^{(k)} = \left(1 + \frac{z}{R_i}\right)u_i(x,y,t) + z\theta_i(x,y,t) + \phi_i^{(k)}(z)\Psi_i(x,y,t)$ $\omega^{(k)} = \omega(x,y,t)$	$u_i^{(k)}$ and $\omega^{(k)}$ are the in−plane displacements θ_i is the average rotation angles of the transverse normal $\phi_i^{(k)}$ and Ψ_i are the piecewise linear zigzag functions and amplitude functions of the RZT	Thin to moderately thick	Domain Decomposition + FEM	Not central (standard beam DOFs)	Focus on numerical strategy (domain decomposition) and elastic boundary conditions rather than kinematic refinement
Belarbi et al. (2022)	FG curved sandwich beam	$u(x,z) = \left(1 + \frac{z}{R}\right)u_0(x) - z.\frac{\partial w_0}{\partial x} + f(z).\phi_x(x)$ $w(x,z) = w_0(x)$	$f(z) = z.\left(1 - \frac{3z^2}{2h^2} + \frac{2z^4}{5h^4}\right)$	Moderately thick to thick	FEM (refined beam element)	3 unknowns	Reduce computational cost while retaining

							shear accuracy in FG sandwich beams
Ansari & Kumar (2019)	FG CNT-reinforced doubly curved shell (truncated cone)	$u(x,y,z) = \left(1+\frac{z}{R_x}\right)u_0(x,y) + z.\theta_x(x,y) + z^2.\xi_x(x,y) + z^3.\zeta_x(x,y)$ $v(x,y,z) = \left(1+\frac{z}{R_y}\right)v_0(x,y) + z.\theta_y(x,y) + z^2.\xi_y(x,y) + z^3.\zeta_y(x,y)$ $w(x,y,z) = w_0(x)$	θ_x; θ_y the bending rotations defined at the mid- plane about the y and x axes ξ_x; ζ_x; ξ_y and ζ_y are higher order terms of Taylor's series expansion	Moderately thick	Analytical + numerical validation	5 unknowns (typical HSDT)	Capture curvature and CNT-FG effects with improved shear representation
Arefi et al. (2022)	FG doubly curved nanoshell	$\bar{u} = u - z.\frac{\partial w_b}{\partial x_1} - f(z).\frac{\partial w_s}{\partial x_1}$ $\bar{v} = v - z.\frac{\partial w_b}{\partial x_2} - f(z).\frac{\partial w_s}{\partial x_2}$ $\bar{w} = w_b + w_s + g(z).\chi$	$f(z) = z - \frac{h}{\pi.\sin\left(\frac{\pi z}{h}\right)}$ $g(z) = 1 - f'(z) = \cos\left(\frac{\pi z}{h}\right)$ w_b, w_s are bending and shear components of transverse displacements and χ is thickness stretching parameter.	Thick / nanoscale	Analytical (Navier-type / closed-form)	>5 unknowns (HSDT + stretching)	Account for transverse shear and normal deformation in FG nanoshells
Tran et al. (2021)	FG porous nanoshell on elastic foundation	$U_1(x,y,z,t) = \left(1+\frac{z}{R_1}\right)u_0(x,y,t) - z.\frac{\partial w^b}{\partial x} - f(z)\frac{\partial w^s}{\partial x}$ $U_2(x,y,z,t) = \left(1+\frac{z}{R_2}\right)v_0(x,y,t) - z.\frac{\partial w^b}{\partial y} - f(z)\frac{\partial w^s}{\partial y}$ $U_3(x,y,z,t) = w^b + w^s$	Four f(z) used: • Hyperbolic sine function follows Soldatos [27]: $f(z) = z - h\sinh\left(\frac{z}{h}\right) + z.\cosh\left(\frac{1}{2}\right)$ • Sinusoidal function follows Touratier [26]: $f(z) = z - \frac{h}{\pi}.\sin\left(\frac{\pi.z}{h}\right)$ • New sinusoidal function follows Mechab : $f(z) = z - \frac{z.\cos\left(\frac{1}{2}\right)}{\cos\left(\frac{1}{2}\right)-1} + \frac{h.\sin\left(\frac{z}{h}\right)}{\cos\left(\frac{1}{2}\right)-1}$ • New hyperbolic sin function follows Mechab [63] : $f(z) = \frac{2.z.\sinh\left(\frac{z^2}{h^2}\right)}{2.\sinh\left(\frac{1}{4}\right) + \cosh\left(\frac{1}{4}\right)}$	Thick / nanoscale	Analytical (nonlocal elasticity)	4 unknowns	Balance accuracy and efficiency while including nonlocality, porosity, and foundation effects
Avhad & Sayyad (2020)	FG sandwich beam curved in elevation	$u(x,z) = \left(1+\frac{z}{R}\right)u_0(x) - z.\frac{\partial w_0}{\partial x} + \left(z - \frac{4.z^3}{3.h^2}\right)\phi_x(x) + \left(z - \frac{16.z^5}{5.h^4}\right)\Psi_x(x)$ $w(x,z) = w_0(x) + \left(1 - \frac{4.z^2}{h^2}\right)\phi_z(x) + \left(1 - \frac{16.z^4}{h^4}\right)\Psi_z(x)$		Moderately thick to thick	Analytical + FEM	4–5 unknowns (HSDT-type)	Improve static response prediction of curved FG sandwich beams

3. Layerwise Methods Review:

Layerwise methods are advanced models that allow to accurately analyse the interlaminar effects and kinematic discontinuities between layers of stratified composite structures, insufficiently described by classical ESL approaches. These models can be classified into two categories: Integral Layerwise (ILWM) and Discrete Layerwise (DLWM) models. ILWMs rely on a single global displacement field, enhanced by layer-dependent functions (such as zigzag functions) that allow capturing abrupt slope variations at interfaces while maintaining a reduced number of unknowns and moderate numerical cost. On the other hand, DLWM integrates distinct displacement fields for each layer, which ensures a nearly exact description of mechanical fields across the thickness despite the significant increase in the number of unknowns and computational complexity.

Thus, ILWM offers an efficiency-precision compromise adapted to global analyses, while DLWM is preferred for the detailed study of local interlaminar stresses and delamination phenomena.

3.1. Discrete Layerwise Methods (DLWM):

The layer-by-layer kinematic description of Discrete Layerwise (DLWM) models allows an accurate representation of displacement fields and interlaminar stresses, but this has a high computational cost.

Van Do and Lee [64]suggest an isogeometric analysis of classical beams and laminated plates ensuring exceptional field continuity and high precision in free bending and vibration, specifically designed for thin to moderately thick laminated plates. Alipour and Shariyat [65]have analyzed the free vibrations of circular and annular sandwich plates with auxetic cores for isotropic and orthotropic faces as illustrated in figure 3. They used a multilayer theory, Hamilton’s principle to derive the equations and a Taylor transformation for resolution.

