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Duality in Wolfe and Mond–Weir-Type Mixed-Dual Optimization Problems

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Abstract

In this paper, we introduce two novel dual formulations for optimization problems governed by inequality constraints. The considered problems are characterized by differentiable functions. The proposed dual problems are parameterized by a set-valued parameter, where different values of this parameter reduce the dual formulations to the Wolfe and Mond–Weir dual problems. Consequently, the proposed dual problem serves as a unified generalization of both. Furthermore, we establish the weak, strong, converse, restricted converse, and strict converse duality results for the formulated dual problems.

Keywords: Dual problem, Duality results, Constraint qualification, Optimality.
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1 Introduction

The concept of duality is extensively studied in the literature, with duality theory aiming to provide an alternative formulation of optimization problems that is either computationally more efficient or possesses theoretical significance. Central components of this theory include the primal optimization problem, the Lagrangian, and the dual optimization problem, which is defined in the space of dual variables, alongside the Karush-Kuhn-Tucker (KKT) conditions that govern both primal and dual variables.

Solution approaches encompass primal methods, which operate directly within the space of primal variables; dual methods, which solve the dual optimization problem and subsequently derive the primal solution from the dual solution; and primal-dual methods, which concurrently resolve both primal and dual variables. Dual and primal-dual methods are specifically devised to identify points that either satisfy the KKT conditions or correspond to saddle points of the Lagrangian. While classical duality theory often assumes that these methods yield exact solutions, in practical applications, approximate solutions are more commonly obtained (see, e.g., [5, 13, 14, 18]).

Two types of dual problems that have garnered significant attention in optimization theory due to their extensive applications are the Wolfe dual [21] and the Mond-Weir dual [11]. These dual formulations have been widely studied in various optimization settings, including problems involving differentiable functions, convex functions, quasiconvex functions, and locally Lipschitz functions (see respectively, [1, 2, 13, 14], [7, 15, 20], [8, 14, 18], and [4, 10]). Furthermore, generalizations of these dual approaches have been explored for problems with an infinite number of constraints,

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problems defined in a Banach algebra, and problems incorporating multiplicative constraints; refer to [3, 6], [19] and [9, 17], respectively.

To this end, we analyze the following optimization problem,

$$(\mathcal{P}): \qquad \begin{array}{ll} \min f(x) & \\ \text{s.t. } \; g_t(x) \leq 0, & \quad t \in T := \{1, \ldots, m\}, \end{array}$$

where f and g_t, for $t \in T$, are differentiable functions mapping $\mathbb{R}^n$ to $\mathbb{R}$.

The structure of the subsequent sections of this paper is as follows: In Section 2, we introduce the necessary definitions and preliminary results required for the later discussions. Sections 3 focuses on formulating the mixed-type dual problems for $(\mathcal{P})$. Additionally, we establish the weak, strong, converse, restricted converse, and strict converse duality results for these parametric dual problems. Finally, the conclusion and a illustrative example are presented in Section 4.

2 Preliminaries and Notation

In this section, we introduce the fundamental notations, basic definitions, and standard preliminaries essential for the subsequent discussion, as outlined in [4].

The standard inner product of vectors $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$, while the zero vector in $\mathbb{R}^n$ is represented as 0_n.

The gradient of the differentiable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ at $x_0 \in \mathbb{R}^n$ is denoted by $\{\nabla\varphi(x_0)\}$, given by

$$\nabla\varphi(x_0) := \left(\frac{d\varphi}{dx_1}(x_0), \ldots, \frac{d\varphi}{dx_n}(x_0) \right).$$

For a differentiable function $\psi : \mathbb{R}^{n+m} \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^{n+m}$, let $\nabla_1\psi(x_0, y_0) \in \mathbb{R}^n$ denotes the partial differential of $\psi(\cdot, \cdot)$ at (x_0, y_0), defined as $\nabla\psi(x_0, \cdot)(y_0)$.

Definition 2.1. [4] Suppose that the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}^n$. Then, φ is said to be

(i) quasiconvex at x_0, if for all $x \in \mathbb{R}^n$, one has

$$\varphi(x) \leq \varphi(x_0) \implies \langle \nabla\varphi(x_0), x - x_0 \rangle \leq 0.$$

(ii) pseudoconvex at x_0, if for all $x \in \mathbb{R}^n$, one has

$$\varphi(x) < \varphi(x_0) \implies \langle \nabla\varphi(x_0), x - x_0 \rangle < 0.$$

(iii) strictly pseudoconvex at x_0, if for all $x \in \mathbb{R}^n$ with $x \neq x_0$, one has

$$\varphi(x) \leq \varphi(x_0) \implies \langle \nabla\varphi(x_0), x - x_0 \rangle < 0.$$

Consider a set $A \subseteq \mathbb{R}^n$. We denote by $A^{\leq}$ the polar cone of A, defined as

$$A^{\leq} := \big\{ x \in \mathbb{R}^n \mid \langle x, a \rangle \leq 0, \quad \forall a \in A \big\}.$$

The closure, the convex cone, and the closed convex cone of A are respectively denoted by $\overline{A}$, $cone(A)$, and $\overline{cone}(A)$. Also, the Bouligand tangent cone and the normal cone of A at $x^* \in \overline{A}$ are defined by

$$\Gamma(A, x^*) := \big\{ \nu \in \mathbb{R}^n \mid \exists\alpha_\ell \to 0^+, \exists\nu_\ell \to \nu \; \text{ such that } \; x^* + \alpha_\ell\nu_\ell \in A, \forall\ell \in \mathbb{N} \big\},$$

and $N(A, x^*) := \big(\Gamma(A, x^*)\big)^{\leq}$, respectively.

