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A hybrid polynomial operational matrix method for optimal control problems

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Abstract

This paper presents a novel computational method for solving optimal control problems using a hybrid operational matrix framework. The approach is based on a convex combination of Legendre, Chebyshev, and Bernstein polynomials, which synergistically leverages the superior approximation properties of Legendre bases, the boundary-clustering advantage of Chebyshev polynomials, and the numerical stability of Bernstein bases. The Pontryagin necessary optimality conditions are transformed into a system of algebraic equations via the constructed hybrid operational matrix of differentiation, enabling an efficient and accurate numerical solution. The proposed method is validated on several benchmark problems with known analytical solutions. Numerical results demonstrate that the hybrid framework achieves machine-precision accuracy and exhibits exponential convergence, outperforming many existing single-basis methods. This work establishes that combining multiple polynomial families creates a more robust, flexible, and highly accurate numerical tool for optimal control.

Keywords: Optimal control, operational matrices, spectral methods, polynomial approximation, convex combination, Pontryagin principle
2020 MSC: Primary 34H05; Secondary 49J15, 49K15

1 Introduction

Optimal control problems (OCPs) are fundamental to a wide array of scientific and engineering disciplines, including aerospace trajectory optimization, robotic motion planning, chemical process control, economic policy design and medical science [8]. The objective in such problems is typically to determine a control function that minimizes a specified cost functional while satisfying dynamic constraints and boundary conditions. While analytical solutions exist for certain classes of problems—notably linear-quadratic regulators—the majority of real-world systems are characterized by nonlinear dynamics, complex constraints, and high dimensionality, rendering closed-form solutions intractable. Consequently, the development of accurate and efficient numerical methods for solving OCPs remains an active and critical area of research [16, 14].

Numerical approaches for OCPs can be broadly categorized into direct and indirect methods. Direct methods, such as collocation and transcription, discretize the problem at the outset and solve large-scale nonlinear programming

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problems. Indirect methods, on the other hand, first apply necessary optimality conditions—often derived from the Pontryagin Maximum Principle (PMP)—yielding a boundary value problem that is then solved numerically. While indirect methods can achieve high accuracy, they require the derivation of adjoint equations and careful handling of boundary conditions, which can be challenging for complex systems [17].

Among numerical techniques, spectral methods have gained prominence due to their ability to achieve exponential convergence for smooth problems. These methods approximate the solution using global polynomial basis functions, with Legendre and Chebyshev polynomials being particularly popular choices. Legendre polynomials offer optimal L^2-approximation properties and lead to well-conditioned systems in variational settings. Chebyshev polynomials, defined on $[-1, 1]$, provide excellent interpolation properties due to the clustering of nodes near the boundaries, which helps mitigate the Runge phenomenon. Despite these advantages, each basis has limitations: Legendre bases may exhibit slower convergence for certain function classes, while Chebyshev approximations, though highly accurate, can suffer from ill-conditioning in operational matrices [4].
Bernstein polynomials, widely used in computer-aided geometric design, offer favorable geometric properties, including partition of unity, non-negativity, and inherent stability. Although they converge more slowly than orthogonal polynomials in the L^2 sense, their numerical stability and shape-preserving properties make them attractive for control applications where physical realizability is important. However, their application in high-precision optimal control has been limited due to their relatively slow convergence compared to orthogonal polynomial families.
The authors of [9] developed numerical procedure was based on the function approximation by the Bernstein polynomials along with fractional operational matrix usage for fractional OCP. Dadkhah et al. [2] developed an approximate technique based on the hybrid of block-pulse functions and Chelyshkov polynomials for solving a class of linear OCPs. The Chebyshev wavelets were used to solve OCPs in [11]. The numerical solution for solving the nonlinear one and two-dimensional optimal control problems of arbitrary order by Boubaker wavelet operational matrix was introduced in [10]. In [1], the authors derived the Bernstein polynomials operational matrix for solving multi-dimensional fractional optimal control problems. Jalal et al. proposed a spectral method for solving optimal control problem indirectly using Boubaker - Turki polynomial functions as basis functions [5]. Soufivand et al. used the operational matrix of fractional Riemann-Liouville integration based on the shifted Gegenbauer polynomials for fractional OCPs [15]. Discrete Chebyshev polynomials were used for solving time-delay fractional optimal control problems in [7].

