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Additive results for gπ-Hirano inverses in Banach algebras

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Abstract

An element a in a Banach algebra $\mathcal{A}$ has generalized π-Hirano inverse if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a^n - ab \in \mathcal{A}^{qnil} \text{ for some } n \in \mathbb{N}.$$

New additive results for the generalized π-Hirano inverse of an element in a Banach algebra are presented. Applications to operator matrices are thereby obtained.

Keywords: generalized π-Hirano inverse; additive property; operator matrix; spectral idempotent.
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1 Introduction

Throughout the paper, X is a Banach space, $\mathcal{L}(X)$ is a Banach algebra of bounded linear operators on X and $\mathcal{A}$ denotes an arbitrary Banach algebra. $U(\mathcal{A})$ is the set of all units in $\mathcal{A}$. The commutant of $a \in \mathcal{A}$ is defined by $\{x \in \mathcal{A} \mid xa = ax\}$ and is denoted by $comm(a)$. Following Koliha [12], an element a in $\mathcal{A}$ has g-Drazin inverse if there exists $b \in \mathcal{A}$ such that, $ab = ba, b = bab$ and $a - a^2b \in \mathcal{A}^{qnil}$. Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 - ax \in U(\mathcal{A}) \text{ for any } x \in comm(a)\}$ and it is the set of all quasinilpotent elements in $\mathcal{A}$. As is well known (see [11, Theorem 9.5.3]),

$$a \in \mathcal{A}^{qnil} \Leftrightarrow \lim_{n\to\infty} \parallel a^n \parallel^{\frac{1}{n}} = 0$$

The preceding b, if exists, is unique, and is denoted by a^d. We call a^d the g-Drazin inverse of a, use $\mathcal{A}^d$ to denote the set of all elements have g-Drazin inverses in $\mathcal{A}$. It was proved that $a \in \mathcal{A}^d$ if and only if there exists an idempotent $p \in \mathcal{A}$ such that $ap = pa$, $a + p$ is invertible and $ap \in \mathcal{A}^{qnil}$, i.e., $a \in \mathcal{A}$ is quasipolar (see [13, Theorem 4.2]). The element $a^\pi = 1 - aa^d$ is called the spectral idempotent of a.

An element $a \in \mathcal{A}$ has gs-Drazin inverse if there exists $b \in \mathcal{A}$ such that $b = bab, ab = ba$ *and* $a - ab \in \mathcal{A}^{qnil}$, (see [3]). Equvalently, an element a in a Banach algebra $\mathcal{A}$ has gs-Drazin inverse if and only if it can be written as the sum of an idempotent and a quasinilpotent that commute. The gs-Drazin inverse is unique if exists and the gs-Drazin inverse of a is denoted by a^{gs}. $\mathcal{A}^{gs}$ stands for the set of all gs-Drazin invertible elements in $\mathcal{A}$. Also an element a

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in $\mathcal{A}$ has g-Hirano inverse if there exists $b \in \mathcal{A}$ such that, $ab = ba, b = bab$ and $a^2 - ab \in \mathcal{A}^{qnil}$, [4]. In this paper we introduce a generalization of generalized Hirano inverse, called generalized π-Hirano inverse (gπH-inverse in short) that is, an element a in a Banach algebra $\mathcal{A}$ has generalized $\pi-$Hirano inverse if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a^n - ab \in \mathcal{A}^{qnil} \text{ for some } n \in \mathbb{N}.$$

This inverse is unique if exists and will be denoted by $a^{g\pi H}$ and $\mathcal{A}^{g\pi H}$ stands for the set of all generalized π-Hirano invertible elements in $\mathcal{A}$.