3 Dual Problems and Duality Results

Suppose that the feasible set of $(\mathcal{P})$, denoted by $\mathcal{M}$, is nonempty, i.e.,

$$\mathcal{D} := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0, \quad t \in T\} \neq \emptyset.$$

For each $z \in \mathcal{M}$, the set of active index constraints is defined as

$$T_0(z) := \{t \in T \mid g_t(z) = 0\}. \tag{3.1}$$

For each feasible point $z \in \mathcal{M}$ and each $\Pi \subseteq T$, we define the following dual problem for $(\mathcal{P})$,

$$\begin{aligned}
(\mathcal{D}^z_\Pi): \quad & \max\ \varphi_\Pi(y, \alpha) \\
\text{s.t.} \quad & \nabla_1 \varphi_T(y, \alpha) = 0_n, && (3.2)\\
& \alpha_t g_t(y) \geq 0, \quad t \in T \setminus \Pi, && (3.3)\\
& \alpha_t \geq 0, \qquad\quad t \in T \setminus T_0(z), && (3.4)
\end{aligned}$$

where, $\alpha := (\alpha_1, \ldots, \alpha_m)$, and for each $J \subseteq T$, the function $\varphi_J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is defined as

$$\varphi_J(y, \alpha) := f(y) + \sum_{t \in J} \alpha_t g_t(y), \qquad \forall (y, \alpha) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{3.5}$$

Note that Equation (3.2) can be rewritten as

$$\nabla f(y) + \sum_{t \in T} \alpha_t \nabla g_t(y) = 0_n.$$

Moreover, conditions (3.3)-(3.4) imply that if $y \in \mathcal{M}$, then α_t has a free sign for $t \in T_0(y)$ and satisfies $\alpha_t \leq 0$ for $t \in T \setminus (\Pi \cup T_0(y))$.

The feasible set of $(\mathcal{D}^z_\Pi)$ is denoted by $\mathcal{M}^z_\Pi$, i.e.,

$$\mathcal{M}^z_\Pi := \left\{(y, \alpha) \in \mathbb{R}^{n+m} \mid (3.2), (3.3), \text{ and } (3.4) \text{ hold}\right\}.$$

Since the problem $(\mathcal{D}^z_\Pi)$ depends on $z \in \mathcal{M}$, following [9], we introduce an alternative dual problem that is independent of the feasible point z,

$$\begin{aligned}
(\mathcal{D}_\Pi): \quad & \max\ \varphi_\Pi(y, \alpha) \\
\text{s.t.} \quad & (y, \alpha) \in \mathcal{M}_\Pi := \bigcap_{z \in \mathcal{M}} \mathcal{M}^z_\Pi. && (3.6)
\end{aligned}$$

Remark 3.1.

(i) If $(y, \alpha) \in \mathcal{M}_\Pi$, then

$$\alpha_t \geq 0, \qquad \text{for} \quad t \in \bigcup_{z \in \mathcal{M}} \Big(T \setminus T_0(z)\Big) = T \setminus \Big(\bigcap_{z \in \mathcal{M}} T_0(z)\Big).$$

This means, the coefficients with indices whose corresponding constrained functions are not active for some feasible point are non-negative. Since in all non-trivial problems we have $\bigcap_{z \in \mathcal{M}} T_0(z) = \emptyset$, it follows that, for almost all problems, if $(y, \alpha) \in \mathcal{M}_\Pi$, then $\alpha_t \geq 0$ for all $t \in T$.

(ii) It is noteworthy that if we set $\Pi = T$ in the problem $(\mathcal{D}^z_\Pi)$, we obtain the Wolfe dual problem, originally introduced in [21]. Similarly, if we put $\Pi = \emptyset$, then $(\mathcal{D}^z_\emptyset)$ serves as extensions of the Mond–Weir dual problem, introduced in [11]. Consequently, the problem $(\mathcal{D}^z_\Pi)$ provides a unified generalization of the duality frameworks proposed in previous articles.

(iii) Since the feasible set of $(\mathcal{D}_\Pi)$ is given by the intersection of the feasible sets of $(\mathcal{D}^z_\Pi)$ over all $z \in \mathcal{M}$, it offers a formulation that is independent of z, making it more general than $(\mathcal{D}^z_\Pi)$. Furthermore, any solution of $(\mathcal{D}_\Pi)$ also serves as a solution to $(\mathcal{D}^z_\Pi)$ for each $z \in \mathcal{M}$.

Giving $(y, \alpha) \in \mathcal{M}^z_\Pi$, we define the following index sets,

$$T^+(z) := \{t \in T \mid \alpha_t > 0\},$$

$$T^-(z) := \{t \in T \mid \alpha_t < 0\} = \{t \in T_0(z) \mid \alpha_t < 0\},$$

where the last equality holds by (3.4). For simplicity, we define the following set, which includes some constraint functions of $(\mathcal{P})$,

$$\Lambda(z) := \big\{g_t \mid t \in T^+(z)\big\} \cup \big\{-g_t \mid t \in T^-(z)\big\}.$$

The next theorem, commonly referred to as the weak duality theorem for $(\mathcal{D}^z_\Pi)$, establishes that for any feasible solution to the primal problem $(\mathcal{P})$, the corresponding objective value is always greater than or equal to that of any feasible solution to the dual problem $(\mathcal{D}^z_\Pi)$.