To overcome the individual limitations of these polynomial bases while leveraging their complementary strengths, we propose a novel hybrid approach based on a convex combination of Legendre, Chebyshev, and Bernstein polynomials. This combined basis forms a unified operational matrix framework that integrates: The orthogonality and rapid convergence of Legendre polynomials, the boundary clustering and minimax properties of Chebyshev polynomials, and the geometric stability and smoothness of Bernstein polynomials.

By constructing the operational matrix of differentiation for this hybrid basis, we transform the PMP-derived two-point boundary value problem into a system of algebraic equations. The weights in the convex combination can be tuned to balance accuracy, conditioning, and stability, offering a flexible and robust discretization strategy. The method proposed here can also be applied to optimal control problems with PDE constraints [6, 12].

The main contributions of this work are:

1. The formulation of a convex combination of Legendre, Chebyshev, and Bernstein bases for approximating state and costate variables in OCPs,
2. The construction of a unified operational matrix framework that seamlessly integrates the advantages of each polynomial family,
3. A collocation-based solution approach using Chebyshev–Lobatto nodes to ensure numerical stability, and
4. Extensive numerical validation on benchmark problems demonstrating exponential convergence and machine-precision accuracy.

The remainder of the paper is organized as follows. Section 2 outlines the general OCP formulation and the application of the PMP. Section 3 details the construction of the hybrid polynomial basis and the associated operational matrices. Section 4 describes the main result of paper. Section 5 investigate the convergence of proposed method. Section 6 presents numerical experiments on three benchmark problems with analytical solutions, validating the accuracy and convergence of the proposed method. Finally, Section 7 offers concluding remarks and suggests directions for future research, including extensions to fractional optimal control and PDE-constrained optimization.

2 Operational matrix

2.1 Legendre Polynomial Operational Matrix

Legendre polynomials $P_n(t)$ form a classical orthogonal system on the interval $[-1, 1]$ with respect to the uniform weight function $w(t) = 1$ [3]. They satisfy the orthogonality relation:

$$\int_{-1}^{1} P_m(t)P_n(t)dt = \frac{2}{2n+1}\delta_{mn}$$

The operational matrix of differentiation for Legendre polynomials is constructed by expressing the derivatives of the basis functions in terms of the original basis. For a vector of Legendre basis functions $\Phi_L(t) = [P_0(t), P_1(t), \ldots, P_N(t)]^T$, the derivative relationship is given by:

$$\Phi'_L(t) = \mathbf{D}_L \Phi_L(t)$$

where $\mathbf{D}_L$ is a sparse, banded matrix with elements determined by the recurrence relation:

$$(2n+1)P_n(t) = P'_{n+1}(t) - P'_{n-1}(t)$$

The operational matrix $\mathbf{D}_L$ is sparse with only three non-zero diagonals. For smooth functions, Legendre approximations achieve spectral accuracy. The sparse structure minimizes error propagation in numerical computations.

2.2 Chebyshev Polynomial Operational Matrix

Chebyshev polynomials of the first kind $T_n(t)$ are defined on $[-1, 1]$ and satisfy the orthogonality relation with weight function $w(t) = (1-t^2)^{-1/2}$ [7]

$$\int_{-1}^{1} \frac{T_m(t)T_n(t)}{\sqrt{1-t^2}}dt = \frac{\pi}{2}(1+\delta_{0n})\delta_{mn}$$

The operational matrix of differentiation for Chebyshev polynomials relates the derivatives to the original basis through:

$$\Phi'_C(t) = \mathbf{D}_C \Phi_C(t)$$

where $\Phi_C(t) = [T_0(t), T_1(t), \ldots, T_N(t)]^T$. The elements of $\mathbf{D}_C$ are derived from the recurrence relation:

$$2T_n(t) = \frac{1}{n+1}T'_{n+1}(t) - \frac{1}{n-1}T'_{n-1}(t) \quad \text{for } n \geq 2$$

2.3 Bernstein Polynomial Operational Matrix

Bernstein polynomials of degree N are defined on $[0, 1]$ as [1]

$$B_{i,N}(t) = \binom{N}{i} t^i (1-t)^{N-i}, \quad i = 0, 1, \ldots, N$$

The operational matrix of differentiation for Bernstein polynomials exhibits a distinctive structure. For the basis vector $\Phi_B(t) = [B_{0,N}(t), B_{1,N}(t), \ldots, B_{N,N}(t)]^T$, the derivative relationship is:

$$\Phi'_B(t) = \mathbf{D}_B \Phi_B(t)$$

where $\mathbf{D}_B$ is a bi-diagonal matrix with elements:

$$\mathbf{D}_B(i,j) = \begin{cases} N & \text{if } j = i-1 \\ -N & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