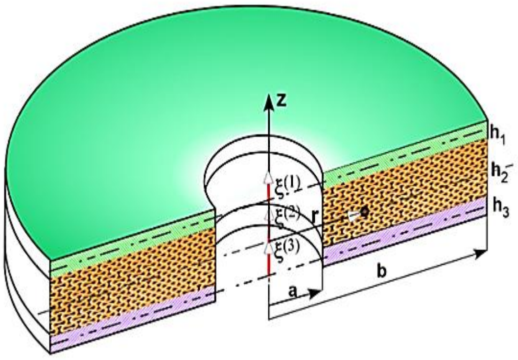

Fig. 3. Geometric configuration of a circular/annular composite sandwich plate with an auxetic core, showing the layer thicknesses, radial dimensions and the coordinate system through thickness used in Alipour and Sharyat Model [65]

Amabili and Reddy [66]use DLWM in the context of nonlinear mechanics by introducing a layerwise theory of third order in thickness capable of capturing both the effects of large deformations and transverse shear in thick sandwich plates. More recently, Gao et al. [67]have advanced a layerwise theory of the third compressible order, which explicitly ensures the continuity of transverse shear stresses between layers. This represents a significant improvement for the accurate analysis of core/skin interfaces. For functional gradient structures (FGM), Pandey and Pradyumna [68] apply a high layerwise theory to FGM sandwich plates demonstrating that the DLWM approach remains relevant to represent both continuous variation of material properties and interlaminar effects, when equivalent monolayer models are insufficient. Their study focused on two sandwich structures the first with FGM Core, homogeneous faces and the second with FGM Faces, homogeneous core as represented in figure 4

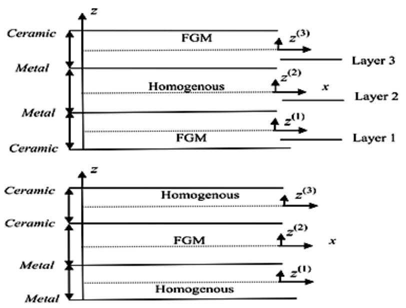

Fig. 4. Configurations used by Pandey and Pradyumna

Table 5. Review of Discret layerwise methods (DLWM)

Reference	Studied structure	Equation	f(z)	Studied loading / response	Thin / thick structure	Number of unknowns /layer	Analysis method	Main contribution / novelty
Van Do & Lee (2021)	Classical laminated composite plates	$u(x,y,z)=u_0(x,y)+\sum_{J=1}^{N}U^J(x,y)\times\phi^J(z)$ $v(x,y,z)=v_0(x,y)+\sum_{J=1}^{N}V^J(x,y)\times\phi^J(z)$ $w(x,y,z)=w(x,y)$	$\phi^J(z)=\frac{z_2-z}{z_2-z_1},\quad z_1<z<z_2\quad for\ J=1$ $\phi^J(z)=\begin{cases}\frac{z-z_{J-1}}{z_J-z_{J-1}}, & z_{J-1}<z<z_J\\ \frac{z_{J+1}-z}{z_{J+1}-z_J}, & z_J<z<z_{J+1}\end{cases}\quad for\ J=2,\dots,N-1$ $\phi^J(z)=\frac{z-z_{N-1}}{z_N-z_{N-1}},\quad z_{N-1}<z<z_N\ \ for\ J=N$	Bending, free vibration (natural frequencies)	Thin to moderately thick	5	IGA (Isogeometric Analysis)	Coupling of DLWM with IGA, providing high-order continuity of displacement fields and improved accuracy for laminated plates
Alipour & Shariyat (2017)	Circular and annular sandwich plates with auxetic cores	$u_1=u_0+\left(\xi^{(1)}+\frac{h_1}{2}\right)\Psi_r^{(1)}+\frac{h_2}{2}\Psi_r^{(2)};\ \frac{-h_1}{2}\le\xi^{(1)}\le\frac{h_1}{2}$ $u_2=u_0+\xi^{(2)}\Psi_r^{(2)};\ \frac{-h_2}{2}\le\xi^{(2)}\le\frac{h_2}{2}$ $u_3=u_0+\left(-\frac{h_2}{2}\right)\Psi_r^{(2)}+\left(\xi^{(3)}-\frac{h_3}{2}\right)\Psi_r^{(3)};\ \frac{-h_3}{2}\le\xi^{(3)}\le\frac{h_3}{2}$ $w=w_0$	ξ local coordinate $\Psi_r^{(k)}$ is the total rotation (sum of the global φ_r and local $\chi_r^{(k)}$ rotations) of the kth layer of the plate	Free vibration	Moderately thick to thick	5	Analytical	First analytical DLWM formulation for sandwich plates with auxetic cores, highlighting interlaminar effects
Amabili & Reddy (2021)	Multilayer sandwich plates	- Top: $U_T=u_T+z_T\Phi_{1T}+z_T^2\Psi_{1T}$ $V_T=v_T+z_T\Phi_{2T}+z_T^2\Psi_{2T}$ $W_T=w_T$ - Bottom: $U_B=u_B+z_B\Phi_{1B}+z_B^2\Psi_{1B}$ $V_B=v_B+z_B\Phi_{2B}+z_B^2\Psi_{2B}$ $W_B=w_B$ - Core: $U_C=u_c+z_c\Phi_{1C}+z_C^2\Psi_{1C}+z_C^3\gamma_{1C}$ $V_C=v_c+z_c\Phi_{2C}+z_C^2\Psi_{2C}+z_C^3\gamma_{2C}$ $W_C=w_c+z_c\chi_1+z_C^2\chi_2+z_C^3\chi_3$	Φ_1 *and* Φ_2 *are* rotations of the normal to the middle plane about the $\alpha2$ and $\alpha1$ axes ψ_1 *and* ψ_2 *are functions* obtained by using the condition that the transverse shear stresses $\tau_{13}=G_{13}\gamma_{13}$ and $\tau_{23}=G_{23}\gamma_{23}$	Nonlinear bending, nonlinear vibration	Thick	7	Analytical + FEM validation	Development of a nonlinear layerwise third-order theory, extending DLWM to large-deflection sandwich plate mechanics
Gao et al. (2024)	Laminated sandwich plates	$u^k=u_0+A^k.\phi_x+z.(-\frac{\partial w_0}{\partial x}+B^k.\phi_x)+\left(z-\frac{4z^3}{3h^2}\right).\phi_x$ $v^k=v_0+C^k.\phi_y+z.(-\frac{\partial w_0}{\partial y}+D^k.\phi_y)+\left(z-\frac{4z^3}{3h^2}\right).\phi_y$ $w^k=w_0$	Ak, Bk, Ck and Dk are parameters their formulas given by [69]	Bending, vibration	Thick	7	FEM	Proposal of a compressible third-order DLWM enforcing exact transverse shear stress continuity at interfaces
Pandey & Pradyumna (2018)	Functionally graded (FGM) sandwich plates	$u^{(2)}(x,y,z)=u_0(x,y)+z^{(2)}\theta_x^{(2)}+L_1$ $v^{(2)}(x,y,z)=v_0(x,y)+z^{(2)}\theta_y^{(2)}+L_2$ $w^{(2)}(x,y,z)=w_0(x,y)$	where $u0$, $v0$ and $w0$ are the displacements of the middle plane along x, y and z directions, respectively The parameters $\theta_x^{(i)}$ and $\theta_y^{(i)}$ are the rotations of normal to middle plane of ith layer about y and x axes, respectively. $L_1=(z^{(2)})^2u^*+(z^{(2)})^3\theta_x^*$ $L_2=(z^{(2)})^2v^*+(z^{(2)})^3\theta_y^*$ u^*, v^*, θ_x^* and θ_y^* are the higher-order terms in the Taylor's series expansion and represent higher-order transverse cross sectional deformation modes of the middle layer	Bending, free vibration	Moderately thick to thick	5	FEM	Application of higher-order DLWM to FGM sandwich plates, capturing both material gradation and interlaminar behavior