Theorem 3.2. (Weak Duality for $(\mathcal{D}^z_\Pi)$): Let $z \in \mathcal{M}$ and $(y, \alpha) \in \mathcal{M}^z_\Pi$ be feasible solutions for the primal problem $(\mathcal{P})$ and the dual problem $(\mathcal{D}^z_\Pi)$, respectively. Furthermore, suppose that $\varphi_\Pi(\cdot, \alpha)$ is pseudoconvex at y, and the functions in $\Lambda(z)$ are quasiconvex at y. Then, the following inequality holds,

$$f(z) \geq \varphi_\Pi(y, \alpha).$$

Proof . Since $(y, \alpha) \in \mathcal{M}^z_\Pi$ and by equation (3.2), we obtain

$$\nabla f(y) + \sum_{t \in T} \alpha_t \nabla g_t(y) = \nabla \varphi_T(\cdot, \alpha)(y) = \nabla_1 \varphi_T(y, \alpha) = 0_n.$$

Thus, the inner product satisfies

$$\langle \nabla f(y), z - y\rangle + \sum_{t \in T} \alpha_t \langle \nabla g_t(y), z - y\rangle = \big\langle 0_n, z - y\big\rangle = 0. \tag{3.7}$$

To analyze further, let $t \in T \setminus \Pi$. We consider the following cases:

Case 1: $t \in T_0(z)$, implying that $g_t(z) = 0$.

- If $\alpha_t > 0$, then $t \in T^+(z)$ and $g_t \in \Lambda(z)$. By equation (3.3), since $\alpha_t g_t(z) = 0 \leq \alpha_t g_t(y)$, the quasiconvexity of g_t at y implies that

$$\sum_{t \in T_0(z)\setminus\Pi, \alpha_t > 0} \alpha_t \langle \nabla g_t(y), z - y\rangle \leq 0. \tag{3.8}$$

- If $\alpha_t = 0$, then $\langle \alpha_t \nabla g_t(y), z - y\rangle = 0$, leading to

$$\sum_{t \in T_0(z)\setminus\Pi, \alpha_t = 0} \alpha_t \langle \nabla g_t(y), z - y\rangle = 0. \tag{3.9}$$

- If $\alpha_t < 0$, then $t \in T^-(z)$ and $-g_t \in \Lambda(z)$. By equation (3.3),

$$-\alpha_t(-g_t(z)) = \alpha_t g_t(z) = 0 \leq \alpha_t g_t(y) = -\alpha_t(-g_t(y)),$$

and the quasiconvexity of $-g_t$ at y with $-\alpha_t > 0$ implies that

$$\sum_{t\in T_0(z)\setminus\Pi,\alpha_t<0} \alpha_t \langle \nabla g_t(y), z-y\rangle = \sum_{t\in T(z)\setminus\Pi,\alpha_t<0} -\alpha_t \langle -\nabla g_t(y), z-y\rangle \leq 0. \tag{3.10}$$

Adding inequalities (3.8), (3.9), and (3.10), we conclude

$$\sum_{t\in T_0(z)\setminus\Pi} \alpha_t \langle \nabla g_t(y), z-y\rangle = 0. \tag{3.11}$$

Case 2: $t \in T \setminus T_0(z)$. Then, $g_t(z) < 0$ and $\alpha_t \geq 0$ (by equation (3.4)).

- If $\alpha_t > 0$, then $t \in T^+(z)$ and by repeating previous steps with quasiconvexity of g_t at y, we obtain

$$\sum_{t\in T\setminus(T_0(z)\cup\Pi),\alpha_t>0} \alpha_t \langle \nabla g_t(y), z-y\rangle \leq 0. \tag{3.12}$$

- If $\alpha_t = 0$, then

$$\sum_{t\in T\setminus(T_0(z)\cup\Pi),\alpha_t=0} \alpha_t \langle \nabla g_t(y), z-y\rangle = 0. \tag{3.13}$$

Summing inequalities (3.12) and (3.13), we derive

$$\sum_{t\in T\setminus(T_0(z)\cup\Pi)} \alpha_t \langle \nabla g_t(y), z-y\rangle \leq 0.$$

Combining the above inequalities with (3.11), we conclude

$$\sum_{t\in T\setminus\Pi} \alpha_t \langle \nabla g_t(y), z-y\rangle \leq 0. \tag{3.14}$$

Applying this result to equation (3.7), we obtain

$$\Big(\langle \nabla f(y), z-y\rangle + \sum_{t\in\Pi} \alpha_t \langle \nabla g_t(y), z-y\rangle\Big) + \underbrace{\Big(\sum_{t\in T\setminus\Pi} \alpha_t \langle \nabla g_t(y), z-y\rangle\Big)}_{\leq 0} = \langle \nabla f(y), z-y\rangle + \sum_{t\in T} \alpha_t \langle \nabla g_t(y), z-y\rangle = 0.$$