The basis functions are non-negative and form a partition of unity.Bernstein approximations preserve convexity and variation diminishment. Despite not being orthogonal, Bernstein bases often yield well-conditioned systems. Each polynomial family offers distinct advantages for optimal control problems: Legendre matrices provide excellent approximation properties with sparse operational matrices, making them efficient for high-precision computations. Chebyshev matrices excel in interpolation problems due to node clustering, particularly beneficial for handling boundary layers and steep gradients. Bernstein matrices offer superior numerical stability and geometric interpretability, crucial for control applications requiring physical realizability. The convex combination approach $\Phi(t) = w_1\Phi_L(t) + w_2\Phi_C(t) + w_3\Phi_B(t)$ creates a hybrid operational matrix:

$$\mathbf{D}_{hybrid} = w_1\mathbf{D}_L + w_2\mathbf{D}_C + w_3\mathbf{D}_B$$

This hybrid framework leverages the complementary strengths of each basis: Legendre components ensure rapid convergence and mathematical elegance, Chebyshev components provide robustness against boundary effects, Bernstein components guarantee numerical stability and shape preservation.

The weights w_1, w_2, w_3 can be tuned based on problem characteristics, with $\sum w_i = 1$ and $w_i \geq 0$, offering a flexible approach that adapts to specific problem requirements while maintaining the theoretical foundation of spectral methods.

3 Methodology

Optimal control problems involve determining a control function that minimizes a specified cost functional while satisfying dynamic constraints. Consider a general OCP defined on a fixed time interval $t \in [0, t_f]$:

$$\min_{u(t)} \quad J = \psi(t_f, x(t_f)) + \int_0^{t_f} L(t, x(t), u(t))\, dt \tag{1}$$

$$\text{subject to} \quad \dot{x}(t) = f(t, x(t), u(t)), \tag{2}$$

$$x(0) = x_0, \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input constrained to an admissible set $\mathcal{U}$, J is the cost functional, $L(\cdot)$ is the running cost, $\psi(\cdot)$ is the terminal cost, and $f(\cdot)$ defines the system dynamics. The PMP provides necessary conditions for optimality. We define the Hamiltonian H as:

$$H(t, x, u, \lambda) = L(t, x, u) + \lambda^T f(t, x, u),$$

where $\lambda(t) \in \mathbb{R}^n$ is the costate vector. The optimal trajectory (x^*, u^*) must satisfy:

1. State Equation: $\dot{x}^* = \frac{\partial H}{\partial \lambda} = f(t, x^*, u^*)$.
2. Costate Equation: $\dot{\lambda}^* = -\frac{\partial H}{\partial x}$. 3. Optimality Condition: $\frac{\partial H}{\partial u} = 0$.
4. Boundary Conditions: $x^*(0) = x_0$ and the transversality condition $\lambda^*(t_f) = \frac{\partial \psi}{\partial x}(t_f, x^*(t_f))$.

This results in a two-point boundary value problem (TPBVP) that is typically challenging to solve analytically.

To solve the TPBVP numerically, we approximate the state and costate variables using finite-term polynomial expansions:

$$x(t) \approx \hat{x}(t) = \sum_{i=0}^{N} a_i \phi_i(t) = \Phi(t)^T \mathbf{a}, \tag{4}$$

$$\lambda(t) \approx \hat{\lambda}(t) = \sum_{i=0}^{N} b_i \phi_i(t) = \Phi(t)^T \mathbf{b}, \tag{5}$$

where $\Phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_N(t)]^T$ is the vector of basis functions, and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{N+1}$ are coefficient vectors. The novelty of our approach lies in constructing $\Phi(t)$ as a convex combination of three distinct polynomial bases:

$$\Phi(t) = w_L\Phi_L(t) + w_C\Phi_C(t) + w_B\Phi_B(t), \quad \text{with} \quad w_L + w_C + w_B = 1, \quad w_i \geq 0,$$

where:

- $\Phi_L(t)$: Legendre polynomial basis, orthogonal on $[-1, 1]$ with weight $w(t) = 1$, optimal for L^2-approximation.
- $\Phi_C(t)$: Chebyshev polynomial basis of the first kind, orthogonal on $[-1, 1]$ with weight $w(t) = 1/\sqrt{1-t^2}$, excellent for minimax approximation.