3.2. Independent Layerwise Models (ILWM):

3.2.1. Laminated Beams and Plates

Zhen & Wanji (2016) [70] have developed a high-order global zigzag model based on the variational theorem of Hu-Washizu, which improves the energy coherence and numerical robustness of multilayer beams.

Other subsequent contributions, including Nath & Mishra (2018) [71] and Si & Zhang (2022) [72], propose improved versions of zigzag theories, with the aim of improving the representation of transverse shear effects and thermo-mechanics without using expensive discrete layerwise models. Ren et al. (2021) [73] apply zigzag models of order three to both symmetric and asymmetric laminates, which reaffirms the kinematic generality of ILWM.

More recently, the work of Sorrenti & Gherlone (2023) [74] and Si et al. (2023) [75] have shown an advance in the ILWM framework by introducing mixed and refined formulations, optimized for thick plates by making an explicit trade-off between interlaminar accuracy and computational efficiency. ILWM's adaptability to advanced functional structures can be demonstrated through applications like Chanda & Sahoo (2020) [76] that incorporate piezoelectric layers using a trigonometric zigzag theory. A zigzag shell model developed by Gao et al. (2024) [77] is based on Reissner's mixed variational theorem. Thanks to the model, inter-layer slope discontinuities are captured with a reduced number of unknowns, which improves interlaminar stress accuracy compared to ESL models and avoids the cost of discrete layerwise formulations.

These studies demonstrate that zigzag models have evolved in a logical manner, from basic formulas to optimized, physically rich, and digitally efficient models. Furthermore, they advocate for ILWM to be a valid alternative to traditional ESL/HSDT and fully discrete DLWM.

Table 6. Review of Independent Layerwise Models (ILWM) applied to laminated beams and plates

Reference	Studied structure	Governing equations	f(z)	Studied loading / response	Thin / Thick structure	Number of unknowns	Analysis method	Main contribution / novelty
Zhen & Wanji (2016)	Multilayered composite beams	$u = u_0 + zu_1 + z^2u_2 + z^3u_3 + M^k(z)u_z$ $w = w_0$	$\mathrm{M^k(z) = (-1)^k\zeta_k}$ Where $\mathrm{\zeta_k = a_k z - b_k}$; $\mathrm{a_k = \frac{2}{z_{k+1} - z_k}; b_k = \frac{z_{k+1} + z_k}{z_{k+1} - z_k}}$	Bending, static response	Thick	6	Analytical	Energetically consistent global zigzag beam model via Hu–Washizu theorem
Nath & Mishra (2018)	Laminated composite plates	$u(x,y,z) = u_k(x,y) + z.\Psi_k^*(x,y) + z^2.\xi(x,y) + z^3.\eta(x,y)$ $w(x,y,z) = w_0(x,y) + z.w_1^k(x,y) + z^2.w_2^k(x,y) + z^3.w_3^k(x,y)$	w_1^k, w_2^k and w_3^k are taken so as to satisfy continuity of transverse displacement. Ψ_k^* shear rotation components	Static bending	Moderately thick to thick	7	Analytical	Improved transverse shear accuracy without shear correction factors
Ren et al. (2021)	Symmetric & asymmetric laminated beams	$U(x,z) = u_0(x) - z.\frac{\partial\omega_0}{\partial x} + f(z).\gamma(x)$ $\omega(x,z) = \omega_0(x)$	$f(z) = A_k + z.B_k + z^2C_k + z^3D_k$ Ak, Bk, Ck and Dk are the coefficients per layer for the k th layer of the laminated beam.	Bending, vibration	Thick	6	Analytical	Unified formulation valid for symmetric and asymmetric laminates
Si & Zhang (2022)	Laminated composite plates	$\mathrm{u^k(x,y,z) = u_0(x,y) + u_3(x,y).\phi_1^k(z) + \frac{\partial\omega_0}{\partial x}.\phi_2^k(z) + \frac{\partial\omega_1}{\partial x}.\phi_3^k(z) + \frac{\partial\omega_2}{\partial x}.\phi_4^k(z)}$ $\mathrm{v^k(x,y,z) = v_0(x,y) + v_3(x,y).\Psi_1^k(z) + \frac{\partial\omega_0}{\partial y}.\Psi_2^k(z) + \frac{\partial\omega_1}{\partial y}.\Psi_3^k(z) + \frac{\partial\omega_2}{\partial y}.\Psi_4^k(z)}$ $\mathrm{w^k(x,y,z) = w_0(x,y) + w_1(x,y).z + w_2(x,y).z^2}$	The expressions of $\phi k\, i$ and $\psi k\, i$ may be reviewed in the ref [8]	Mechanical & thermal loading	Thick	7–8	Analytical	Accurate thermo-mechanical response with reduced kinematic complexity
Chanda & Sahoo (2020)	Laminated plates with piezoelectric layer	$U(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x} + \sum_{i=1}^{n_u-1}(z - z_i^u)H(z - z_i^u)\alpha_{xu}^i + \sum_{j=1}^{n_l-1}(z - z_j^l)H(-z + z_j^l)\alpha_{xl}^j + [g(z) + z\,\Omega_x]\beta_x$ $V(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y} + \sum_{i=1}^{n_u-1}(z - z_i^u)H(z - z_i^u)\alpha_{yu}^i + \sum_{j=1}^{n_l-1}(z - z_j^l)H(-z + z_j^l)\alpha_{yl}^j + [g(z) + z\,\Omega_y]\beta_y$ $W(x,y,z) = w_0(x,y)$	$\mathrm{\beta_x; \beta_y}$ are the rotations of the transverse normal to the mid plane about the y and x axis respectively H represents Heaviside unit step function $\alpha_{xu}^i, \alpha_{xl}^j, \alpha_{yu}^i\ et\ \alpha_{yl}^j$ are the change in slopes defined at the interface of ith and jth layer corresponding to the upper and lower layers respectively $g(z) = zsec(\frac{rz}{h})$	Static electromechanical response	Moderately thick	7	Analytical	Introduction of trigonometric zigzag for smart composite plates
Sorrenti & Gherlone (2023)	Thick multilayered composite plates	$U^{(k)}(X) = u(x) + z\theta(x) + z^2\chi(x) + z^3\omega(x) + \varphi^{(k)}(z)\Psi(x)$ $U_3(X) = w^{(0)}(x) + zw^{(1)}(x) + z^2w^{(2)}(x)$	u_α(x) and θ_α(x) are the global uniform displacements and rotations of the normal to the reference plane about the positive *x2* and the negative *x1* directions, respectively; ψ_α(x) are the zigzag rotations, whereas χ_α(x), ω_α(x), $w^{(1)}$ and $w^{(2)}$ are the additional kinematic unknowns that take into account the effect of non-linear distribution of in-plane and transverse displacements along the thickness direction	Static & dynamic response	Thick	~9	FEM (mixed formulation)	High-accuracy interlaminar stress prediction with mixed variables
Si et al. (2023)	Laminated composite plates	$u_l^k(x,y,z) = N_1^k(z).\overline{u}_k(x,y) + N_2^k(z).\overline{u}_{k+1}(x,y)$ $v_l^k(x,y,z) = N_1^k(z).\overline{v}_k(x,y) + N_2^k(z).\overline{v}_{k+1}(x,y)$	$N_1^k(z) = \frac{1}{4}(\zeta_k(z) - 1)^2$ $N_2^k(z) = \frac{1}{4}(\zeta_k(z) + 1)^2$ $\zeta_k(z) = \frac{2}{h^{(k)}}.z - \frac{z_k + z_{k+1}}{h^{(k)}}$	Bending, vibration	Thin to thick	6–7	Analytical + FEM validation	Computationally efficient zigzag model with accuracy close to layerwise

