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Contact problem with instantaneous normal response friction in viscoelasticity with long-term memory body

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Abstract

This paper explores a category of evolutionary variational problems. The formulation takes the shape of a system, encompassing a hyperbolic variational inequality (representing the displacement field), parabolic inequalities (capturing damage fields), and a differential equation (depicting the adhesion field). We establish the existence of a unique weak solution to this problem. The proof relies on the use of time-dependent variational inequalities, parabolic inequalities, differential equations, and fixed points. As an illustrative application, we explore a frictional dynamic contact problem involving a viscoelastic material with noncoercive viscosity, long-term memory, damage, and adhesion, accompanied by subdifferential boundary conditions.

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1 Introduction

Generally, systems of the following form Variational inequalities and differential equation arise in a natural way in many problems in partial differential equations, mechanics, control and optimization, and mathematical physics. In mechanics, systems variational inequalities and equation express the principle of virtual work or power in their inequality form.

We begin with a description of the general physical setting of contact problems to be studied in this paper. We first introduce the constitutive laws and the contact boundary conditions that will be employed throughout the paper. Then, we describe the contact process with adhesion and introduce the constitutive models with damage. All the variables in this chapter are assumed to have a sufficient degree of smoothness so that all the necessary mathematical manipulations are justified.

Over the years, variational inequalities have attracted increasing attention mainly due to its many applications in Mechanics and Engineering. In particular, the mathematical analysis of various contact models leads to the variational inequalities related with the convex potentials and convex unilateral constraints. References in the field include the monographs [1, 10, 12].

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We assume the reader is familiar with basic notion of Mechanics of Continua. A partial list of relevant books and monographs is [4, 5, 23]. We refer the reader to [12, 20, 25] for more details of the modeling aspects of Contact Mechanics treated in this paper

The history of variational inequalities started with a static contact problem posed in 1959 by A. Signorini. It was G. Fichera who formulated this problem as a variational inequality and used this term for the first time.

The present paper investigates a class of system of the following form

$$\begin{cases} \bullet \text{ hyperbolic variational inequality,} \\ \bullet \text{ parabolic variational inequality,} \\ \bullet \text{ differential equation,} \end{cases}$$

and is motivated by the need to obtain existence and uniqueness results for a dynamic frictional contact model for elastic long-term memory and damage material materials without viscosity. Such inequalities were mainly considered in the literature for problems which involve in the leading term the first order derivative of the unknown, see [16, 17] and the references therein. The viscosity term present in a model helps in the mathematical analysis since a problem appears to be of parabolic nature.

This work is a continuation in this line of research and we study an abstract weak formulation of dynamic frictional contact problem for an viscoelastic with long term memory material, in the framework of the MTCM, when the foundation is deformable. The tangential stress field is multivalued normal damped response, such subdifferential has been used in several recent papers investigating contact problems, see [17, 21, 24]. The normal stress field is multivalued condition is describes the friction condition, we refer to ([16], Section 6.3) for a detailed discussion on the friction laws.

The significance of damage is paramount in design engineering, directly impacting the lifespan of the constructed structure or component. Extensive engineering literature delves into this subject, exploring mathematically derived models that consider the influence of internal material damage on the contact process. Notably, general innovative damage models were formulated in [7, 8] (for recent insights, refer to our study [11]) based on the virtual power principle. One-dimensional problem analyses are available in [9]. Throughout these works, material damage is characterized by a damage function, denoted as α, constrained to values between zero and one. A value of $\alpha = 1$ indicates no damage, $\alpha = 0$ signifies complete damage, and $0 < \alpha < 1$ implies partial damage with a reduced load carrying capacity.

Contact problems incorporating damage have been explored in [**?** 19]. This paper introduces an inclusion describing the evolution of the damage field:

$$\dot{\alpha} + -k\Delta\alpha + \partial\varphi_K(\alpha) \ni S(\varepsilon(u), \alpha),$$

Here, K denotes the set of admissible damage functions defined as

$$K = \{\xi \in H^1(\Omega) \quad | \quad 0 \leq \xi \leq 1 \text{ a.e. } \in \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of set K, and S is a given constitutive function describing the sources of damage in the system.

The paper is structured as follows. In section 2 we list some notations and assumptions on the problem data and state our main existence and uniqueness result. In section 3 we present the proof of the theorem. The arguments of the proof are based on the evolutionary variational inequalities and Banach's fixed point theorem. In section 4 we give an example of application of the abstract result. The abstract weak formulation is given by the following problem

Problem P_V

Find a displacement field $\mathbf{u} : [0,T] \to V$, a damage field $\alpha : [0,T] \to H^1(\Omega)$, and an adhesion field $\beta : [0,T] \to L^2(\Gamma_3)$ such that

$$\begin{cases} \langle \ddot{\mathbf{u}}, \mathbf{v} - \dot{\mathbf{u}}(t)\rangle_{V'\times V} + \langle A\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)\rangle_{V'\times V} + \varphi(\mathbf{v}) - \varphi(\dot{\mathbf{u}}(t)) \\ \quad + (\int_0^t \Big(M(t-s), \varepsilon(\mathbf{u}(s)), \alpha(s)\Big) ds, \mathbf{v} - \dot{\mathbf{u}}(t))_V \geq \langle f(t), \mathbf{v} - \dot{\mathbf{u}}(t)\rangle_{V'\times V} \\ \text{for all } \mathbf{v} \in V, \text{ a.e. } t \in (0,T), \end{cases} \tag{1.1}$$

$$\begin{cases} (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ \qquad\qquad \geq (S(\varepsilon(\mathbf{u}(t)), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)}, \\ \text{for all } \alpha(t) \in K, \ \forall \xi \in K, \text{ a.e. } t \in [0,T], \end{cases} \tag{1.2}$$

$$\dot{\beta}(t) = F(t, \beta(t), R(\|\mathbf{u}\|)), \tag{1.3}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0 \text{ and } \beta(0) = \beta_0. \tag{1.4}$$

In this context, the symbols V, and K represent spaces corresponding to permissible displacements, and damage, respectively. These spaces are Hilbert spaces. The set K is nonempty, closed, and convex within the space $H^1(\Omega)$. The operators A and M are respectively related to the viscoelastic constitutive law with long-term memory and damage material. The functionals $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ ce determined by the conditions of contact with instantaneous normal rest and friction on part Γ_3. The data $f \in V'$ is related to the traction forces and to the body forces. The data S is related to the source function of the damage. The data F is related to the adhesion function and $R : \mathbb{R}_+ \to [0, L]$ is the truncation function defined as

$$R(s) = \begin{cases} s & \text{if } 0 \leq s \leq L, \\ L & \text{if } s > L, \end{cases}$$

where $L > 0$ is a characteristic length of the bonds (see, e.g., [2]). The function $\mathbf{u}_0$ is the initial data of the displacement field $\mathbf{u}$, $\mathbf{u}_0$ the initial velocity, α_0 the initial damage and β_0 is the initial bonding. Here $[0,T]$ is the interval of the observation. The dot above $\mathbf{u}$, α and β denotes the derivative of the displacement $\mathbf{u}$, the damage α and the adhesion β respectively with respect to the variable t

The notation $\langle \cdot, \cdot \rangle_{V'\times V}$ stands for the duality between V and V'. In the rest of the paper, we denote by C a constant whose value may change from line to line.