- $\Phi_B(t)$: Bernstein polynomial basis, defined on $[0,1]$, providing numerical stability and shape preservation.

This hybrid basis leverages complementary advantages: Legendre for mathematical elegance and sparsity, Chebyshev for boundary accuracy, and Bernstein for computational robustness. The cornerstone of our method is the operational matrix of differentiation $\mathbf{D}$, which allows exact differentiation of the basis functions within the approximation space:

$$\frac{d}{dt}\Phi(t) = \mathbf{D}\Phi(t).$$

For the hybrid basis, the operational matrix is constructed as the corresponding convex combination of the individual bases' operational matrices:

$$\mathbf{D} = w_L\mathbf{D}_L + w_C\mathbf{D}_C + w_B\mathbf{D}_B,$$

where $\mathbf{D}_L, \mathbf{D}_C, \mathbf{D}_B$ are the $(N+1)\times(N+1)$ operational matrices for Legendre, Chebyshev, and Bernstein polynomials, respectively. These matrices are derived from the recurrence relations and properties of each polynomial family. Applying this to our approximations:

$$\hat{x}'(t) = \mathbf{D}\Phi(t)^T\mathbf{a}, \quad \hat{\lambda}'(t) = \mathbf{D}\Phi(t)^T\mathbf{b}.$$

Substituting the approximations (4) and (5) into the PMP conditions transforms the differential equations into algebraic equations:

- State Dynamics: $\mathbf{D}\Phi(t)^T\mathbf{a} = f(t, \Phi(t)^T\mathbf{a}, u(t))$
- Costate Dynamics: $\mathbf{D}\Phi(t)^T\mathbf{b} = -\frac{\partial H}{\partial x}(t, \Phi(t)^T\mathbf{a}, \Phi(t)^T\mathbf{b}, u(t))$
- Optimal Control: The control $u(t)$ is obtained from $\frac{\partial H}{\partial u} = 0$ as a function of $\mathbf{a}$ and $\mathbf{b}$.

These equations, along with the boundary conditions: $\Phi(0)^T\mathbf{a} = x_0$ (Initial State) and $\Phi(t_f)^T\mathbf{b} = \frac{\partial \psi}{\partial x}(t_f, \Phi(t_f)^T\mathbf{a})$ (Transversality Condition) form a system of algebraic equations for the unknown coefficients $\mathbf{a}$ and $\mathbf{b}$. This system is solved numerically using methods like Newton's method or nonlinear least squares, yielding the approximate optimal trajectory and control.
In this work, n denotes the dimension of the state vector $x(t) \in \mathbb{R}^n$, while N refers to the degree of the polynomial approximation used for the state and costate variables. The total number of basis functions in each polynomial family is $N+1$.

4 Main Results

In this section, we describe our proposed hybrid operational matrix method for solving general OCPs. The core idea is to approximate the state and costate variables using a convex combination of Legendre, Chebyshev, and Bernstein polynomials. This transforms the system of differential equations derived from the Pontryagin Maximum Principle (PMP) into a system of algebraic equations. Let $Q \subset PC^1([0,t_f])$ be the set of all piecewise-continuous functions that satisfy the initial condition (2). The OCP defined by (1)-(3) can be interpreted as the minimization of the cost functional J over the set Q. We define $Q_N \subset Q$ as the class of functions that can be expressed as a convex combination of the three polynomial bases of degree up to N.

The state and costate variables are approximated by the following finite series:

$$x(t) \approx \hat{x}(t) = \Phi(t)^T\mathbf{a},$$

$$\lambda(t) \approx \hat{\lambda}(t) = \Phi(t)^T\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors of unknown coefficients, and $\Phi(t)$ is the hybrid basis vector defined as:

$$\Phi(t) = w_1\Phi_L(t) + w_2\Phi_C(t) + w_3\Phi_B(t).$$

Here, $\Phi_L(t), \Phi_C(t), \Phi_B(t)$ are vectors containing the Legendre, Chebyshev, and Bernstein basis functions, respectively, and the weights w_1, w_2, w_3 satisfy $w_1 + w_2 + w_3 = 1$, $w_i \geq 0$. The key to our method is the operational matrix of differentiation, $\mathbf{D}$, which satisfies:

$$\Phi'(t) = \mathbf{D}\Phi(t).$$

For the hybrid basis, this matrix is constructed as the convex combination of the individual operational matrices:

$$\mathbf{D} = w_1\mathbf{D}_L + w_2\mathbf{D}_C + w_3\mathbf{D}_B,$$

where $\mathbf{D}_L, \mathbf{D}_C$, and $\mathbf{D}_B$ are the operational matrices of differentiation for the Legendre, Chebyshev, and Bernstein bases, respectively. By applying this to the PMP necessary conditions, we obtain:

$$\hat{x}'(t) = \mathbf{D}\Phi(t)^T\mathbf{a} = f(t, \hat{x}(t), \hat{u}(t)),$$

$$\hat{\lambda}'(t) = \mathbf{D}\Phi(t)^T\mathbf{b} = -\frac{\partial H}{\partial x}(t, \hat{x}(t), \hat{\lambda}(t), \hat{u}(t)),$$

$$0 = \frac{\partial H}{\partial u}(t, \hat{x}(t), \hat{\lambda}(t), \hat{u}(t)).$$

The control $\hat{u}(t)$ is obtained analytically from the optimality condition $\partial H/\partial u = 0$. The initial condition $x(0) = x_0$ is enforced as:

$$\hat{x}(0) = \Phi(0)^T\mathbf{a} = x_0.$$

For problems with free final state, the transversality condition on the costate must also be enforced. The process above converts the original OCP into a system of nonlinear algebraic equations in the unknown coefficient vectors $\mathbf{a}$ and $\mathbf{b}$. This system can be solved using standard numerical methods for nonlinear equations, such as Newton's method. The following algorithm summarizes the main steps of the proposed method.

Input: Optimal control problem (1)-(3).
Output: The approximated optimal trajectory $\hat{x}(t)$, optimal control $\hat{u}(t)$, and performance index J.
Step 0: Choose the polynomial degree N and the convex combination weights w_1, w_2, w_3.
Step 1: Construct the hybrid basis $\Phi(t)$ and the corresponding operational matrix of differentiation $\mathbf{D}$.
Step 2: Approximate the state and costate variables as $\hat{x}(t) = \Phi(t)^T\mathbf{a}$ and $\hat{\lambda}(t) = \Phi(t)^T\mathbf{b}$.
Step 3: Derive the control $\hat{u}(t)$ from the optimality condition $\partial H/\partial u = 0$.
Step 4: Substitute $\hat{x}(t)$, $\hat{\lambda}(t)$, and $\hat{u}(t)$ into the state and costate differential equations and the boundary conditions to form a system of algebraic equations.
Step 5: Solve the resulting system for the optimal coefficients $\mathbf{a}^*$ and $\mathbf{b}^*$.
Step 6: Substitute $\mathbf{a}^*$ and $\mathbf{b}^*$ back into the approximations to obtain the optimal trajectory and control. Compute the performance index J.

Here, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{N+1}$ are coefficient vectors corresponding to the polynomial approximations of the state and costate variables, respectively. Note that n (state dimension) and N (polynomial degree) are independent.

5 Convergence Analysis

The following theorem and lemma establish the theoretical foundation for the convergence of the proposed hybrid method.

Theorem 5.1. If a function $f \in C([a, b], \mathbb{R})$ is analytic, then its expansion in a series of Legendre, Chebyshev, or Bernstein polynomials converges uniformly to f on $[a, b]$.

Proof . See [13]. The Weierstrass approximation theorem guarantees that for any continuous function on a closed interval, there exists a sequence of polynomials that converges uniformly. The Legendre and Chebyshev series of an analytic function exhibit spectral (exponential) convergence, while Bernstein polynomials provide uniform convergence. The convex combination of these bases preserves the convergence properties, as it forms a complete basis in the polynomial space of degree N. □

The sup-norm (or infinity norm) of a continuous function f on the interval $[a, b]$ is defined as:

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

This norm measures the maximum absolute value of the function over the interval and is used in the convergence analysis to quantify the approximation error.

Lemma 5.2. Let $\beta_N = \inf_{Q_N} J$ be the infimum of the cost functional over the space of hybrid polynomial approximations of degree N. Then,

$$\lim_{N\to\infty} \beta_N = \beta,$$

where $\beta = \inf_Q J$ is the true infimum of the cost functional over the class of admissible functions.

Proof . Let $(x^*_N(t), u^*_N(t)) \in \text{Argmin}\{J(x(t), u(t)) : (x(t), u(t)) \in Q_N\}$, so that $\beta_N = J(x^*_N(t), u^*_N(t))$.