Gao et al. (2024)	**Curved laminated composite shells**	$u_i^{(k)} = \left(1+\frac{z}{R_i}\right)u_i(x,y,t) + z\theta_i(x,y,t) + \phi_i^{(k)}(z)\Psi_i(x,y,t)$ $\omega^{(k)} = \omega(x,y,t)$	$u_i^{(k)}$ and $\omega^{(k)}$ are the in−plane displacements θ_i is the average rotation angles of the transverse normal $\phi_i^{(k)}$ and Ψ_i are the piecewise linear zigzag functions and amplitude functions of the RZT	**Static and mechanical response of composite shells**	**Moderately thick to thick**	**Limited set of global unknowns (ILWM zigzag-type)**	**Analytical formulation with numerical validation**	**Proposal of a new ILWM zigzag shell model capturing interlaminar slope discontinuities with higher accuracy than ESL models and lower cost than DLWM formulations**

3.2.2. Sandwich Structures

A gradual methodological evolution has been implemented to improve the kinematic inter-layer representation of sandwich structures, while controlling the numerical cost. By applying layerwise theories coupled with isogeometric analysis, we were able to accurately describe the discontinuities of displacement and stress across the thickness, which was a significant advance in this research. In this context, Liu and Jeffers [78] as well as Thai et al. [79] have demonstrated that combining layerwise formulations with IGA provides excellent accuracy for laminated sandwich plates and FGM. Particularly for bending and transverse shear. Thai et al. [60] then advanced a layerwise HSDT theory of type C0 that agrees with standard finite elements to decrease the complexity associated with strictly layerwise formulations while maintaining an accurate prediction of interlaminar stresses. At the same time, several authors have elaborated refined zigzag theories as an intermediate solution between ESL and complete zigzag theories. Garg et al. [80] developed a trigonometric zigzag theory that is perfectly suited for hygro-thermo-mechanical loads, showing remarkable improvement in stress fields. Garg and Chalak [81] and Chen and Huang [82] have extended these methods to sandwich beams, suggesting high-level zigzag theories that can accurately capture local shear and bending variations. Finally, Garg and Chalak [83]extended these models to right and left sandwich plates in hygro-thermo-mechanical environments, with explicit integration of transverse stresses.

3.2.3. FGM Structures

Ren et al. [84] advanced a theory of high-order shear deformation, which is explicitly in agreement with the laws of gradation of material properties. This allows a more accurate representation of the displacement and stress fields in thick FGM plates. In parallel, Garg et al. [85] expanded these concepts to CNT-reinforced FGM sandwich beams, examining free bending and vibration of symmetrical and non-symmetrical configurations with a flexible core emphasizing the joint impact of material gradation, nanotubular reinforcement, and structural asymmetry. In the continuity of refined zigzag approaches, Garg et al. [86] developed an HOZT method that fully uses transverse displacements to analyse the bending of FGM sandwich plates, significantly improving transverse stress prediction compared to classical ESL models

Table 7. Review of Independent Layerwise Models (ILWM) applied to Sandwich and FGM structures