In Section 2, we present new conditions on two elements $a, b \in \mathcal{A}^{g\pi H}$ under which $a+b$ in $\mathcal{A}$ has gπ-Hirano inverse. In Section 3, we consider the gπ-Hirano inverse of a 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \mathcal{L}(X), B \in \mathcal{L}(X,Y), C \in \mathcal{L}(Y,X), D \in \mathcal{L}(Y)$ have gπ-Hirano inverses and X, Y are complex Banach spaces. Here, M is a bounded operator on $X \oplus Y$. The g-Drazin and g-Hirano inverses of such linear oprator matrices were extensively studied (See [4, 5, 6, 16] respectivly). We present the gπ-Hirano inverse for a 2×2 oprator matrix M under a number of diffrent conditions, which generalize [21, Theorem 2.1 and Theorem 2.2]. Now we will state some auxiliary lemmas.

Lemma 1.1. Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:

(1) *a has gs-Drazin inverse;*
(2) *There exists* $e^2 = e \in comm(a)$ *such that* $a - e \in \mathcal{A}^{qnil}$*;*
(3) $a - a^2 \in \mathcal{A}^{qnil}$. In this case, $a^{gs} = a^d$.

Proof . See [10, Theorem 3.2], [3, Theorem 3.2]. □

Lemma 1.2. Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$. If $a, b \in \mathcal{A}$ have g-Drazin inverses, then $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A})$ has g-Drazin inverse.

Proof . See [9, Lemma 2.2]. □

Lemma 1.3. Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$. If $a, b \in \mathcal{A}$ have gs-Drazin inverses, then $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A})$ has gs-Drazin inverse.

Proof . This is obvious by Lemma 1.2 and Lemma 1.1. □

2 Additive results

The purpose of this section is to establish the new results for the gπ-Hirano inverse of the $a+b$ which will be used in the sequel. We begin with the following lemma.

Lemma 2.1. Let $a, b \in \mathcal{A}^{qnil}$. If $ab^2 = 0, aba = 0$ then $a + b \in \mathcal{A}^{qnil}$.

Proof . Since $a, b \in \mathcal{A}^{qnil}$, we see that, a, b has g-Drazin inverse and $a^d = 0, b^d = 0$. In view of [20, Theorem 2.1], we see that $(a+b)^d = 0$. Therefore, $a + b = (a+b) - (a+b)^2(a+b)^d \in \mathcal{A}^{qnil}$. This completes the proof. □

Theorem 2.2. Let $a, b \in \mathcal{A}^{gs}$. If $a^3b = 0, bab = 0$ and $ba^2b = 0$, then $a + b \in \mathcal{A}^{g\pi H}$.

Proof . Set

$$M = \begin{pmatrix} a^3 + a^2b + aba + ab^2 & a^2b^2 + ab^3 \\ a^2 + ab + ba + b^2 & a^2b + ab^2 + b^3 \end{pmatrix}.$$

Then,

$$M = \begin{pmatrix} a^2b+aba+ab^2 & a^2b^2+ab^3 \\ 0 & a^2b+ab^2 \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2+ab+ba+b^2 & b^3 \end{pmatrix}$$
$$:= G+F.$$

We see that $G^2=0$ and $FGF=0$. Moreover, we have,

$$F = \begin{pmatrix} a^3 & 0 \\ a^2+ab+ba+b^2 & b^3 \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ a^2+ba & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b^2+ab & b^3 \end{pmatrix}$$
$$:= H+K.$$

It is easily verified that,

$$H = \begin{pmatrix} a^3 & 0 \\ a^2+ba & b^3 \end{pmatrix} = \begin{pmatrix} a^2 \\ a+b \end{pmatrix} \begin{pmatrix} a & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} a & 0 \end{pmatrix}\begin{pmatrix} a^2 \\ a+b \end{pmatrix} = a^3$, by [2, Lemma 3.1], $a^3 \in \mathcal{A}^{gs}$, it follows by Cline's formula that H has gs-Drazin inverse, see [3, Corollary 2.5]. Also we have

$$K = \begin{pmatrix} 0 & 0 \\ b^2+ab & b^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b^2+ab & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b^3 \end{pmatrix} = K_1+K_2,$$

where K_1 is nilpotent, hence it is gs-Drazin invertible and b^3 is gs-Drazin invertible, so K_2 is gs-Drazin invertible, and $K_1K_2 = K_2K_1 = 0$ so K has gs-Drazin inverse. Clearly, $HK=0$, by [3, Corolary 3.3] F has gs-Drazin inverse.