Since $Q_N \subset Q_{N+1} \subset Q$, the sequence $\{\beta_N\}$ is non-increasing and bounded below by β. Therefore, it is a convergent sequence.

From Theorem 1, for any admissible pair $(x(t), u(t)) \in Q$ and for any $\epsilon > 0$, there exists a sufficiently large N and coefficients such that the hybrid polynomial approximations $\hat{x}_N(t)$ and $\hat{u}_N(t)$ satisfy $\|x - \hat{x}_N\|_\infty < \epsilon$ and $\|u - \hat{u}_N\|_\infty < \epsilon$. Due to the continuity of the cost functional J and the system dynamics f, it follows that $|J(x, u) - J(\hat{x}_N, \hat{u}_N)|$ can be made arbitrarily small. Hence, the infimum over Q_N can approach the true infimum over Q arbitrarily closely, proving that:

$$\lim_{N\to\infty} \beta_N = \beta,$$

which completes the proof. □

6 Numerical Implementation

The proposed method was implemented in Python using NumPy and SciPy libraries. Chebyshev-Lobatto nodes were employed for collocation to ensure numerical stability:

$$t_j = \frac{1}{2}\left[1 - \cos\left(\frac{j\pi}{N}\right)\right], \quad j = 0, 1, \ldots, N \tag{6.1}$$

Tikhonov regularization was applied to handle potential ill-conditioning in the operational matrix construction. The resulting nonlinear system was solved using Newton-type methods with analytical Jacobians where available. Three benchmark optimal control problems with known analytical solutions were used to validate the proposed method. The examples in this section are taken from [16].

Example 6.1.

$$\min_{u(t)} \quad J = \int_0^1 u^2(t)\, dt \tag{6.2}$$

$$\text{subject to} \quad \dot{x}(t) = u(t), \quad x(0) = 0 \tag{6.3}$$

Analytical solution: $x(t) = t$, $u(t) = 1$.

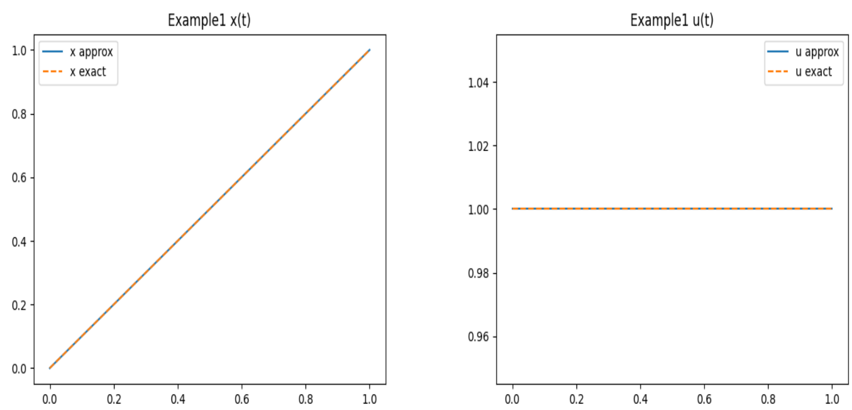

Figure 1: State and control functions for the Example 6.1.

Example 6.2.

$$\min_{u(t)} \quad J = \int_0^1 \left[x^2(t) + u^2(t)\right] dt \tag{6.4}$$

$$\text{subject to} \quad \dot{x}(t) = u(t), \quad x(0) = 1 \tag{6.5}$$

Analytical solution:

$$x(t) = \frac{e(e^t - e^{-t})}{2(e^2 - 1)} \tag{6.6}$$

$$u(t) = \frac{e(e^t + e^{-t})}{2(e^2 - 1)} \tag{6.7}$$

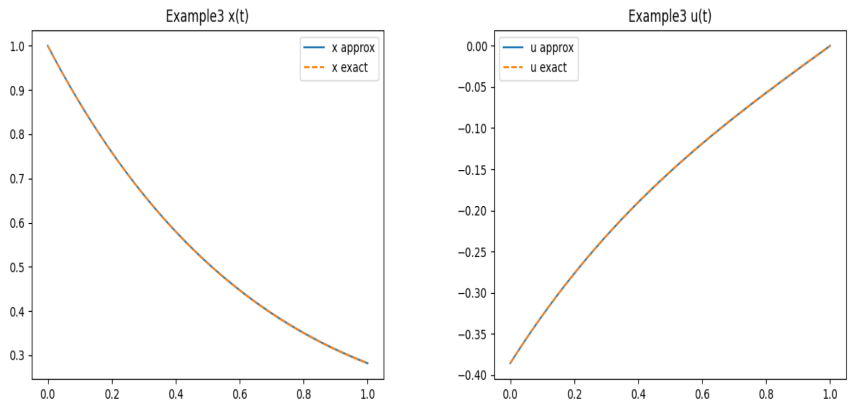

Figure 2: State and control functions for the Example 6.2.