Sandwich structures

Ref.	Studied structure	Governing equations	f(z)	Studied loading / response	Thin / Thick structure	Number of unknowns	Analysis method	Main contribution / novelty
Liu & Jeffers (2017)	**Laminated composite and FGM sandwich plates**	$\mathbf{u(x,y,z)=\sum_{i=1}^{N_{cp}} u_i(x,y)N^i(z)}$ $\mathbf{v(x,y,z)=\sum_{i=1}^{N_{cp}} v_i(x,y)N^i(z)}$ $\mathbf{w(x,y,z)=\sum_{i=1}^{N_{cp}} w_i(x,y)N^i(z)}$	**where Ncp denotes the number of control points in the thickness direction.** $N^i(z)$ **is the B-spline basis function for the discretization of the through-thickness displacements**	**Static bending, free vibration**	**Moderately thick to thick**	**(3N) per layer**	**Isogeometric Analysis (IGA)**	**Accurate layerwise IGA framework capturing interlaminar continuity and through-thickness stresses in sandwich and FGM plates**
Thai et al. (2016)	**Laminated composite and sandwich plates**	$\mathbf{u^k(x,y,z)=u_0(x,y)-z.\frac{\partial w_0(x,y)}{\partial x}+\left(A^k+z.B^k+f(z)\right).\phi x}$ $\mathbf{v^k(x,y,z)=v_0(x,y)-z.\frac{\partial w_0(x,y)}{\partial y}+\left(C^k+z.D^k+f(z)\right).\phi y}$ $\mathbf{w(x,y,z)=w_0(x,y)}$	$\mathbf{A^k, B^k, C^k}$ **and** $\mathbf{D^k}$ **are constants determined from continuity conditions**	**Static bending, free vibration**	**Thick and moderately thick**	**Higher than ESL, reduced vs full LW**	**IGA + generalized HSDT**	**Unified generalized layerwise HSDT compatible with IGA for improved shear and stress prediction**
Thai et al. (2017)	**Laminated composite and sandwich plates**				**Moderately thick**	**Reduced compared to classical LW**	**FEM (standard (C0) elements)**	**C0-compatible layerwise HSDT enabling efficient implementation without special continuity requirements**
Garg et al. (2019)	**Laminated composite and sandwich plates**	$\mathbf{u(x,y,z)=u_0(x,y)-z.\frac{\partial w_0}{\partial x}+\sum_{i=1}^{n_u-1}(z-z_i^u).H(z-z_i^u).\alpha_i^{xu}+\sum_{j=1}^{n_l-1}\left(z-z_j^l\right).H(-z+z_j^l).\alpha_j^{xl}+[g(z)+\Omega_x.z].\beta_x}$ $\mathbf{v(x,y,z)=v_0(x,y)-z.\frac{\partial w_0}{\partial y}+\sum_{i=1}^{n_u-1}(z-z_i^u).H(z-z_i^u).\alpha_i^{yu}+\sum_{j=1}^{n_l-1}\left(z-z_j^l\right).H(-z+z_j^l).\alpha_j^{yl}+\left[g(z)+\Omega_y.z\right].\beta_y}$ $w(x,y,z)=w_0(x,y)$	$\mathbf{\alpha^{xu}, \alpha^{xl},\ \alpha^{yu}, \alpha^{yl}}$**, depend upon the layer considered and indicates the** **slope of the concerned layer** $\mathbf{H(z-z_i^u)}$ **and** $\mathbf{H\left(-z+z_j^l\right)}$ **are the Heaviside step functions** **nu and nl represents the number of layers above and below the reference plane respectively.** **Ωx and Ωy are constant determined by applying the traction free conditions at top and bottom surface** **g(z) is a trigonometric function describing the non-linear variation of shear**	**Static bending under hygro-thermo-mechanical loading**	**Thick plates**	**Low (ESL + zigzag enrichment)**	**Analytical / FEM-based**	**Introduction of trigonometric zigzag function improving transverse shear and stress accuracy under coupled fields**

Chen & Huang (2023)	**Sandwich composite beams**	$\mathbf{u_x^{(k)} = u_0 + z.\theta_x + \psi_x(x).\phi_x^{(k)}(z) + P_x^{(k)}.\gamma_0 + Q_x^{(k)}\psi_x(x)}$ $\mathbf{u_z = w(x)}$	$\mathbf{P_x^{(k)} = a_{1x}^{(k)}.(\alpha_x^{(k)})^2 + a_{2x}^{(k)}.(\alpha_x^{(k)})^3}$ $\mathbf{Q_x^{(k)} = b_{1x}^{(k)}.(\alpha_x^{(k)})^2 + b_{2x}^{(k)}.(\alpha_x^{(k)})^3}$ **$a_1^{(k)}$, $a_2^{(k)}$, $b_1^{(k)}$, $b_2^{(k)}$ are constants** **γ 0 = θx + uz,x** $\boldsymbol{\alpha_x^{(k)} = z - \frac{z^{(k+1)} + z^{(k)}}{2}}$	**Static bending**	**Thick beams**	**Reduced (beam-type zigzag model)**	**Analytical / FEM**	**Novel refined zigzag theory capturing local shear effects with high accuracy for sandwich beams**
Garg & Chalak (2021)	**Laminated sandwich beams**	$\mathbf{U(x) = u^{(0)} + z.\varphi^{(x)} + z^2.\mu^{(x)} + z^3.\xi^{(x)} + z^4.\Psi^{(x)} + \sum_{i=1}^{N^{(u)}-1}(z - z_i^{(u)})H(z - z_i^{(u)}).\phi_i^{(x_u)} + \sum_{j=1}^{N^{(l)}-1}(z - z_j^{(l)})H(-z + z_j^{(l)}).\phi_j^{(x_l)}}$ $\mathbf{U(y) = v^{(0)} + z.\varphi^{(y)} + z^2.\mu^{(y)} + z^3.\xi^{(y)} + z^4.\Psi^{(y)} + \sum_{i=1}^{N^{(u)}-1}(z - z_i^{(u)})H(z - z_i^{(u)}).\phi_i^{(y_u)} + \sum_{j=1}^{N^{(l)}-1}(z - z_j^{(l)})H(-z + z_j^{(l)}).\phi_j^{(y_l)}}$ $\mathbf{U(z) = w^{(0)} + z.\varphi^{(z)} + z^2.\mu^{(z)} + z^3.\xi^{(z)} + z^4.\Psi^{(z)} + \sum_{i=1}^{N^{(u)}-1}(z - z_i^{(u)})H(z - z_i^{(u)}).\phi_i^{(z_u)} + \sum_{j=1}^{N^{(l)}-1}(z - z_j^{(l)})H(-z + z_j^{(l)}).\phi_j^{(z_l)}}$	**$u^{(0)}$, $v^{(0)}$ and $w^{(0)}$ are the displacement of any point on the mid-plane about x- y- and z-axis** **$\varphi^{(x)}$, $\varphi^{(y)}$ and $\varphi^{(z)}$ are the rotation of the mid-plane about y-, x- and z-axis respectively** **$\mu^{(x)}$; $\xi^{(x)}$; $\Psi^{(x)}$; $\mu^{(y)}$; $\xi^{(y)}$; $\Psi^{(y)}$; $\mu^{(z)}$; $\xi^{(z)}$ and $\Psi^{(z)}$ represents higher order unknowns** **Φ denotes the slope of the ith/jth layer for upper and lower layers** **H represents Heaviside unit step function**	**Static bending, stress analysis**	**Thick beams**	**Moderate (zigzag-enhanced ESL)**	**Analytical / FEM**	**Development of a novel higher-order zigzag beam theory balancing accuracy and computational efficiency**
Garg & Chalak (2021)	**Non-skew and skew laminated composite and sandwich plates**			**Hygro-thermo-mechanical loading, transverse stresses**	**Thick plates**	**Moderate**	**FEM**	**Extension of zigzag plate theory to skew geometries with explicit transverse stress variation modeling**
FGM structures								
Ren et al. (2022)	***Functionally graded plates***	$\boldsymbol{u(x,y,z) = u_0(x,y) - z.w_{0,x}(x,y) - \frac{1}{3}.z^3 w_{1,x}(x,y) + \sum_{i=1}^{n} g_i(z).\beta_{ix}(x,y)}$ $\boldsymbol{v(x,y,z) = v_0(x,y) - z.w_{0,y}(x,y) - \frac{1}{3}.z^3 w_{1,y}(x,y) + \sum_{i=1}^{n} g_i(z).\beta_{iy}(x,y)}$ $\boldsymbol{w(x,y,z) = w_0(x,y) + z^2 w_1(x,y)}$	***gi(z) are the shear strain shape functions for the RHSDTs*** ***βix and βiy denote shear related rotational angle variables at the reference surface of the FG plate***	***static bending, free vibration***	***Moderately thick to thick***	***Higher than FSDT, lower than full LW***	***Analytical / FEM***	***Development of refined HSDTs explicitly conforming to continuous material property variation through thickness***
Garg et al. (2022)	***Symmetric and unsymmetric FGM CNT-reinforced sandwich beams with soft core***	$\boldsymbol{U(x) = u^{(0)} + z.\varphi^{(x)} + z^2.\mu^{(x)} + z^3.\xi^{(x)} + z^4.\Psi^{(x)} + \sum_{i=1}^{N^{(u)}-1}(z - z_i^{(u)})H(z - z_i^{(u)}).\phi_i^{(x_u)} + \sum_{j=1}^{N^{(l)}-1}(z - z_j^{(l)})H(-z + z_j^{(l)}).\phi_j^{(x_l)}}$ $\boldsymbol{U(y) = v^{(0)} + z.\varphi^{(y)} + z^2.\mu^{(y)} + z^3.\xi^{(y)} + z^4.\Psi^{(y)} + \sum_{i=1}^{N^{(u)}-1}(z - z_i^{(u)})H(z - z_i^{(u)}).\phi_i^{(y_u)} + \sum_{j=1}^{N^{(l)}-1}(z - z_j^{(l)})H(-z + z_j^{(l)}).\phi_j^{(y_l)}}$	***$u^{(0)}$, $v^{(0)}$ and $w^{(0)}$ are the displacement of any point on the mid-plane about x- y- and z-axis*** ***$\varphi^{(x)}$, $\varphi^{(y)}$ and $\varphi^{(z)}$ are the rotation of the mid-plane about y-, x- and z-axis respectively*** ***$\mu^{(x)}$; $\xi^{(x)}$; $\Psi^{(x)}$; $\mu^{(y)}$; $\xi^{(y)}$; $\Psi^{(y)}$; $\mu^{(z)}$; $\xi^{(z)}$ and $\Psi^{(z)}$ represents higher order unknowns*** ***Φ denotes the slope of the ith/jth layer for upper and lower layers***	***Bending, free vibration***	***Thick beams***	***Moderate (beam HSDT/zigzag)***	***Analytical / FEM***	***Coupled modeling of CNT reinforcement, FGM gradation, soft core effects, and structural asymmetry***
Garg et al. (2022)	***Functionally graded***			***Static bending, transverse stresses***	***Thick plates***	***Moderate (zigzag-enriched ESL)***	***FEM***	***HOZT including transverse displacement effects for***