$$\begin{aligned} M &= \left(\begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & b \end{pmatrix}\right)^3 \\ &= \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}^3 \\ &= \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}\begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}\begin{pmatrix} a & ab \\ 1 & b \end{pmatrix} \\ &= \begin{pmatrix} a^2+ab & a^2b+ab^2 \\ a+b & ab+b^2 \end{pmatrix}\begin{pmatrix} a & ab \\ 1 & b \end{pmatrix} \\ &= \begin{pmatrix} (a^2+ab)a+a^2b+ab^2 & (a^2+ab)ab+(a^b+ab^2)b \\ a^2+ba+ab+b^2 & a^2b+bab+ab^2+b^3 \end{pmatrix} \end{aligned}$$

By using $a^3b = bab = ba^2b = 0$, we have,

$$\begin{aligned} \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}^3 &= \begin{pmatrix} a^3+aba+a^2b+ab^2 & a^3b+abab+a^2b^2+ab^3 \\ a^2+ba+ab+b^2 & a^2b+bab+ab^2+b^3 \end{pmatrix} \\ &= \begin{pmatrix} a^3+aba+a^2b+ab^2 & a^2b^2+ab^3 \\ a^2+ba+ab+b^2 & a^2b+ab^2+b^3 \end{pmatrix}. \end{aligned}$$

It follows that $\begin{pmatrix} a \\ 1 \end{pmatrix}\begin{pmatrix} 1 & b \end{pmatrix}$ has gπ-Hirano inverse, then by Cline's formula $\begin{pmatrix} 1 & b \end{pmatrix}\begin{pmatrix} a \\ 1 \end{pmatrix} = a+b$ has gπ-Hirano inverse. □

Corollary 2.3. Let $a, b \in \mathcal{A}^{gs}$. If $ab = ba = 0$, then $a+b \in \mathcal{A}^{gs}$

Proof . This is obvious by Theorem 2.2. □

In [20], Wang et al. considered the g-Drazin inverse of $P+Q$ under the conditions that $Q^2P+QP^2 = 0, PQP = 0$ and $QPQ = 0$ for bounded linear operators P and Q. We now extend this result to gπ-Hirano inverse in a Banach algebra.

Theorem 2.4. Let $a, b \in \mathcal{A}^{gs}$. If $a^2b + ab^2 = 0, ab^3 = 0$ and $aba = 0$, then $a + b \in \mathcal{A}^{g\pi H}$ for some $n \in \mathbb{N}$.

Proof . Set

$$M = \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & bab + b^3 \end{pmatrix}.$$

Then

$$\begin{aligned} M &= \begin{pmatrix} 0 & 0 \\ 0 & bab \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= F + G. \end{aligned}$$

We see that $F^3 = 0$ and by hypothesis, we easily check that,

$$F^3G = 0,\ GFG = 0,\ GF^2G = 0.$$

As in the proof Theorem 2.2, G has gπ-Hirano inverse. Since F is nilpotent, it has gs-Drazin inverse, by Theorem 2.2, M has gπ-Hirano inverse. Similary to the proof in Theorem 2.2, $a + b$ has gπ-Hirano inverse. □

Corollary 2.5. Let $a, b \in \mathcal{A}^{gs}$. If $ab = 0$, then $a + b \in \mathcal{A}^{gs}$.

Proof . This is obvious by Theorem 2.4. □

Lemma 2.6. Let $a, b \in \mathcal{A}^{qnil}$. If $aba = 0, bab = 0, a^2b^2 = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^{qnil}$.