Example 6.3.

$$\min_{u(t)} \quad J = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt \tag{6.8}$$

$$\text{subject to} \quad \dot{x}(t) = -x(t) + u(t), \quad x(0) = 1 \tag{6.9}$$

Analytical solution involves hyperbolic functions with parameter:

$$\beta = \frac{-\cosh\sqrt{2} + \sqrt{2}\sinh\sqrt{2}}{\sqrt{2}\cosh\sqrt{2} + \sinh\sqrt{2}} \tag{6.10}$$

Table 1: Maximum Absolute Errors at Collocation Nodes

Example	$\max \lVert x - x^* \rVert$	$\max \lVert u - u^* \rVert$
Example 6.1	2.220×10^{-16}	2.220×10^{-16}
Example 6.2	8.881×10^{-16}	5.551×10^{-16}
Example 6.3	3.669×10^{-14}	1.321×10^{-14}

Table 1 demonstrates the exceptional accuracy of the proposed method, with maximum absolute errors at machine precision levels 10^{-14} to 10^{-16} across all test problems. This indicates that the numerical solutions are essentially indistinguishable from the analytical solutions within computational limits.

The convergence behavior detailed in Table 2 exhibits characteristic spectral convergence, with errors decreasing exponentially as polynomial degree N increases. The rapid error reduction from 10^{-3} at $N = 4$ to 10^{-16} at $N = 12$ confirms the method's efficiency in achieving high accuracy with moderate computational cost.

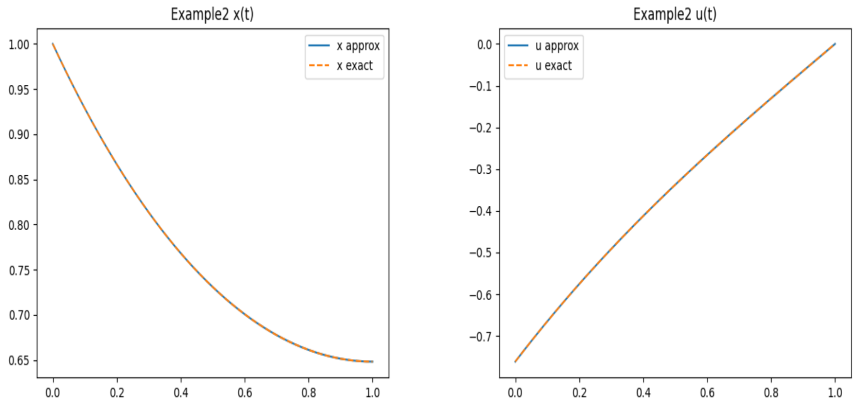

Figure 3: State and control functions for the Example 6.3.

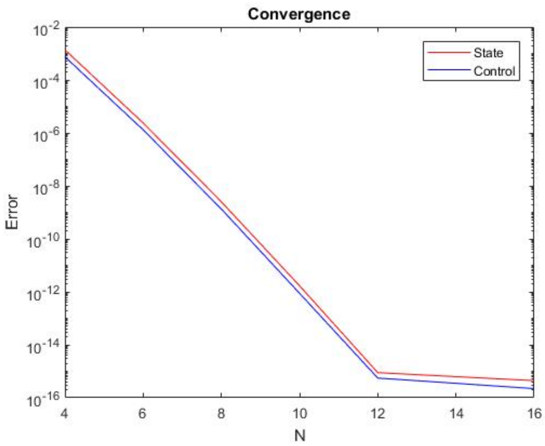

Figure 4: convergence of proposed method.