sandwich plates	$\boldsymbol{U(z) = w^{(0)} + z.\varphi^{(z)} + z^2.\mu^{(z)} + z^3.\xi^{(z)} + z^4.\Psi^{(z)} + \sum_{i=1}^{N^{(u)}-1} \left(z - z_i^{(u)}\right) H\left(z - z_i^{(u)}\right).\phi_i^{(z_u)} + \sum_{j=1}^{N^{(l)}-1} \left(z - z_j^{(l)}\right) H\left(-z + z_j^{(l)}\right).\phi_j^{(z_l)}}$	***H represents Heaviside unit step function***	***improved prediction of transverse stresses in FGM sandwich plates***

4. Carrera Unified Formulation

The theoretical framework called Carrera Unified Formulation (CUF) aims to ensure the mechanical modelling of multilayer structures such as beams, plates, and composite shells, sandwiches, or FGM, whose essential principle is to describe in a unified and systematic way the displacement fields through the thickness. The (CUF) does not propose a specific kinematic theory (ESL, zigzag, or layer-wise) but describes displacements as a generic development depending on the thickness, whether polynomial or not, and whose order can be adjusted freely. This method allows for generating simple monolayer equivalent models and refined layer-wise models using the same basic formulation, while maintaining the same matrix structure for the equations.

The works of Carrera et al [87] are based on the Carrera Unified Formulation (CUF), which provides a unified hierarchical framework for the systematic creation of equivalent monolayer, zigzag, or layer-wise models applied to plates, shells, sandwich structures, FGM, and complex cases including piezoelectricity or delamination. In this same logic, Pagani et al. [88]use MITC9 elements to ensure numerical robustness with respect to shear and locking. Daraei et al. [89] recently extended these theoretical frameworks using alternative numerical schemes (finite strip, boundary discontinuous method) to improve computational efficiency while maintaining a detailed description of the layer-wise. The research of Castañeda et al. [90] focuses on the optimization of shear theories for sandwich beams. While Mantari et al. [91] introduce non-polynomial displacement fields for FGM plates.