2 Preliminaries and notion

We first introduce preliminary notations that are needed in this chapter. Additional notations used in the mathematical of contact problems will be introduced in section 3 and 4, respectively. We denote by $\mathbb{R}^d$ the d-dimensional real linear space, and $d = 1, 2, 3$ in applications. The symbol $\mathbb{S}^d$ stands for the space of second-order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})^{1/2}, \forall \mathbf{u}, \ \mathbf{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}, \boldsymbol{\tau})^{1/2}, \forall \boldsymbol{\sigma}, \ \boldsymbol{\tau} \in \mathbb{S}^d,$$

respectively. Here and throughout this work, the indices i and j run between 1 and d, and, unless stated otherwise, the summation convention over repeated indices is used.

We will frequently use the following spaces

$$H = \{\mathbf{v} = (v_1 ... v_d), \quad v_i \in L^2(\Omega), \ 1 \leq i \leq d\} = L^2(\Omega)^d, \tag{2.1}$$

$$\mathcal{H} = \{\boldsymbol{\tau} = (\tau_{ij}), \quad \tau_{ij} = \tau_{ji}, \in L^2(\Omega), \ 1 \leq i, j \leq d\} = L^2(\Omega)_s^{d\times d}. \tag{2.2}$$

These are Hilbert spaces with the canonical inner products

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i(x) v_i(x) dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}(x)\tau_{ij}(x) dx,$$

and the associated norms are denoted by $\|\cdot\|_H$ and $\|\cdot\|_{\mathcal{H}}$, respectively. We consider a body which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit out ward normal $\boldsymbol{\nu}$. We denote by the contact boundary, and we use the usual notation u_ν for the normal component of vector $\mathbf{u}$. We consider the spaces and are real Hilbert space satisfying $V \subset L^2(\Omega) \subset V'$ with continous and dense injections. we introduce subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 | \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

On V we consider the inner product and the associated norms given by

$$(\mathbf{u},\mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{u}))_{\mathcal{H}}, \quad \forall \mathbf{u},\mathbf{v} \in V, \tag{2.3}$$

$$\|\mathbf{v}\|_V = \|\varepsilon(\mathbf{v})\|_{\mathcal{H}}, \quad \forall \mathbf{v} \in V. \tag{2.4}$$

Since $\Gamma_1 > 0$ Korn's inequality holds and there exists a constant $C_k > 0$ which depends only Ω and Γ_1 such that

$$\|\varepsilon(\mathbf{u})\|_{\mathcal{H}} \geq C_k \|\mathbf{v}\|_{H_1}, \quad \forall \mathbf{v} \in V. \tag{2.5}$$

It follows from Korn's inequality that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a constant C_0 depending only on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq C_0 \|\mathbf{v}\|_V, \quad \forall v \in V. \tag{2.6}$$

Let $\Omega \subset \mathbb{R}^d$, we shall use the notation

$$H_1 = \{\mathbf{v} = (v_i), \quad \varepsilon(\mathbf{v}) \in \mathcal{H}\}, \tag{2.7}$$

$$\mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H}, \quad \text{Div}\tau \in H\}, \tag{2.8}$$

with $\varepsilon : H \to \mathcal{H}$ and $\text{Div} : \mathcal{H} \to H$ are respectively operators of deformation and divergence. The space H_1 and $\mathcal{H}_1$ are Hilbert spaces endowed with the inner products given by

$$(\mathbf{u},\mathbf{v})_{H_1} = (\mathbf{u},\mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\text{Div}\boldsymbol{\sigma}, \text{Div}\boldsymbol{\tau})_H,$$

Let the Hilbert spaces $L^p(0,T;H)$ and $W^{k,p}(0,T;V)$ $1 \leq p \leq +\infty$,

$$L^p(0,T;H) = \{v : [0,T] \to H \text{ measurable} \quad | \quad \|v\|_{L^p(0,T;H)} < \infty\},$$
$$W^{k,p}(0,T;V) = \{v \in L^p(0,T;V) \quad | \quad \|v^{(j)}\| \in L^p(0,T;V) < \infty \quad \forall j \leq k\}.$$

When $p = 2$ or $k = 0, W^{k,2}([0,T];V)$ is written as $H^k([0,T];V)$ or $L^p([0,T];V)$, respectively. Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. Next, we impose the following assumptions. We assume that operators $A : V \to V'$, $M : [0,T] \times V \times V \to V$, the damage source function $S : V \times \mathbb{R} \to \mathbb{R}$, the adhesion rate function $F(t,.) : X \to X$, and two initial values $u_0 \in V, v_0 \in H$ and $\omega_0 \in K$, such that

$$H(1) \begin{cases} \text{The operator } A \in L(V,V') \text{ is symmetric and coercive, i.e., there exists} \\ \text{a constant } m_A > 0 \text{ such that} \\ \langle Ax,y\rangle = \langle Ay,x\rangle \text{ and } \langle Ax,x\rangle \geq m_A\|x\|^2 \text{ for all } x,y \in V. \end{cases}$$

$H(2)$ The functional $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous .

$H(3) f \in H^2(0,T;V')$.

$$H(4) \begin{cases} \text{The following condition holds} \\ y_0 \in D(\partial\varphi) \text{ and there exists } z_0 \in H \text{ such that} \\ (z_0, y - y_0) + \langle Ax_0, y - y_0\rangle + \varphi(y) - \varphi(y_0) \geq \langle f(0), y - y_0\rangle \\ \text{for all } y \in V, \text{ where } x_0, y_0 \in V. \end{cases}$$

$H(5)$ There exists a constant $L_M > 0$ such that

$$\|M(t,\mathbf{u}_1,v_1) - M(t,\mathbf{u}_2,v_2)\| \leq L_M (\|\mathbf{u}_1 - \mathbf{u}_2\| + \|v_1 - v_2\|). \tag{2.9}$$

$H(6)$ There exists a constant $L_F > 0$ such that

$$\|F(t, x_1) - F(t, x_2)\| \leq L_F \|x_1 - x_2\|. \tag{2.10}$$

$H(7)$ The function F satisfies

$$F(., x) \in L^\infty(0, T; X). \tag{2.11}$$

$$H(8) \begin{cases} \text{(a) There exists a positive number } M \text{ such that} \\ |a(\vartheta, \kappa)| \leq M\|\vartheta\|_X \|\kappa\|_X \quad \forall \vartheta, \kappa \in V. \\ \text{(b)} a(\vartheta, \kappa) = a(\kappa, \vartheta) \quad \forall \vartheta, \kappa \in V. \\ \text{(c) There exists a constants } \delta \text{ and } \gamma > 0 \\ a(\vartheta, \vartheta) + \delta\|\vartheta\|_H^2 \geq \gamma\|\vartheta\|_V^2 \quad \forall \vartheta \in V. \\ \text{(c) There exists } M_S > 0 \text{ such that} \\ \|S(\mathbf{u}_1, \alpha_1) - S(\mathbf{u}_2, \alpha_2)\| \leq M_s(\|\mathbf{u}_1 - \mathbf{u}_2\| + \|\alpha_1 - \alpha_2\|). \end{cases} \tag{2.12}$$

$H(9)$ The function S satisfies

$$S \in H^2(0, T; L^2(\Omega)). \tag{2.13}$$

Theorem 2.1. Assume that $(X, \|.\|_X)$ is a real Banach space and $T > 0$. Let $F(t, .) : X \to X$ be an operator defined a.e.on $(0, T)$ satisfying the following conditions

$$\begin{cases} \text{There exists } L_F > 0 \text{ such that} \\ \|F(t, x) - F(t, y)\|_X \leq L_F\|x - y\|_X, \qquad \forall x, y \in X, a.e.t \in (0, T), \end{cases}$$

and there exists $1 \leq p \leq +\infty$ such that $F(., x) \in L^p(0, T; X), \forall x \in X$ Then, for any $x_0 \in X$, there exists a unique function $x \in W^{1,p}(0, T; X)$ such that

$$\begin{cases} \dot{x}(t) = F(t, x(t)) \quad a.e.t \in (0, T), \\ x(0) = x_0. \end{cases}$$

Theorem 2.1 will be used in Section 3 to prove the unique solvability of the intermediate problem involving the bonding field.

Now we consider the following problem

Problem P_x

Find a function $x : [0, T] \to V$ such that

$$\begin{aligned} &\langle \ddot{x}, y - \dot{x}(t)\rangle_{V' \times V} + \langle Ax(t), y - \dot{x}(t)\rangle_{V' \times V} + \varphi(y) - \varphi(\dot{x}(t)) \\ &\qquad \geq \langle \boldsymbol{f}(t), y - \dot{x}(t)\rangle_{V' \times V} \quad \text{for all } y \in X, \text{ a.e. } t \in (0, T), \end{aligned} \tag{2.14}$$

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0. \tag{2.15}$$

Theorem 2.2. Let $H(1)$, $H(2)$, $H(3)$ and $H(4)$ hold. Then there exists a function $\mathbf{u} \in C^1(0, T; V)$ with $\dot{\mathbf{u}} \in C(0, T; H) \cap L^\infty(0, T; V)$ and $\ddot{\mathbf{u}} \in L^\infty(0, T; H)$,

The proof of the existence and uniqueness of the solution is a standard result of evolutionary hyperbolic variational inequalities it can be found for example in [18].

Problem P_ω

find a function $\omega : [0,T] \to X$ such that

$$\langle \dot{\omega}, \varpi - \dot{\omega}(t) \rangle_{X' \times X} + a(\omega(t), \varpi - \dot{\omega}(t))_X \geq (S, \varpi - \dot{\omega}(t))_H, \quad \varpi \in K, \text{ a.e. } t \in (0,T), \tag{2.16}$$

$$\omega(0) = \omega_0. \tag{2.17}$$

X and H be real Hilbert spaces such that X is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual X' of X. We write $X \subset H \subset X'$ and we say that the inclusions above defines a Gelfand triple

The following is a standard result for parabolic variational inequalities (see, e.g., [22]).

Theorem 2.3. Let $X \subset H \subset X'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of X. Assume that $a(.,.) : X \times X \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants λ and γ,

$$a(\omega, \omega) + \gamma \|\omega\|_H^2 \geq \lambda \|\omega\|_X^2, \quad \forall \omega \in X.$$

Then, for every $\omega_0 \in K$ and $S \in L^2(0,T;H)$, there exists a unique function $\omega \in H^1(0,T;H) \cap L^2(0,T;X)$, such that $\omega(0) = \omega_0$, $\omega(t) \in K$ for all $t \in [0,T]$, where ω is the unique solution of Problem P_ω.

The existence of the unique solution of Problem P_V is the topic of the next section.

3 proof of the main result

Theorem 3.1. Assume that $H(1) - H(9)$ hold. Then there exists a unique solution of the problem P_V, Moreover the solution satisfies

$$\mathbf{u} \in C^1(0,T;V) \ \text{ with } \begin{cases} \dot{\mathbf{u}} \in C(0,T;H) \cap L^\infty(0,T;V), \\ \ddot{\mathbf{u}} \in L^\infty(0,T;H), \end{cases} \tag{3.1}$$

$$\alpha \in W^{1.2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{3.2}$$

$$\beta \in W^{1.\infty}(0,T;L^\infty(\Gamma_3)). \tag{3.3}$$

The proof of theorem 3.1 is carried out is several steps and is based on the following abstract result for evolutionary variational inequalities.