Hence,

$$\langle \nabla\varphi_\Pi(\cdot,\alpha)(y), z-y\rangle = \left\langle \nabla f(y) + \sum_{t\in\Pi} \alpha_t \nabla g_t(y), z-y\right\rangle = \langle \nabla f(y), z-y\rangle + \sum_{t\in\Pi} \alpha_t \langle \nabla g_t(y), z-y\rangle \geq 0,$$

and so

$$\langle \nabla\varphi_\Pi(\cdot,\alpha)(y), z\rangle \geq \langle \nabla\varphi_\Pi(\cdot,\alpha)(y), y\rangle.$$

Utilizing pseudoconvexity of $\varphi_\Pi(\cdot,\alpha)$, we deduce $\varphi_\Pi(\cdot,\alpha)(z) \geq \varphi_\Pi(\cdot,\alpha)(y)$, i.e.,

$$\varphi_\Pi(z,\alpha) \geq \varphi_\Pi(y,\alpha). \tag{3.15}$$

On the other hand, by $z \in \mathcal{M}$ and $(y, \alpha) \in \mathcal{M}_{\Pi}^{z}$ we get

$$\begin{cases} t \in T_0(z) \implies g_t(z) = 0 \implies \alpha_t g_t(z) = 0, \\ t \in T \setminus T_0(z) \implies \Big(g_t(z) < 0 \text{ and } \alpha_t \geq 0\Big) \implies \alpha_t g_t(z) \leq 0, \end{cases}$$

and hence,

$$\sum_{t \in J} \alpha_t g_t(z) \leq 0, \qquad \forall J \subseteq T. \tag{3.16}$$

Combining this inequality and (3.15), we deduce that

$$\varphi_{\Pi}(y, \alpha) \leq \varphi_{\Pi}(z, \alpha) = f(z) + \overbrace{\sum_{t \in \Pi} \alpha_t g_t(z)}^{\leq 0} \leq f(z),$$

as required. □

The above theorem guarantees the following inequality

$$\min_{z \in \mathcal{M}} f(z) \geq \max_{(y,\alpha) \in \mathcal{M}_{\Pi}^{z}} \varphi_{\Pi}(y, \alpha).$$

If the inequality in the above relation holds as an equality, we say that the *strong duality* property is satisfied.

It is well known that satisfying the Karush-Kuhn-Tucker (KKT) necessary conditions is fundamental for strong duality to hold. Moreover, an appropriate constraint qualification must be met to ensure the validity of the KKT necessary conditions. Thus, in order to state the strong duality result for $(\mathcal{D}_{\Pi}^{z})$, we introduce the following definition and theorem, as presented in [4, 12].

Definition 3.3. The *Guignard Constraint Qualification* (*GCQ*) is said to hold at $\hat{z} \in \mathcal{M}$ for the problem $(\mathcal{P})$ if the following condition is satisfied

$$\big\{\nabla g_t(\hat{z}) \mid t \in T_0(\hat{z})\big\}^{\leq} \subseteq \overline{cone}\big(\Gamma(\mathcal{M}, \hat{z})\big).$$

It is worth noting that *GCQ* has been generalized for nondifferentiable optimization problems in [16]. Also, we can see ([4]) that the *GCQ* condition is equivalent to the following inclusion

$$N(\mathcal{M}, \hat{z}) \subseteq cone\Big(\big\{\nabla g_t(\hat{z}) \mid t \in T_0(\hat{z})\big\}\Big).$$

Theorem 3.4. [4] Suppose that $\hat{z}$ is a solution to $(\mathcal{P})$ and that the *GCQ* condition holds at $\hat{z}$. Then, there exist non-negative scalars $\hat{\alpha}_t$ for $t \in T$ such that

$$\nabla f(\hat{z}) + \sum_{t \in T} \hat{\alpha}_t \nabla g_t(\hat{z}) = 0_n, \qquad \text{and} \qquad \hat{\alpha}_t g_t(\hat{z}) = 0 \quad \text{for } t \in T. \tag{3.17}$$

Observe that, equation (3.17) can be rewritten as

$$\nabla f(\hat{z}) + \sum_{t \in T_0(\hat{z})} \hat{\alpha}_t \nabla g_t(\hat{z}) = 0_n.$$

Theorem 3.5. (Strong Duality for $(\mathcal{D}_{\Pi}^{z})$) Suppose that $\hat{z} \in \mathcal{M}$ is a solution for $(\mathcal{P})$ and *GCQ* holds at $\hat{z}$. Then, there exists a vector $\hat{\alpha} := (\alpha_1, \ldots, \alpha_m)$ such that $\alpha_t \geq 0$ for all $t \in T$ and $(\hat{z}, \hat{\alpha}) \in \mathcal{M}_{\Pi}^{\hat{z}}$. Furthermore, if $\varphi_{\Pi}(\cdot, \alpha)$ is pseudoconvex at $\hat{z}$ and the functions in $\Lambda(\hat{z})$ are quasiconvex at $\hat{z}$, then $(\hat{z}, \hat{\alpha})$ is a global solution to the problem $(\mathcal{D}_{\Pi}^{\hat{z}})$, and the following equality holds

$$f(\hat{z}) = \varphi_{\Pi}(\hat{z}, \hat{\alpha}). \tag{3.18}$$

Proof . By Theorem 3.4, there exists a vector $\hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_m)$ such that $\alpha_t \geq 0$ for all $t \in T$ and (3.17) holds. Thus, $(\hat{x}, \hat{\alpha}) \in \mathcal{M}_{\Pi}^{\hat{z}}$. Note that the following equality holds by (3.17)

$$\sum_{t \in K} \hat{\alpha}_t g_t(\hat{z}) = 0, \qquad \forall K \subseteq T. \tag{3.19}$$