Table 8. Review of Carrera Unified Formulation

Reference	Studied structure	Governing equations	f(z)	Studied loading / response	Thin / Thick structure	Number of unknowns	Analysis method	Main contribution / novelty
Francesco Tornabene (2016)	Doubly-curved laminated composite shells and panels	$U_1^{(k)} = F_0^{(k)}.u_1^{(k0)} + F_1^{(k)}.u_1^{(k1)} + F_2^{(k)}.u_1^{(k2)} + \cdots + F_N^{(k)}.u_1^{(kN)} + F_{N+1}^{(k)}.u_1^{(k(N+1))}$ $U_2^{(k)} = F_0^{(k)}.u_2^{(k0)} + F_1^{(k)}.u_2^{(k1)} + F_2^{(k)}.u_2^{(k2)} + \cdots + F_N^{(k)}.u_2^{(kN)} + F_{N+1}^{(k)}.u_2^{(k(N+1))}$ $U_3^{(k)} = F_0^{(k)}.u_3^{(k0)} + F_1^{(k)}.u_3^{(k1)} + F_2^{(k)}.u_3^{(k2)} + \cdots + F_N^{(k)}.u_3^{(kN)} + F_{N+1}^{(k)}.u_3^{(k(N+1))}$	thickness functions $F_\tau^{(k)}$: $F_\tau^{(k)} = \begin{cases} \frac{P_0 - P_1}{2} = \frac{1 - z_k}{2} \; for \; \tau = 0 \\ P_{\tau+1} - P_{\tau-1} \; for \; \tau = 1,2 \ldots, N \\ \frac{P_0 + P_1}{2} = \frac{1 + z_k}{2} \; for \; \tau = 5 = N + 1 \end{cases}$ Legendre polynomials P_τ defined in the article	Free vibration (natural frequencies, mode shapes)	Thin to thick	Layer-dependent; increases with order and number of layers	Analytical + numerical solution	General high-order LW model for curved shells with high accuracy in vibration
Ernesto Carrera & Valvano (2017)	Laminated composite shells with embedded piezoelectric layers	$u^k(x,y,z) = F_0(z).u_0^k(x,y) + F_1(z).u_1^k(x,y) + \cdots + F_N(z).u_N^k(x,y)$ $v^k(x,y,z) = F_0(z).v_0^k(x,y) + F_1(z).v_1^k(x,y) + \cdots + F_N(z).v_N^k(x,y)$ $w^k(x,y,z) = F_0(z).w_0^k(x,y) + F_1(z).w_1^k(x,y) + \cdots + F_N(z).w_N^k(x,y)$	Fi are the thickness functions depending only on z	Static and dynamic electromechanical response	Thin to thick	Variable, hierarchical (CUF)	FEM (shell elements)	Unified treatment of piezoelectric laminated shells within CUF
Alessandro Pagani et al. (2018)	Laminated composites and sandwich plates			Static bending and free vibration	Thin to thick	Variable, depends on CUF order	FEM (MITC9 plate elements)	Robust variable-kinematic MITC9 elements avoiding shear locking
Ying Yan et al. (2017)	Laminated, box and sandwich beams	$\boldsymbol{u^k(x,y,z,t) = F_\tau(x,y).u_\tau^k(y,t) \quad \tau = 1,2,\ldots,M}$	- L3 configuration: F$_1$=1-r-s; F$_2$=r ; F$_3$=s - L4 configuration: $F_\tau = \frac{1}{4}(1 + rr_\tau)(1 + ss_\tau)$ - L9 configuration: $F_\tau = \frac{1}{4}(r^2 + rr_\tau)(s^2 + ss_\tau), \quad \tau = 1,3,5,7$ $F_\tau = \frac{1}{2}s_\tau^2(s^2 - ss_\tau)(1 - r^2) + \frac{1}{2}r_\tau^2(r^2 - rr_\tau)(1 - s^2), \quad \tau = 2,4,6,8$ $F_\tau = (1 - r^2)(1 - s^2), \quad \tau = 9$ where r and s from -1 to +1	Free vibration (exact solutions)	Thin to thick	Proportional to number of layers	Exact / semi-analytical solutions	Benchmark exact solutions for LW beam vibration
S. K. Kumar et al. (2018)	Delaminated plates and shells			Modal analysis (natural frequencies, modes)	Thin to thick	Variable, CUF-dependent	FEM (MITC9 shell)	CUF framework extended to delamination modeling
Behzad Daraei et al. (2022)	Composite laminated plates	$\boldsymbol{u(x,y,z,t) = F_\tau(x,z).u_\tau(y,t) \quad \tau = 1,2,\ldots,N}$	$F_\tau(x,z)_k^{(p_x,p_z)} = \begin{cases} (-1)^{p_z}(\frac{2x}{b})^{p_x} \text{ if } z < z_{k-1} \\ (1)(\frac{2x}{b})^{p_x}(\frac{2\zeta}{h_k})^{p_z} \text{ if } z_{k-1} < z < z_k \\ (1)^{p_z}(\frac{2x}{b})^{p_x} \text{ if } z > z_k \end{cases}$ px, pz are the polynomial orders and ζ is the subdomain thickness coordinate	Free vibration	Thin to moderately thick	Reduced compared to full FEM	Finite Strip Method (FSM)	Efficient CUF-FSM formulation for vibration analysis
R. W. Laureano & Mantari (2025)	Multilayered composite plates	$u^k = F_b(\zeta)u_b^k(x,y) + F_r(\zeta)u_r^k(x,y) + F_t(\zeta)u_t^k(x,y)$ $v^k = F_b(\zeta)v_b^k(x,y) + F_r(\zeta)v_r^k(x,y) + F_t(\zeta)v_t^k(x,y)$ $w^k = F_b(\zeta)w_b^k(x,y) + F_r(\zeta)w_r^k(x,y) + F_t(\zeta)w_t^k(x,y)$	- $\zeta_k = \frac{2z_k}{h_k}$ - $Fb = \frac{\zeta_k - 1}{2}$	Static and dynamic response	Thin to thick	Layer-dependent	Boundary Discontinuous Method (BDM)	Novel LW formulation solved via BDM (plates)

<table>
<tr><td>R. W. Laureano & Mantari (2025)</td><td>Multilayered composite shells</td><td></td><td>- $F2 = \frac{3\zeta_k^2-3}{2}$
- $F3 = \frac{5\zeta_k^3-5\zeta_k}{2}$
- $F4 = \frac{35\zeta_k^4-42\zeta_k^2+7}{8}$
- $Fb = \frac{\zeta_k+1}{2}$</td><td>Static and dynamic response</td><td>Thin to thick</td><td>Layer-dependent</td><td>Boundary Discontinuous Method (BDM)</td><td>Extension of BDM-LW approach to shell structures</td></tr>
<tr><td>W. M. Castañeda et al. (2019)</td><td>Sandwich beams</td><td>$\boldsymbol{u(x,y,z) = F_\tau(y,z).u_\tau(x) \quad \tau = 1,2,\dots,M}$</td><td>$F_\tau$ function of the polynomial order N [92]
$$\boldsymbol{M = \frac{(N+1)(N+2)}{2}}$$</td><td>Static bending and vibration</td><td>Thick</td><td>Few, optimized set</td><td>Analytical + FEM</td><td>Identification of optimal shear deformation theories</td></tr>
<tr><td>José L. Mantari et al. (2016)</td><td>Functionally graded plates</td><td>$u_x = f_1u_{x_1} + f_2u_{x_2} + f_3u_{x_3} + f_4u_{x_4}+f_5u_{x_5}$
$u_y = f_1u_{y_1} + f_2u_{y_2} + f_3u_{y_3} + f_4u_{y_4}+f_5u_{y_5}$
$u_z = f_1u_{z_1} + f_2u_{z_2} + f_3u_{z_3} + f_4u_{z_4}+f_5u_{z_5}$</td><td><table>
<tr><td></td><td>f_1</td><td>f_2</td><td>f_3</td><td>f_4</td><td>f_5</td></tr>
<tr><td>pol</td><td>1</td><td>Z</td><td>Z^2</td><td>Z^3</td><td>Z^4</td></tr>
<tr><td>sinh</td><td>1</td><td>Z</td><td>$\cosh (\frac{z}{h})$</td><td>$\sinh (\frac{z}{h})$</td><td>$\cosh^2 (\frac{z}{h}$</td></tr>
<tr><td>exp</td><td>1</td><td>Z</td><td>$e^{(\frac{z}{h})^2}$</td><td>$z.e^{(\frac{z}{h})^2}$</td><td>$z^2.e^{(\frac{z}{h})^2}$</td></tr>
<tr><td>tan</td><td>1</td><td>Z</td><td>$\sec (\frac{z}{5h})$</td><td>$\mathrm{tg} (\frac{z}{5h})$</td><td>$\sec^2 (\frac{z}{5h}$</td></tr>
<tr><td>sin</td><td>1</td><td>Z</td><td>$\cos (\frac{z\pi}{h})$</td><td>$\sin (\frac{z\pi}{h})$</td><td>$\cos^2 (\frac{z\pi}{h}$</td></tr>
<tr><td>Sin*</td><td>1</td><td>z</td><td>$\cos (\frac{z}{h})$</td><td>$\sin (\frac{z}{h})$</td><td>$\cos^2 (\frac{z}{h})$</td></tr>
</table></td><td>Static analysis</td><td>Thin to thick</td><td>Variable, CUF-based</td><td>FEM</td><td>Non-polynomial CUF displacement fields for improved FGM accuracy</td></tr>
</table>