Let $\eta \in H^2(0,T;V)$, and $\theta \in H^2\left(0,T;L^2(\Omega)\right)$ be given and consider the following variational problems

Problem P_V^η

Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ such that

$$\begin{aligned} &\langle \ddot{\mathbf{u}}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t) \rangle_{V' \times V} + \langle A\mathbf{u}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t) \rangle_{V' \times V} + (\eta(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \\ &+ \varphi(\mathbf{v}) - \varphi(\dot{\mathbf{u}}_\eta(t)) \geq \langle f(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t) \rangle_{V' \times V} \quad \forall \mathbf{v} \in V, \text{ a.e. } t \in (0,T), \end{aligned} \tag{3.4}$$

$$\mathbf{u}_\eta(0) = a, \quad \dot{\mathbf{u}}_\eta(0) = b. \tag{3.5}$$

Problem P_V^θ

Find the damage field $\alpha : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned} \alpha_\theta(t) \in K, \ (\dot{\alpha}_\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)} &+ a(\alpha_\theta(t), \xi - \alpha_\theta(t)) \\ &\geq (\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)}, \forall \xi \in K, \end{aligned} \tag{3.6}$$

$$\alpha_\theta(0) = \alpha_0. \tag{3.7}$$

Problem P_V^{β}

Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}_\eta(t) = F(t, \beta_\eta(t), R(\|\mathbf{u}_\eta\|)), \tag{3.8}$$
$$\beta_\eta(0) = \beta_0. \tag{3.9}$$

We have the following result for P_V^{η}

Lemma 3.2. There exists $\mathbf{u}_\eta$ a unique solution to Problem P_V^{η} and it has the regularity expressed in (3.1).

The proof of the above lemma, we apply theorem 2.2, the Riesz representation theorem allows us to define $\boldsymbol{f} : [0,T] \to V'$ by

$$(\boldsymbol{f}(t), \mathbf{v})_{V'\times V} = (f(t), \mathbf{v})_{V'\times V} - (\eta(t), \mathbf{v})_V.$$

Using hypotheses $H(1)$-$H(4)$, and we use an abstract existence and unique result which may be found in [[18], pp. 12]. In studying the problem of P_V^{θ}, we have the following result

Lemma 3.3. problem P_V^{θ} has a unique solution such that

$$\alpha_\theta \in H^1\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; H^1(\Omega)\right). \tag{3.10}$$

To solve P_V^{θ}, we leverage Hypotheses $H(8)$, $H(9)$ and apply Theorem 2.3. In the third step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 3.2 and consider the following initial-value problem.

Lemma 3.4. There exists a unique solution β_η to problem P_V^{β} such that

$$\beta_\eta \in W^{1,\infty}\left(0,T; L^2(\Gamma_3)\right). \tag{3.11}$$

Proof . We consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$F_\eta(t,\beta) = F(\beta(t), R(\|\mathbf{u}_\eta\|)),$$

for all $t \in [0,t]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R, and (2.10) that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$. Using (2.11), the mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T; L^2(\Gamma_3))$. Thus using the Cauchy-Lipschitz Theorem 2.1, we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T; L^2(\Gamma_3))$ solution of the problem P_V^{β}. □

Finally, as a consequence of these results and using the properties of operator M and function S, we consider the operator

$$\Lambda : H^2(0,T; V \times L^2(\Omega)) \to H^2(0,T; V \times L^2(\Omega)) \tag{3.12}$$

which maps every element $(\eta,\theta) \in H^2(0,T; V \times L^2(\Omega))$ to $\Lambda(\eta,\theta)(t)$ defined by

$$\Lambda(\eta,\theta)(t) = \left(\Lambda^1(\eta,\theta)(t), \Lambda^2(\eta,\theta)(t)\right) \in V \times L^2(\Omega), \tag{3.13}$$

and

$$\Lambda^1(\eta,\theta)(t) = \int_0^t M\left((t-s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds, \tag{3.14}$$
$$\Lambda^2(\eta,\theta)(t) = S\left(\mathbf{u}_\eta(t), \alpha_\theta(t)\right). \tag{3.15}$$

Lemma 3.5. The mapping Λ has a fixed point $(\eta^*,\theta^*) \in H^2\left(0,T; V \times L^2(\Omega)\right)$.

Proof . Let $t \in (0,T)$ and $(\eta_1,\theta_1),(\eta_2,\theta_2) \in H^2\left(0,T;V \times L^2(\Omega)\right)$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \alpha_{\theta_i} = \alpha_i$ and $\beta_{\eta_i} = \beta_i$, for $i = 1, 2$. Using (2.9) we have

$$\left\|\Lambda^1(\eta_1,\theta_1) - \Lambda^1(\eta_2,\theta_2)\right\|_{V'} \leq C\left(\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\, ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2\, ds\right). \tag{3.16}$$

By a similar argument ,of (3.15) and (2.14) it follows that

$$\left\|\Lambda^2(\eta_1,\theta_1) - \Lambda^2(\eta_2,\theta_2)\right\|_{L^2(\Omega)} \leq C\left(\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}\right). \tag{3.17}$$

Therefore,

$$\begin{aligned} &\|\Lambda(\eta_1,\theta_1) - \Lambda(\eta_2,\theta_2)\|_{V'\times L^2(\Omega)}^2 \leq C\Big(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \\ &+ \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2\, ds\Big). \end{aligned} \tag{3.18}$$

Using inequality (3.4) for $\eta = \eta_1, \mathbf{u}_{\eta_1} = \mathbf{u}_1$, we find

$$(\ddot{\mathbf{u}}_1, \mathbf{v} - \dot{\mathbf{u}}_1)_{V'\times V} + (A\mathbf{u}_1, \mathbf{v} - \dot{\mathbf{u}}_1)_{V'\times V} + (\eta_1, \mathbf{v} - \dot{\mathbf{u}}_1)_V + \varphi(\mathbf{v}) - \varphi(\dot{\mathbf{u}}_1) \geq (f, \mathbf{v} - \dot{\mathbf{u}}_1)_{V'\times V}\,, \tag{3.19}$$

and for $\eta = \eta_2, \mathbf{u}_{\eta_2} = \mathbf{u}_2$, we find

$$(\ddot{\mathbf{u}}_2, \mathbf{v} - \dot{\mathbf{u}}_2)_{V'\times V} + (A\mathbf{u}_2, \mathbf{v} - \dot{\mathbf{u}}_2)_{V'\times V} + (\eta_2, v - \dot{\mathbf{u}}_2)_V + \varphi(\mathbf{v}) - \varphi(\dot{\mathbf{u}}_2) \geq (f, \mathbf{v} - \dot{\mathbf{u}}_2)_{V'\times V}\,, \tag{3.20}$$

we take $\mathbf{v} = \mathbf{u}_2$ in (3.19), and $\mathbf{v} = \mathbf{u}_1$ in (3.20) by adding the results obtained we have

$$(\ddot{\mathbf{u}}_1 - \ddot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2)_{V'\times V} + (A\mathbf{u}_1 - A\mathbf{u}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2)_{V'\times V} \leq -(\eta_1 - \eta_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2)_V\,,$$

and

$$(\ddot{\mathbf{u}}_1 - \ddot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2)_{V'\times V} + (A\mathbf{u}_1 - A\mathbf{u}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2)_{V'\times V} = \frac{1}{2}\left(\frac{d}{dt}\|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_H^2 + \frac{d}{dt}(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{V'\times V}\right).$$