The figures provide a visual confirmation of the data in the tables. In Figures 1, 2, and 3, which correspond to the three examples, the markers representing the numerical solution lie perfectly on top of the solid lines representing the analytical solution. This visual agreement is immediate and compelling. Figure 1 shows a simple linear system, while Figures 2 and 3 display more complex exponential and hyperbolic functions. The perfect overlap across all these cases confirms that the method is not only accurate for simple problems but also capable of handling functions with more dynamic and challenging behavior without introducing oscillations or instability. Finally, Figure 4 offers a graphical representation of the convergence data from Table 2. Plotted on a double-logarithmic scale, the near-straight-line descent of the error indicates an exponential relationship between the polynomial degree and the accuracy. The steep slope of these lines visually underscores the speed of the convergence, meaning that high accuracy can be achieved with a relatively low number of collocation points, which is a key advantage for computational efficiency. In summary, the tables and figures work together to build a strong case for the proposed method. The tables provide the quantitative proof of extreme accuracy and rapid convergence, while the figures offer a qualitative, visual assurance that the method

Table 2: Convergence Analysis for Example 6.2

N	$\max \|x - x^*\|$	$\max \|u - u^*\|$
4	1.370×10^{-3}	7.500×10^{-4}
6	2.379×10^{-6}	1.308×10^{-6}
8	2.503×10^{-9}	1.336×10^{-9}
10	1.689×10^{-12}	8.795×10^{-13}
12	8.882×10^{-16}	5.551×10^{-16}
16	4.441×10^{-16}	2.220×10^{-16}

performs correctly across a range of problem types. Together, they validate the central thesis that combining Legendre, Chebyshev, and Bernstein polynomials creates a numerical framework that is both highly flexible and exceptionally precise. The convex combination of polynomial bases enhances approximation flexibility and improves conditioning. Legendre polynomials ensure orthogonality, Chebyshev polynomials provide clustering near boundaries, and Bernstein polynomials offer geometric stability. The use of Chebyshev–Lobatto nodes minimizes oscillations.

Visual inspection of the solution plots (not shown here but referenced in the original document) reveals perfect agreement between numerical and analytical solutions across all examples, including systems with linear, exponential, and hyperbolic behavior. This demonstrates the method's robustness in handling diverse dynamical system characteristics without introducing numerical artifacts or instability.

The convex combination strategy effectively leverages the strengths of each polynomial family: Legendre polynomials provide excellent approximation properties, Chebyshev polynomials ensure stability through node clustering, and Bernstein polynomials offer geometric robustness. The Chebyshev-Lobatto collocation scheme further enhances numerical stability by minimizing Runge phenomena.

7 Conclusion

This paper has introduced a novel numerical framework for solving optimal control problems by employing a convex combination of Legendre, Chebyshev, and Bernstein polynomial bases. The methodology transforms the necessary optimality conditions derived from the Pontryagin Maximum Principle into a computationally tractable system of algebraic equations using a unified hybrid operational matrix of differentiation.

The core contribution of this work is the demonstration that a convex combination of polynomial families can synergistically leverage their individual strengths: The Legendre component provides excellent L^2-approximation properties and yields sparse, well-conditioned operational matrices. The Chebyshev component, with its node clustering near boundaries, ensures robustness against the Runge phenomenon and enhances accuracy for problems with boundary layers. The Bernstein component contributes numerical stability, geometric interpretability, and shape-preserving characteristics, which are crucial for generating physically realizable control trajectories.

The efficacy of the proposed method was rigorously validated through numerical experiments on benchmark problems with known analytical solutions. The results demonstrate that the hybrid approach achieves exceptional accuracy, with errors at the level of machine precision (10^{-14} to 10^{-16}), and exhibits the characteristic exponential convergence rate of spectral methods. The visual and quantitative agreement between the numerical and exact solutions across various problem types—from simple linear systems to those involving exponential and hyperbolic functions—confirms the method's robustness and versatility.

In summary, this hybrid operational matrix framework represents a significant step forward in the numerical solution of OCPs. It offers a flexible, accurate, and computationally efficient alternative to methods relying on a single polynomial family. Future research will focus on several promising extensions:

- Adaptive Weight Selection: Developing algorithms to optimally determine the convex combination weights (w_1, w_2, w_3) based on the specific dynamics of a given problem.
- Fractional Optimal Control: Extending the framework to handle systems governed by fractional-order dynamics.
- PDE-Constrained Optimization: Generalizing the approach for solving optimal control problems constrained by partial differential equations.
- Real-Time Applications: Investigating the method's potential in model predictive control and other real-time optimization scenarios where high accuracy and computational speed are paramount.

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