5. Conclusion

This review revealed that choosing a model for laminated, sandwich, and FGM structures requires an explicit compromise between the necessary precision and the quantities of interest (global displacements, interlaminar constraints, local fields near the interfaces), the admissible numerical cost, and the physical complexity taken into consideration, including anisotropy, heterogeneity, non-linearities, and multi-physical couplings.

Comparative analysis and areas of use of model families:

a) ESL – Equivalent Single Layer

In engineering, ESL models (refined CLPT/FSDT/HSDT) are widely used for the overall response (deflection, stiffness, eigenfrequencies, overall buckling) of thin to moderately thick plates/beams/shells. Their main advantage is their low cost, ability to be implemented directly (analytical, FEM, IGA), and robustness for parametric and optimization studies. Their structural limit lies in the lack of field representation through thickness, particularly the precision of interlaminar stresses (σ_{xz}, σ_{yz}, and σ_{zz}), and interface discontinuities. LECs are therefore appropriate in case of limited accuracy on local fields, or when corrections/overlaps are validated by a reference model.

b) Layerwise DLWM – Discrete Layerwise Models

DLWM can explicitly model the degrees of freedom per layer, thus allowing a faithful description of layer-by-layer variations, kinematic discontinuities, and a notable improvement in interlaminar stresses. Their use is especially beneficial for thick structures, sandwiches with high contrast (rigid faces/flexible core), cases in which the risk of delamination or interfacial rupture needs to be evaluated, and local analyses (support areas, concentrated loads). Their main drawback is the increase in the number of unknowns (that is, proportional to the number of layers), which leads to a higher numerical cost and increased sensitivity to meshing and locking, especially if the formulation is not stabilized.

c) Layerwise ILWM – Integral/Independent Layerwise Models (including zigzag)

The objective of ILWM is to preserve a rich representation through thickness, while limiting the increase in the number of unknowns. Zigzag methods offer an interesting compromise to observe slope discontinuities and interlaminar effects at a lower price than DLWM methods. They are suitable for laminates and sandwiches where the interface effects are important, but where one wishes to remain in a lighter computational framework than DLWM. Their limit is their precision, which is strongly influenced by the choice of thickness functions, continuity assumptions and the treatment of boundary conditions. Some local responses remain less reliable than with a full DLWM.

d) 3D Elasticity

For scale models, 3D elasticity is both a conceptual reference and a basis for validation. When phenomena are inherently three-dimensional, such as high stress concentrations, contact, localized loads, complex geometries, edge effects, or fine heterogeneities, it becomes crucial. The cost is the limiting factor for long structures, parametric analyses, or optimization strategies.

e) Carrera Unified Formulation (CUF)

The CUF has the ability to generate a hierarchy of models (from ESL-type to quasi-3D/layerwise models) through controlled expansion across the thickness. The main advantage of it is methodological: it enables coherent comparisons, orderly refinement (an increase in the rank of expansion), and adaptation to the needs of precision without changing the theoretical framework. The CUF is used in practice for studies that seek an adjustable compromise between cost and fidelity, particularly for sandwich/FGM and thick structures, while enabling the transition towards reference models. The limitations can be attributed to the complexity of implementation and the necessity of careful validation in strongly non-linear or multiphysical cases.

Table 9 gives an overall summary of all the previously mentioned methods

Table 9. Comparison between the different methods mentioned

Method family	**Method**	**Types of structures studied**	**Thin / thick structure (geometric criteria)**	**Heterogeneity representation**	**Accuracy**
ESL	CPT (Classical Plate Theory)	Homogeneous composite plates	Thin: (b/h > 20), (L/h > 20)	Very limited (global homogenization)	Low
	FSDT (First-Order Shear Deformation Theory)	Composite plates and beams	Thin to moderately thick: (10 < b/h < 20)	Limited	Medium
	HSDT (Higher-Order Shear Deformation Theories)	Plates, beams, simple sandwich structures	Thin to thick: (b/h > 5)	Medium	Good
Layerwise	ILWM (Independent / Integral Layerwise Models)	Laminated plates and beams, sandwich structures	Thin to very thick: (b/h ≥2)	High (layer-by-layer variation)	Very high
	DLWM (Discrete Layerwise Models)	Plates, beams, sandwich and multilayer FGM structures	Thin to very thick: (b/h≥ 2)	Very high (explicit interfaces)	Very high
CUF	Carrera Unified Formulation	Beams, plates, sandwich and FGM structures	All regimes: thin to very thick	Variable (ESL or LW depending on kinematic order)	Variable to very high
3D Elasticity	Three-Dimensional Elasticity Theory	Beams, plates, sandwich, FGM, complex structures	All regimes (no thin/thick assumption)	Exact (full 3D variation)	Very high (reference solution)

Future research should focus on analysing geometric non-linearities and non-linear behaviours of materials, as well as multiphysical interactions such as thermo-mechanical, hygro-thermal, and electro-chemical interactions, which are now crucial in composite, sandwich, and functionally gradient material structures. The development of multiscale frameworks that link microstructural characteristics and defects induced by manufacturing processes to macroscopic structural responses is another research direction that aims to improve predictive capacity and sustainability assessment. High-fidelity 3D FEM models can also be used as a reference for calibrating and approving CUF models of various expansion orders. Using this method, it would be possible to identify the minimum level of refinement required to correctly predict the quantities of interest (global displacements, interlaminar stresses, thermal fields). Finally, use machine learning to automatically identify the areas requiring layerwise or 3D processing within a global ESL/CUF model. By adopting this approach, we can automate the transition between fidelity levels, reduce user intervention, and improve the efficiency of analyses of complex structures subject to non-linear loading.

Conflicts of Interest

There is no conflict of interest regarding the publication of this article.

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