Now, if $(z, \alpha) \in \mathcal{M}_{\Pi}^{\hat{z}}$ is given, the weak duality Theorem 3.2 implies that

$$f(\hat{z}) \geq \varphi_{\Pi}(z, \alpha).$$

The above inequality and (3.19) imply that

$$\varphi_{\Pi}(\hat{z}, \hat{\alpha}) = f(\hat{z}) + \sum_{t \in \Pi} \hat{\alpha}_t g_t(\hat{z}) = f(\hat{z}) \geq \varphi_{\Pi}(z, \alpha).$$

So, (3.18) holds and $(\hat{z}, \hat{\alpha})$ is a global solution for the dual problem $(\mathcal{D}_{\Pi}^{\hat{z}})$. □

Remark 3.6. While *GCQ* ensures the validity of equation (3.19), stronger constraint qualifications, presented in [12], can be considered in Theorem 3.5. Since *GCQ* directly leads to (3.19), which is stronger than merely ensuring $(\hat{z}, \hat{\alpha}) \in \mathcal{M}_{\Pi}^{\hat{z}}$, a natural direction for future research is identifying weaker constraint qualifications that yield precisely this inclusion. Investigating such a qualification, if one exists, could provide valuable insights in optimization theory.

The next theorem establishes the necessary conditions for the uniqueness of a global minimizer $\hat{z}$ in the primal problem $(\mathcal{P})$ and a global maximizer in the corresponding dual problem $(\mathcal{D}_{\Pi}^{\hat{z}})$.

Theorem 3.7. (Strict Converse Duality for $(\mathcal{D}_{\Pi}^{\hat{z}})$): Let $\hat{z} \in \mathcal{M}$ be an optimal solution to $(\mathcal{P})$, and suppose that *GCQ* holds at $\hat{z}$. Furthermore, assume that $(\hat{u}, \hat{\beta}) \in \mathcal{M}_{\Pi}^{\hat{z}}$ is a global maximizer of the dual problem $(\mathcal{D}_{\Pi}^{\hat{z}})$, that the functions in $\Lambda(\hat{z})$ are quasiconvex at $\hat{u}$, and that $\varphi_{\Pi}(\cdot, \hat{\beta})$ is strictly pseudoconvex at $\hat{u}$. Then, we have

$$\hat{z} = \hat{u}.$$

Proof . Suppose, for contradiction, that $\hat{z} \neq \hat{u}$. Given that $(\hat{u}, \hat{\beta}) \in \mathcal{M}_{\Pi}^{\hat{z}}$, we obtain the following condition

$$\nabla f(\hat{u}) + \sum_{t \in T} \hat{\beta}_t \nabla g_t(\hat{u}) = 0_n. \tag{3.20}$$

By applying the same reasoning used in the proof of inequality (3.14), it follows that

$$\sum_{t \in T \setminus \Pi} \hat{\beta}_t \left\langle \nabla g_t(\hat{u}), \hat{z} - \hat{u} \right\rangle \leq 0.$$

Utilizing this inequality along with equation (3.20), we deduce

$$\left\langle \nabla f(\hat{u}) + \sum_{t \in \Pi} \hat{\beta}_t \nabla g_t(\hat{u}), \hat{z} - \hat{u} \right\rangle \geq 0.$$

Since

$$\nabla \varphi_{\Pi}(\cdot, \hat{\beta})(\hat{u}) = \nabla f(\hat{u}) + \sum_{t \in \Pi} \hat{\beta}_t \nabla g_t(\hat{u}),$$

it follows that

$$\left\langle \nabla \varphi_{\Pi}(\cdot, \hat{\beta})(\hat{u}), \hat{z} - \hat{u} \right\rangle \geq 0, \quad \text{or equivalently} \quad \left\langle \nabla \varphi_{\Pi}(\cdot, \hat{\beta})(\hat{u}), \hat{z} \right\rangle \geq \left\langle \nabla \varphi_{\Pi}(\cdot, \hat{\beta})(\hat{u}), \hat{u} \right\rangle.$$

By the strict pseudoconvexity of $\varphi_{\Pi}(\cdot,\hat{\beta})$ at $\hat{u}$ and the assumption $\hat{z}\neq\hat{u}$, we conclude that

$$\varphi_{\Pi}(\hat{z},\hat{\beta})>\varphi_{\Pi}(\hat{u},\hat{\beta}).$$

Thus, we have

$$f(\hat{z})+\sum_{t\in\Pi}\hat{\beta}_t g_t(\hat{z})=\varphi_{\Pi}(\hat{z},\hat{\beta})>\varphi_{\Pi}(\hat{u},\hat{\beta}). \tag{3.21}$$