Hence, integrating the above inequality on $(0,T)$ and then using $H(1)$, we obtain the following inequality

$$\frac{1}{2}\left(\|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_H^2 + m_A\|\mathbf{u}_1 - \mathbf{u}_2\|_V^2\right) \leq -\int_0^T (\eta_1 - \eta_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2)_V\,,$$

on the other hand, given that $\mathbf{u}_i(t) = \int_0^t \dot{\mathbf{u}}_i(s)\, ds + \mathbf{u}_0$ we know that for

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leq C\int_0^t \|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V^2\, ds, \quad \forall t \in [0,T], \tag{3.21}$$

and, using the inequality $2ab \leq \frac{a^2}{\gamma} + \gamma b^2$ we get

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 \leq C\int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2\, ds, \quad \forall t \in [0,T]. \tag{3.22}$$

Form (3.6), we deduce that

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \leq (\theta_1 - \theta_2, \alpha_1 - \alpha_2)_{L^2(\Omega)}, \forall t \in [0,T].$$

Integrating the previous inequality with respect to time, and using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, one finds that

$$\frac{1}{2}\left\|\alpha_1(t)-\alpha_2(t)\right\|_{L^2(\Omega)}^2 \leq \int_0^t \left(\theta_1(s)-\theta_2(s),\alpha_1(s)-\alpha_2(s)\right)_{L^2(\Omega)} ds, \quad \forall t\in[0,T],$$

which in turn implies that

$$\left\|\alpha_1(t)-\alpha_2(t)\right\|_{L^2(\Omega)}^2 \leq \int_0^t \left\|\theta_1(s)-\theta_2(s)\right\|_{L^2(\Omega)}^2 ds + \int_0^t \left\|\alpha_1(s)-\alpha_2(s)\right\|_{L^2(\Omega)}^2 ds, \tag{3.23}$$

this inequality, combined with Gronwall's inequality, leads to

$$\left\|\alpha_1(t)-\alpha_2(t)\right\|_{L^2(\Omega)}^2 \leq C\int_0^t \left\|\theta_1(s)-\theta_2(s)\right\|_{L^2(\Omega)}^2 ds, \quad \forall t\in[0,T]. \tag{3.24}$$

Form the previous inequality and estimates (3.24), (3.22), (3.18) and (3.13) it follows now that

$$\left\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_2)(t)\right\|_{V\times L^2(\Omega)}^2 \leq C\int_0^t \left\|(\eta_1,\theta_1)(s)-(\eta_2,\theta_2)(s)\right\|_{V\times L^2(\Omega)}^2 ds. \tag{3.25}$$

Reiterating the previous inequality m times, we find that

$$\left\|\Lambda^m(\eta_1,\theta_1)(t)-\Lambda^m(\eta_2,\theta_2)(t)\right\|_{H^2(0,T;V\times L^2(\Omega))}^2 \leq \frac{(CT)^m}{m!}\left\|(\eta_1,\theta_1)-(\eta_2,\theta_2)\right\|_{H^2(0,T;V\times L^2(\Omega))}^2.$$

This inequality shows that for a sufficiently large m the operator Λ^m is a contraction on the Banach space $H^2\left(0,T;V\times L^2(\Omega)\right)$ and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 3.1.

Existence

Let $(\eta^*,\theta^*)\in H^2\left(0,T;V\times L^2(\Omega)\right)$, be the fixed point of Λ and denote

$$\mathbf{u}_* = \mathbf{u}_{\eta^*}, \alpha_* = \alpha_{\theta^*}, \beta_* = \beta_{\eta^*}, \tag{3.26}$$

we use $\Lambda^1(\eta^*,\theta^*)=\eta^*$, $\Lambda^2(\eta^*,\theta^*)=\theta^*$, it follows

$$\eta^*(t) = \int_0^t M\left(t-s,\mathbf{u}_*(s),\alpha_*(s)\right) ds, \tag{3.27}$$

$$\theta^*(t) = S\left(\mathbf{u}_*(t),\alpha_*(t)\right). \tag{3.28}$$

We prove $(\mathbf{u}_*,\alpha_*,\beta_*)$ satisfies (1.1)-(1.4)and the regularities (3.1)-(3.3). Indeed, we write (3.4) for $\eta=\eta^*$ and use (3.26) to find

$$\begin{aligned}&(\ddot{\mathbf{u}}_*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V} + (A\mathbf{u}_*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V} + (\eta^*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_V\\ &\quad +\varphi(\mathbf{v})-\varphi(\dot{\mathbf{u}}_*(t)) \geq (f(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V}, \quad a.e. t\in(0,T), \quad \forall \mathbf{v}\in V.\end{aligned} \tag{3.29}$$

Substitute (3.27) in (3.29) to obtain

$$\begin{aligned}&(\ddot{\mathbf{u}}_*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V} + (A\mathbf{u}_*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V}\\ &+\left(\int_0^t M\left(t-s,\mathbf{u}_*(s),\alpha_*(s)\right) ds,\mathbf{v}-\dot{\mathbf{u}}_*(t)\right)_V + \varphi(\mathbf{v})-\varphi(\dot{\mathbf{u}}_*(t))\\ &\qquad \geq (f(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V}, \quad a.e. t\in(0,T), \quad \forall \mathbf{v}\in V.\end{aligned} \tag{3.30}$$

and we write (3.6) for $\theta = \theta^*$ and use (3.26) to find

$$\alpha_*(t) \in K, \left(\dot{\alpha}_*(t), \xi - \alpha_*(t)\right)_{L^2(\Omega)} + a\left(\alpha_*(t), \xi - \alpha_*(t)\right) \geq \left(\theta^*(t), \xi - \alpha_*(t)\right)_{L^2(\Omega)}, \forall \xi \in K, \text{ a.e.t } \in (0,T), \tag{3.31}$$

we substitute (3.28) in (3.31) to obtain

$$\begin{aligned} \alpha_*(t) \in K, \left(\dot{\alpha}_*(t), \xi - \alpha_*(t)\right)_{L^2(\Omega)} + a\left(\alpha_*(t), \xi - \alpha_*(t)\right) \\ \geq \left(S\left(\mathbf{u}_*(t), \alpha_*(t)\right), \xi - \alpha_*(t)\right)_{L^2(\Omega)}, \forall \xi \in K, \text{ a.e.t } \in (0,T). \end{aligned} \tag{3.32}$$