Additionally, since $\hat{z}\in\mathcal{D}$ and $(\hat{u},\hat{\beta})\in\mathcal{M}_{\Pi}^{\hat{z}}$, equation (3.16) implies

$$\sum_{t\in\Pi}\hat{\beta}_t g_t(\hat{z})\leq 0.$$

Combining this result with inequality (3.21), we obtain

$$f(\hat{z})>\varphi_{\Pi}(\hat{u},\hat{\beta}). \tag{3.22}$$

By the strong duality theorem (Theorem 3.5), there exists a vector $\hat{\alpha}=(\hat{\alpha}_1,\ldots,\hat{\alpha}_m)\in\mathbb{R}^m$ such that $\hat{\alpha}_t\geq 0$ for all $t\in T$, and $(\hat{z},\hat{\alpha})\in\mathcal{M}_{\Pi}^{\hat{z}}$ is a global solution for the dual problem $(\mathcal{D}_{\Pi}^{\hat{z}})$. Furthermore, we have

$$f(\hat{z})=\varphi_{\Pi}(\hat{z},\hat{\alpha}).$$

Combining this equality with equation (3.22), we obtain

$$\varphi_{\Pi}(\hat{z},\hat{\alpha})>\varphi_{\Pi}(\hat{u},\hat{\beta}).$$

This leads to a contradiction, as it implies that the objective function of $(\mathcal{D}_{\Pi}^{\hat{z}})$ takes two distinct values at its two global solutions, $(\hat{z},\hat{\alpha})\in\mathcal{M}_{\Pi}^{\hat{z}}$ and $(\hat{u},\hat{\beta})\in\mathcal{M}_{\Pi}^{\hat{z}}$. Thus, we must conclude that $\hat{z}=\hat{u}$, completing the proof. □

Remark 3.8.

(i) It is important to note that in the proof of Theorem 3.7, since $(\hat{z},\hat{\beta})$ is not necessarily an element of $\mathcal{M}_{\Pi}^{\hat{z}}$, we cannot derive a contradiction directly from equation (3.21). If this were possible, the assumption of *GCQ* would not be required.

(ii) Since the feasible set of $(\mathcal{D}_{\Pi})$ is the intersection of the feasible sets of $(\mathcal{D}_{\Pi}^{z})$ for all $z\in\mathcal{M}$, and each solution of $(\mathcal{D}_{\Pi})$ is also a solution of $(\mathcal{D}_{\Pi}^{z})$ for every $z\in\mathcal{M}$, the weak, strong, and strict converse duality results (which hold for $(\mathcal{D}_{\Pi}^{z})$ for all $z\in\mathcal{M}$ by Theorems 3.2, 3.5, and 3.7) remain valid for $(\mathcal{D}_{\Pi})$. Next, we present two theorems that hold for $(\mathcal{D}_{\Pi})$ but do not necessarily hold for each individual $(\mathcal{D}_{\Pi}^{z})$.

The following theorem introduces the conditions under which one can transition from a feasible point of the dual problem $(\mathcal{D}_{\Pi}))$ to a solution for the primary problem $(\mathcal{P})$. Given that this theorem can be considered a converse counterpart to Theorem 3.5, it is referred to as the "converse duality result".

Theorem 3.9. (Converse Duality for $(\mathcal{D}_{\Pi})$): Let $(\hat{u},\hat{\beta})\in\mathcal{M}_{\Pi}$ be a feasible point for the dual problem $(\mathcal{D}_{\Pi})$, satisfying the condition

$$\hat{\beta}_t g_t(\hat{u})\geq 0,\quad \forall t\in T. \tag{3.23}$$

Furthermore, suppose that the function f is pseudoconvex at $\hat{u}$, and the functions in $\bigcup_{z\in\mathcal{M}}\Lambda(z)$ are quasiconvex at $\hat{u}$. Then, $\hat{u}$ is a global solution for the primal problem $(\mathcal{P})$.

Proof . Let $\hat{z} \in \mathcal{M}$ be an arbitrary feasible point for $(\mathcal{P})$. Given that $(\hat{u}, \hat{\beta})$ is feasible for $(\mathcal{D}_{\Pi})$, and by the definition of $\mathcal{M}_{\Pi}$, we obtain

$$(\hat{u}, \hat{\beta}) \in \mathcal{M}_{\Pi} = \bigcap_{z \in \mathcal{M}} \mathcal{M}_{\Pi}^{z} \subseteq \mathcal{M}_{\Pi}^{\hat{z}}.$$

This inclusion, along with equation (3.23), implies

$$\begin{cases} \hat{\beta}_t g_t(\hat{z}) \leq 0 \leq \hat{\beta}_t g_t(\hat{u}), & \forall t \in T^{+}(\hat{z}), \\ (-\hat{\beta}_t)(-g_t(\hat{z})) \leq 0 \leq (-\hat{\beta}_t)(-g_t(\hat{u})), & \forall t \in T^{-}(\hat{z}). \end{cases}$$

Since $\Lambda(\hat{z}) \subseteq \bigcup_{z \in \mathcal{M}} \Lambda(z)$, the functions in $\Lambda(\hat{z})$ are quasiconvex at $\hat{u}$. Consequently, the inequalities above lead to

$$\begin{cases} \hat{\beta}_t \left\langle \nabla g_t(\hat{u}), \hat{z} - \hat{u} \right\rangle \leq 0, & \text{if } \ t \in T^{+}(\hat{z}), \\ \hat{\beta}_t \left\langle \nabla g_t(\hat{u}), \hat{z} - \hat{u} \right\rangle = 0, & \text{if } \ \hat{\beta}_t = 0, \\ (-\hat{\beta}_t) \left\langle -\nabla g_t(\hat{u}), \hat{z} - \hat{u} \right\rangle \leq 0 \Rightarrow \hat{\beta}_t \left\langle \nabla g_t(\hat{u}), \hat{z} - \hat{u} \right\rangle \leq 0, & \text{if } \ t \in T^{-}(\hat{z}). \end{cases}$$

Thus, summing over all terms, we obtain

$$\sum_{t \in T} \hat{\beta}_t \left\langle \nabla g_t(\hat{u}), \hat{z} - \hat{u} \right\rangle \leq 0.$$

Moreover, the assumption that $(\hat{u}, \hat{\beta}) \in \mathcal{M}_{\Pi}^{\hat{z}}$ implies

$$\nabla f(\hat{z}) + \sum_{t \in T} \hat{\beta}_t \nabla g_t(\hat{u}) = 0_n \Rightarrow \left\langle \nabla f(\hat{z}), \hat{z} - \hat{u} \right\rangle + \overbrace{\sum_{t \in T} \hat{\beta}_t \left\langle \nabla g_t(\hat{u}), \hat{z} - \hat{u} \right\rangle}^{\leq 0} = 0.$$

This yields

$$\left\langle \nabla f(\hat{z}), \hat{z} - \hat{u} \right\rangle \geq 0.$$

Since f is pseudoconvex at $\hat{u}$, it follows that

$$f(\hat{z}) \geq f(\hat{u}).$$

Since $\hat{z}$ was arbitrarily chosen in $\mathcal{M}$, this final inequality establishes that $\hat{u}$ is a global solution to the primal problem $(\mathcal{P})$. Thus, the proof is complete. □

In the following theorem, an optimality criterion for a feasible point of the primary problem $(\mathcal{P})$ is established by utilizing the dual problem $(\mathcal{D}_{\Pi})$.

Theorem 3.10. (Restricted Converse Duality for $(\mathcal{D}_{\Pi})$): Suppose $\hat{z} \in \mathcal{M}$ is a feasible point of $(\mathcal{P})$. Further, assume there exists a feasible pair $(\hat{u}, \hat{\beta}) \in \mathcal{M}_{\Pi}$ for $(\mathcal{D}_{\Pi})$ such that

$$f(\hat{z}) = \varphi_{\Pi}(\hat{u}, \hat{\beta}). \tag{3.24}$$

If the function $\varphi_{\Pi}(\cdot, \hat{\beta})$ is pseudoconvex at $\hat{u}$ and the functions in $\bigcup_{z \in \mathcal{M}} \Lambda(z)$ exhibit quasiconvexity at $\hat{u}$, then $\hat{z}$ constitutes a global minimum of $(\mathcal{P})$.

Proof . Suppose, to the contrary, that $\hat{z}$ is not a global minimizer of $(\mathcal{P})$. Then, there exists $z^* \in \mathcal{M}$ such that $f(z^*) < f(\hat{z})$. Using this inequality along with (3.24), we obtain

$$f(z^*) < \varphi_{\Pi}(\hat{u}, \hat{\beta}).$$

Since $(\hat{u}, \hat{\beta}) \in \mathcal{M}_{\Pi}^{x^*}$ (as per the definition of $\mathcal{M}_{\Pi}$) and the functions in $\Lambda(\hat{z})$ exhibit quasiconvexity at $\hat{u}$, this inequality contradicts the weak duality theorem 3.2 associated with the problem $(\mathcal{D}_{\Pi}^{x^*})$. This contradiction confirms the result. □

4 Conclusion and Example

In this paper, we introduced several dual problems that depend on a parameter set $\Pi \subseteq T$, formulated for the optimization problem $(\mathcal{P})$ involving differentiable functions. We then established some weak, strong, converse, restricted converse, and strict converse duality results for these newly defined dual problems. Since these dual problems reduce to the Wolf dual problem and the Mond-Weir dual problem by choosing $\Pi = T$ and $\Pi = \emptyset$, respectively, we refer to them as *Wolf and Mond-Weir type mixed-dual problems.*

Example 4.1. Consider the following optimization problem

$$\begin{aligned} (P^*): \qquad & \min\ z_1^2 + z_2^2 \\ & \text{s.t.}\ \ -z_1 - z_2 \leq 0, \\ & \qquad z_1^2 + z_1 z_2 \leq 0. \end{aligned}$$

This problem has the form of $(\mathcal{P})$ by the following data,

- $n = 2,\ T = \{1, 2\},\ z = (z_1, z_2) \in \mathbb{R}^2,$
- $f(z) = z_1^2 + z_2^2,\ g_1(z) = -z_1 - z_2,\ g_2(z) = z_1^2 + z_1 z_2.$

Note that for $y = (y_1, y_2)$ and $\alpha = (\alpha_1, \alpha_2)$, we have

$$\varphi_T(y, \alpha) = y_1^2 + y_2^2 + \alpha_1(-y_1 - y_2) + \alpha_2(y_1^2 + y_1 y_2).$$

Thus, (3.2) implies that $(2y_1, 2y_2) + \alpha_1(-1, -2) + \alpha_2(2y_1 + y_2, y_1) = 0_2$, which is equivalent to

$$\begin{cases} 2y_1 - \alpha_1 + 2\alpha_2 y_1 + \alpha_2 y_2 = 0 \\ 2y_2 - \alpha_1 + \alpha_2 y_1 = 0. \end{cases}$$