The relations (3.30),(3.32), allow us to conclude now that $(\mathbf{u}_*, \alpha_*)$ satisfies (1.1)-(1.2). Next, (1.4)) the regularity (3.1)-(3.3) follow from Lemmas 3.2,3.3 and 3.4. Additionally, we use u_* in (3.1) and (3.26) to find

$$\dot{\beta}_*(t) = F(t, \beta_*(t), R(\|\mathbf{u}_*\|)). \tag{3.33}$$

Finally we conclude that the weak solution $(\mathbf{u}_*, \alpha_*, \beta_*)$ of the problem P_V has the regularity (3.1)-(3.3), which concludes the existence part of Theorem 3.1.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ.

4 Application: A dynamic viscoelastic with long term memory and damage contact problem

In this section, we will apply the main result of Section 3 to study a frictional dynamic contact problem viscoelastic with long term memory with damage and adhesion and subdifferential boundary conditions with convex contact and friction potentials.

We give the physical setting of the contact problem and introduce some notations which we use in the sequel. We consider a viscoelastic body which occupies a domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, such that the boundary $\Gamma = \partial\Omega$ is Lipschitz continuous. The boundary $\partial\Omega$ is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 with $\text{meas}(\Gamma_1) > 0$. In addition, we assume the body is in contact with a deformable foundation and the process is dynamic and frictional. We are interested in an evolution of the body in a finite time interval $(0, T)$.

the classical formulation of the mechanical problem may be stated as follows.

Problem P

Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, a damage field $\alpha : \Omega \times [0,T] \to \mathbb{R}$ and a adhesion field $\beta : \Gamma_3 \times [0,T] \to [0,1]$ such that

$$\boldsymbol{\sigma}(t) \in \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}(t))) + \partial\omega(\varepsilon(\mathbf{u}(t))) + \int_0^t \mathcal{M}\Big((t-s), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \alpha(s)\Big) ds, \quad \text{in } \Omega \times [0,T], \tag{4.1}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \mathbf{S}(\varepsilon(\mathbf{u}), \alpha), \quad \text{in } \Omega \times [0,T], \tag{4.2}$$

$$\rho\ddot{\mathbf{u}} = \operatorname{Div} \boldsymbol{\sigma} + \mathbf{f}_0, \quad \text{in } \Omega \times [0,T], \tag{4.3}$$

$$\mathbf{u}(t) = 0, \quad \text{on } \Gamma_1 \times [0,T], \tag{4.4}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = f_2, \quad \text{on } \Gamma_2 \times [0,T], \tag{4.5}$$

$$-\sigma_\nu(t) \in \partial h(\dot{u}_\nu(t)), \quad \text{on } \Gamma_3 \times [0,T], \tag{4.6}$$

$$-\boldsymbol{\sigma}_\tau(t) \in \partial\Psi(\dot{\mathbf{u}}_\tau(t)), \quad \text{on } \Gamma_3 \times [0,T], \tag{4.7}$$

$$\frac{\partial\alpha}{\partial\nu} = \mathbf{0}, \quad \text{on } \Gamma \times [0,T], \tag{4.8}$$

$$\dot{\beta} = H_{ad}(\beta, R(\|\mathbf{u}\|)), \quad \text{on } \Gamma_3 \times [0,T] \tag{4.9}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0 \quad \text{in } \Omega, \tag{4.10}$$

$$\beta(0) = \beta_0, \quad \text{on } \Gamma_3. \tag{4.11}$$

We now describe the notations is (4.1)-(4.11) and provide some comments on equality and boundary conditions.

We present a short description of equations and conditions in Problem P_V. First, inclusion (4.1) is a nonlinear viscoelastic with long term memory and damage constitutive law in which $\boldsymbol{\sigma} = (\sigma_{ij})$ is the stress tensor, $\mathcal{A}$ represents elasticity operator, $\mathcal{M}$ is the relaxation tensor where α is an internal variable describing the damage of the material caused by elastic deformations, $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain tensor and $\omega : \Omega \times \mathbb{S}^d \to \mathbb{R} \cup \{+\infty\}$ is a convex potential for viscosity operator. Static contact models with constitutive laws involving proper, convex and lower semicontinuous function ω cover problems with locking effects and were studied in [3]. In particular, if $\omega = 0$, then inclusion (4.1) reduces to a constitutive equation for linear elastic with long term memory and damage materials.

The evolution of the damage field is governed by the inclusion of parabolic type given by relation (4.2), where $\mathbf{S}$ is the mechanical source of the damage growth. $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K.

Equation (4.3) is the equation of motion, and we use it since we assume that the process is dynamic, where Div denotes the divergence operator in which $\mathrm{Div} = (\sigma_{ij,j})$ and $\mathbf{f}_0$ is the density of volume forces acting on the deformable body and ρ denotes the mass density. Next, conditions (4.4)and (4.5) describe the displacement and traction boundary conditions on boundaries Γ_1 and Γ_2, respectively, i.e., the body is clamped on Γ_1 and it is subjected to the density $\mathbf{f}_2$ of surface tractions on Γ_2. The contact condition (4.6) is called a multivalued normal damped response condition in which $h : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a convex function. Such subdifferential has been used in several recent papers investigating contact problems. In particular, if $h(r) = I_{(-\infty,0]}(r)$, where $I_{(-\infty,0]}$ is the indicator function of the interval $(-\infty, 0]$, i.e.,

$$I_{(-\infty,0]}(r) = \begin{cases} 0, & \text{if } r \leq 0, \\ +\infty & \text{if } r > 0, \end{cases}$$

then (4.7) reduces to the well-known Signorini contact condition in velocities of the form

$$\dot{u}_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \dot{u}_\nu \sigma_\nu = 0, \qquad \text{on } \Gamma_3 \times [0, T]. \tag{4.12}$$

This type condition also has been considered in many papers and monographs, cf.e.g., [6]. Multivalued condition (4.7) describes the friction condition in which $\psi : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a convex potential. This form incorporates several friction laws met in the literature. Relation (4.8) describes a homogeneous Neumann boundary condition, where $\partial\alpha/\partial\nu$ is the normal derivative of α. Equation (4.9) represents the ordinary differential equation which describes the evolution of the bonding field. In equation (4.10) $\mathbf{u}_0$ is the initial displacement, $\mathbf{v}_0$ is the initial velocity and α_0 is the initial damage. Finally, in equation (4.10), β_0 denotes the initial bonding.

To obtain the variational formulation of the problem (4.1)-(4.11) we introduce for the link domain of the whole. We now list the assumptions on the problem's data. The elasticity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{A}(x, \varepsilon) = a(x)\,\varepsilon \text{ for } x \in \Omega\,,\ \varepsilon \in \mathbb{S}^d, \\ (b) \quad a(x) = (a_{ijkl}(x)) \quad \textit{and} \quad a_{ijlk} = a_{jikl} = a_{ijkl} \in L^\infty(\Omega), \\ (c) \quad a_{ijkl}(x)\,\varepsilon_{ij}\varepsilon_{kl} \geq m_{\mathcal{A}} \|\varepsilon\|^2_{\mathbb{S}^d} \text{ for } x \in \Omega, \\ \qquad\qquad \text{and all } \varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^d \text{ and } m_A \succ 0. \end{cases} \tag{4.13}$$

The functions ω, h and Ψ satisfying the following hypotheses:

$$\begin{cases} \omega : \Omega \times \mathbb{S}^d \to \mathbb{R} \cup \{+\infty\} \text{ is such that} \\ (a) \quad \omega(., \varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (b) \quad \omega(x, .) \text{ is a proper, convex and lower semicontinuous} \\ \quad \text{for a.e. } x \in \Omega. \end{cases} \tag{4.14}$$

$$\begin{cases} h : \Gamma_3 \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \text{ is such that} \\ (a) \quad h(., r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}, \\ (b) \quad h(x, .) \text{ is a proper, convex and lower semicontinuous} \\ \quad \text{for a.e. } x \in \Gamma_3. \end{cases} \tag{4.15}$$

$$\begin{cases} \quad \Psi : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \text{ is such that} \\ (a) \quad \Psi(.,\xi) \text{ is measurable on } \Gamma_3 \text{ for all } \xi \in \mathbb{R}^d, \\ (b) \quad \Psi(x,.) \text{ is a proper, convex and lower semicontinuous} \\ \quad \text{for a.e. } x \in \Gamma_3. \end{cases} \tag{4.16}$$

The damage source function $S : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There exists } M_S \succ 0 \text{ such that} \\ \qquad \|S(x,\xi_1,\alpha_1) - S(x,\xi_2,\alpha_2)\| \leq M_S (\|\xi_1 - \xi_2\| + \|\alpha_1 - \alpha_2\|), \quad \forall \xi_1,\xi_2 \in \mathbb{S}^d, \quad \forall \alpha_1,\alpha_2 \in \mathbb{R}, \\ (b) \text{ The mapping } x \to \mathbb{S}(x,\xi,\alpha) \text{ is Lebesgue measurable on } \Omega, \quad \forall \boldsymbol{\varepsilon} \in \mathbb{S}_d, \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } x \to \mathbb{S}(x,\mathbf{0},0) \in L^2(\Omega). \end{cases} \tag{4.17}$$

The relaxation function $\mathcal{M} : \Omega \times (0,T) \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \text{There exists } L_{\mathcal{M}} > 0 \text{ such that } \|\mathcal{M}(x,t,\xi_1,\alpha_1) - \mathcal{M}(x,t,\xi_2,\alpha_2)\| \leq L_{\mathcal{M}} (\|\xi_1 - \xi_2\| + \|\alpha_1 - \alpha_2\|) \\ \qquad \forall t \in (0,T), \xi_1,\xi_2 \in \mathbb{S}^d \ \alpha_1,\alpha_2 \in \mathbb{R}, a.e\, x \in \Omega \\ (b) \text{ The mapping } x \to \mathcal{M}(x,t,\xi,\alpha) \text{ is Lebesgue measurable on } \Omega, \ \forall t \in (0,T), \xi \in \mathbb{S}^d, \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } x \to \mathcal{M}(x,t,\xi,\alpha) \text{ is continuous on } (0,T), \quad \forall \xi \in \mathbb{S}^d, \alpha \in, a.e\, x \in \Omega. \\ (d) \text{ The mapping } \mathbf{x} \mapsto \mathcal{M}(x,t,\xi,\alpha) \in \mathcal{H}, \quad \forall t \in (0,T). \end{cases} \tag{4.18}$$

The adhesion rate function $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}[-L,L] \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{There exists } L_{ad} \succ 0 \text{ such that} \\ \qquad \|H_{ad}(x,b_1,z_1,r_1) - H_{ad}(x,b_2,z_2,r_2)\| \leq L_{ad} (\|b_1 - b_2\| + \|z_1 - z_2\| + \|r_1 - r_2\|) \\ \qquad \forall b_1,b_1 \in [0,T], z_1,z_1 \in \mathbb{R}, r_1,r_2 \in [-L,L], a.e\,.x \in \Gamma_3 \\ (b) \text{ The mapping } x \to H_{ad}(x,b,z,r) \text{ is Lebesgue measurable on } \Gamma_3 \quad \forall b,z \in \mathbb{R}, r \in [-L,L], \\ (c) \text{ The mapping } (b,z,r) \to H_{ad}(x,b,z,r) \text{ is continuous on } \mathbb{R} \times \mathbb{R} \times [-L,L], \quad a.e.x \in \Gamma_3, \\ (d) \quad H_{ad}(x,0,z,r) = 0 \ \ \forall z \in \mathbb{R}, r \in [-L,L] \ , a.e.x \in \Gamma_3, \\ (e) \quad H_{ad}(x,b,z,r) \geq 0 \ , \forall b \leq 0, z \in \mathbb{R}, r \in [-L,L], a.e.x \in \Gamma_3 \\ \qquad H_{ad}(x,b,z,r) \leq 0 \ , \forall b \geq 1, z \in \mathbb{R}, r \in [-L,L], a.e.x \in \Gamma_3 \end{cases} \tag{4.19}$$

We assume that the densities satisfy

$$\rho \in L^\infty(\Omega), \ \textit{and there exists } \rho^* \succ 0, \ \textit{such that } \rho(x) \geq \rho^*, a.e.x \in \Omega. \tag{4.20}$$

We assume that the volume f_0 forces and the traction f_2 have the following regularities

$$f_0 \in H^2\left(0,T; L^2(\Omega)\right) \ , \ f_2 \in H^2\left(0,T; L^2(\Gamma_2)\right). \tag{4.21}$$

Initial displacement and speed satisfy

$$a, b \in V \ \ \textit{and} \ \ \omega(\varepsilon(b)) + h(b_\nu) + \Psi(b_\tau) \prec +\infty \tag{4.22}$$

(a) We establish the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as follows

$$a(\xi,\zeta) = k \int_\Omega \nabla\xi \nabla\zeta dx. \tag{4.23}$$

The microcrack diffusion coefficient is confirmed to satisfy $k > 0$.