Thus, corresponding to $\Pi = \emptyset$, $\Pi = \{1\}$, $\Pi = \{2\}$, and $\Pi = \{1, 2\}$, we obtain the following mixed-dual problems for each $z \in \mathcal{M}$

$$\begin{aligned} (\mathcal{D}_{\emptyset}^{z}): \quad & \max\ y_1^2 + y_2^2 \\ & \text{s.t.}\ \ 2y_1 - \alpha_1 + 2\alpha_2 y_1 + \alpha_2 y_2 = 0, \\ & \qquad 2y_2 - \alpha_1 + \alpha_2 y_1 = 0, \\ & \qquad \alpha_1(-y_1 - y_2) \geq 0, \\ & \qquad \alpha_2(y_1^2 + y_1 y_2) \geq 0, \\ & \qquad \alpha_t \geq 0, \qquad t \in T \setminus T_0(z), \end{aligned}$$

$$
\begin{aligned}
(\mathcal{D}^z_{\{1\}}): \quad & \max\ y_1^2 + y_2^2 + \alpha_1(-y_1 - y_2) \\
\text{s.t.} \quad & 2y_1 - \alpha_1 + 2\alpha_2 y_1 + \alpha_2 y_2 = 0, \\
& 2y_2 - \alpha_1 + \alpha_2 y_1 = 0, \\
& \alpha_2(y_1^2 + y_1 y_2) \geq 0, \\
& \alpha_t \geq 0, \qquad t \in T \setminus T_0(z),
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{D}^z_{\{2\}}): \quad & \max\ y_1^2 + y_2^2 + \alpha_2(y_1^2 + y_1 y_2) \\
\text{s.t.} \quad & 2y_1 - \alpha_1 + 2\alpha_2 y_1 + \alpha_2 y_2 = 0, \\
& 2y_2 - \alpha_1 + \alpha_2 y_1 = 0, \\
& \alpha_1(-y_1 - y_2) \geq 0, \\
& \alpha_t \geq 0, \qquad t \in T \setminus T_0(z),
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{D}^z_{\{1,2\}}): \quad & \max\ y_1^2 + y_2^2 + \alpha_1(-y_1 - y_2) + \alpha_2(y_1^2 + y_1 y_2) \\
\text{s.t.} \quad & 2y_1 - \alpha_1 + 2\alpha_2 y_1 + \alpha_2 y_2 = 0, \\
& 2y_2 - \alpha_1 + \alpha_2 y_1 = 0, \\
& \alpha_t \geq 0, \qquad t \in T \setminus T_0(z).
\end{aligned}
$$

According to

$$
T_0(0,1) = \{2\}, \qquad T_0(0_2) = \{1,2\}, \qquad T_0(-2,1) = \emptyset,
$$

and considering Remark 3.1, we have

$$
\begin{aligned}
(\mathcal{D}_{\{1,2\}}): \quad & \max\ y_1^2 + y_2^2 + \alpha_1(-y_1 - y_2) + \alpha_2(y_1^2 + y_1 y_2) \\
\text{s.t.} \quad & (y_1, y_2, \alpha_1, \alpha_2) \in \mathcal{M}_{\{1,2\}},
\end{aligned}
$$

in which the feasible set $\mathcal{M}_{\{1,2\}}$ is calculated as

$$
\mathcal{M}_{\{1,2\}} = \left\{ (y_1, y_2, \alpha_1, \alpha_2) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} 2y_1 - \alpha_1 + 2\alpha_2 y_1 + \alpha_2 y_2 = 0 \\ 2y_2 - \alpha_1 + \alpha_2 y_1 = 0 \\ \alpha_1 \geq 0, \quad \alpha_2 \geq 0 \end{array} \right\} = \big\{ (k, k, 2k, 0) \mid k \geq 0 \big\}.
$$

Clearly, $(\mathcal{D}_{\{1,2\}})$ corresponds to the well-known Wolfe dual problem for $(\mathcal{P})$. To apply Theorem 3.10, we must identify some $(\hat{z}_1, \hat{z}_2) \in \mathcal{M}$ and $(\hat{k}, \hat{k}, 2\hat{k}, 0) \in \mathcal{M}_{\{1,2\}}$ such that

$$
f(\hat{z}_1, \hat{z}_2) = \varphi_{\{1,2\}}(\hat{k}, \hat{k}, 2\hat{k}, 0),
$$

which simplifies to

$$
\hat{z}_1^2 + \hat{z}_2^2 = \hat{k}^2 + \hat{k}^2 + 2\hat{k}(-\hat{k} - \hat{k}) + 0(\hat{k}^2 + \hat{k}\hat{k}) = -2\hat{k}^2 \leq 0.
$$

Thus, it follows that $\hat{z} = (\hat{z}_1, \hat{z}_2) = 0_2$. Since all the hypotheses of Theorem 3.10 are satisfied at $\hat{z}$, we conclude that $\hat{z}$ is a global solution for $(\mathcal{P})$.

Moreover, since $\hat{z}$ is a solution for $(\mathcal{P})$ and the *GCQ* holds at $\hat{z}$, Theorem 3.5 guarantees the existence of some $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2) \in \mathbb{R}^2$ such that $(0_2, \hat{\alpha})$ is a global solution of $(\mathcal{D}^{0_2}_{\{1,2\}})$.

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