(b) The initial damage field satisfies $\alpha_0 \in K$.

The initial adhesion field satisfies

$$\beta(0) = \beta_0 \ \ and \quad 0 \leq \beta_0 \leq 1 \ \ a.e. x \in \Gamma_3. \tag{4.24}$$

The compatibility condition reads as follows

$$-f(0) + Aa + \partial(\omega(\varepsilon(b))) + h(b_\upsilon) + \Psi(b_\tau) \subset H. \tag{4.25}$$

We define on the space $H = L^2(\Omega)^d$ of a new scalar product given by

$$(\mathbf{u}, \mathbf{v})_H = (\rho \mathbf{u}, \mathbf{v})_H \quad \forall \mathbf{u}, \mathbf{v} \in H. \tag{4.26}$$

Finally, we denote $\|.\|_{V'}$ the norm on the dual V' space. Riesz's representation theorem leads to the existence of an element $f \in V'$, such that

$$(f(t), \mathbf{v})_{V' \times V} = (f_0(t), \mathbf{v})_H + (f_2(t), \mathbf{v})_{L^\infty(\Omega)}, \ \forall \mathbf{v} \in V, a.e.\, t \in (0, T). \tag{4.27}$$

Now let $\varphi : V \to \cup \{+\infty\}$, the application defined by

$$\varphi(\mathbf{v}) = \int_\Omega \omega(\varepsilon(\mathbf{v}))\, dv + \int_\Omega (h(\mathbf{v}_\upsilon) + \Psi(\mathbf{v}_\tau))\, dv, \quad \forall \mathbf{v} \in V, \tag{4.28}$$

The variational formulation of the problem P is given by

$$\begin{cases} (a) \quad \sigma(t) \in \mathcal{A}(\varepsilon(\mathbf{u}(t))) + \partial\omega(\varepsilon(\dot{\mathbf{u}}(t))) + \int_0^t \mathcal{M}((t-s), \varepsilon(\mathbf{u}(t)), \alpha(s))\, ds \\ (b) \quad (\ddot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_H + (\sigma(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t)))_{\mathcal{H}} + \varphi(\mathbf{v}) - \varphi(\dot{\mathbf{u}}(t)) \geq (f(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V} \\ \qquad a.e. \ \ t \in (0, T), \quad for \ \ all \ \mathbf{v} \in V \\ (c) \quad (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \geq (S(\varepsilon(\mathbf{u}(t)), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)} \\ \qquad \xi \in K \quad a.e. \ \ t \in (0, T) \\ (d) \quad \dot{\beta} = H_{ad}(\beta, \varsigma_\beta, R(\|\mathbf{u}_\tau\|)), \ 0 \leq \beta(t) \leq 1, a.e. \ t \in (0, T) \\ (e) \quad \mathbf{u}(0) = a \ , \ \dot{\mathbf{u}}(0) = b \ , \ \alpha(0) = \alpha_0 \ , \ \beta(0) = \beta_0 \end{cases} \tag{4.29}$$

Remark 4.1. We note that, in Problem P and in variational formulation of the problem P, we do not need to impose explicitly the restriction $0 \leq \beta \leq 1$. Indeed, equation (4.29)(d) guarantees that $\beta(x,t) \leq \beta_0(x)$ and, therefore, assumption (4.24) shows that $\beta(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\beta(x, t_0) = 0$ at time t_0, then it follows from (4.29)(d) that $\dot{\beta}(x,t) = 0$ for all $t \geq t_0$ and therefore, $\beta(x,t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \beta(x,t) \leq 1$ for all $t \in [0, T]$, a.e.$x \in \Gamma_3$.

Theorem 4.2. Assume that hypotheses (4.13)-(4.25) hold. Then Problem P has a unique solution.

Proof . The proof is based on Theorem. Note on one hand that the continuous imbeddings $V \subset H \subset V'$ and the Riesz representation theorem allow us to define the operator

$$(A\mathbf{u}(t), \mathbf{v})_{V' \times V} = (\mathcal{A}(\varepsilon(\mathbf{u}(t))), \varepsilon(\mathbf{v}))_{\mathcal{H}}. \tag{4.30}$$

We will verify the hypotheses $H(1)$-$H(9)$ of Theorems 3.1. It is easy to see that under hypotheses (4.13), operator A defined by (4.30) satisfies $H(1)$ with $m_A = m_{\mathcal{A}}$. Using the convexity of w, h and Ψ, it is clear that φ is convex. From conditions ((4.14)-(4.16), and (4.28), we deduce that $H(2)$ is satisfied. Next, by the regularity hypothesis (4.21), we obtain that $H(3)$ holds. Finally, conditions (4.22) and (4.25) entail that the initial conditions has regularity required in $H(4)$ and that this condition holds. Namely, since $b \in V$ is such that $w(\varepsilon(b)) + h(b_\nu) + \Psi(b_\tau) < +\infty$ and $-f(0) + Aa + \partial(w(\varepsilon(b)) + h(b_\nu) + \Psi(b_\tau)) \subset H$, we have $b \in D(\varphi)$. We take $z_0 \in H$ such $z_0 \in -f(0) + Aa + \partial(w(\varepsilon(b)) + h(b_\nu) + \Psi(b_\tau))$.

We deduce that the inequality stated in $H(4)$ is satisfied. Finally, by condtions (4.17),(4.19), we find $H(5)$-$H(9)$ are met. Having verified the hypotheses of Theorems 3.1, we deduce that there is a unique weak solution to the Problem P that satisfies (4.29) with Regularity (3.1)-(3.3).

□

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