Proof . Set

$$M = \begin{pmatrix} a^3 + a^2b + ab^2 & a^3b \\ a^2 + ab + ba + b^2 & a^2b + ab^2 + b^3 \end{pmatrix}.$$

Then,

$$\begin{aligned} M &= \begin{pmatrix} a^2b + ab^2 & a^3b \\ 0 & a^2b + ab^2 \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= G + F. \end{aligned}$$

We see that $G^4 = 0$ and $GF = 0$. Since $a, b \in \mathcal{A}^d$, it follows by [12, Theorem 5.4] that $a^3, b^3 \in \mathcal{A}^d$. By virtue of Lemma 1.2, F has g-Drazin inverse. According to [4, Theorem 2.2], M has g-Drazin inverse. In light of [4, Theorem 2.2] , $(a + b)^d = 0$. This implies that $a + b \in \mathcal{A}^{qnil}$.□

Theorem 2.7. Let $a, b \in \mathcal{A}^{gs}$. If $aba = 0, bab = 0, a^2b^2 = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^{g\pi H}$.

Proof . Set

$$M = \begin{pmatrix} a^3 + a^2b + ab^2 & a^3b \\ a^2 + ab + ba + b^2 & a^2b + ab^2 + b^3 \end{pmatrix}.$$

Then,

$$\begin{aligned} M &= \begin{pmatrix} a^2b + ab^2 & a^3b \\ 0 & a^2b + ab^2 \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= G + F. \end{aligned}$$

We see that $G^4 = 0$ and $GF = 0$. Moreover, we have,

$$\begin{aligned} F &= \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ a^2 + ba & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b^2 + ab & b^3 \end{pmatrix} \\ &:= H + K. \end{aligned}$$

It is easily verified that

$$H = \begin{pmatrix} a^3 & 0 \\ a^2 + ba & b^3 \end{pmatrix} = \begin{pmatrix} a^2 \\ a+b \end{pmatrix} \begin{pmatrix} a & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} a & 0 \end{pmatrix} \begin{pmatrix} a^2 \\ a+b \end{pmatrix} = a^3 \in \mathcal{A}^{g\pi}$-H it follows by Clin's formula that H has gπ-Hirano inverse, (See [3, Corollary 2.5].

Similarly, K has gπ-Hirano inverse. Clearly, $HK = 0$, by Corollary 2.5, F has gπ-Hirano inverse. As $G^{gs} = 0$ by Corollary 2.5, again, M has gπ-Hirano inverse, obviously $M = \left(\begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & b \end{pmatrix} \right)^3$, it follows that $\begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & b \end{pmatrix}$ has gπ-Hirano inverse, then by Cline's formula that $\begin{pmatrix} 1 & b \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = a + b$ has gπ-Hirano inverse. □

Lemma 2.8. Let $a, b \in \mathcal{A}^{qnil}$. If $a^2 b = 0$ and $bab = 0$, then $a + b \in \mathcal{A}^{qnil}$.

Proof . Let $x = b^2 + ab$ and $y = ba + a^2$. Then $(b+a)^2 = x + y$. Clearly, $a^2, ba, ab, b^2 \in \mathcal{A}^{qnil}$. Since $b^2(ab) = 0$ and $a^2(ba) = 0$, it follows by [4, Lemma 3.2] that $x, y \in \mathcal{A}^{qnil}$. Moreover, $yx = (ba + a^2)b^2 + (b + a)a^2 b = 0$, and so $x + y \in \mathcal{A}^{qnil}$ by [4, Lemma 3.2]. Therefore $a + b \in \mathcal{A}^{qnil}$, as asserted. □

Example 2.9. Let

$$P = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} -I & -I & 0 \\ 0 & 0 & -I \\ I & I & I \end{pmatrix}$$

where I is the identity operator on a Banach space X. Here P, Q has gs-Drazin inverse and $P^2Q + PQ^2 = 0, PQ^3 = 0$ and $PQP = 0$. Then by Theorem 2.4, $P + Q$ has gπ-Hirano inveres but $QPQ \neq 0$.

3 Splitting approach

To illustrate the preceding results, we are concerned with the gπ-Hirano inverse for a operator matrix. Here the operator matrix M is

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \mathcal{L}(X)^{gs}, D \in \mathcal{L}(Y)^{gs}, B \in \mathcal{L}(X,Y)$ *and* $C \in \mathcal{L}(Y,X)$. Using different splitting approach, we will obtain various conditions for the gπ-Hirano inverse of M.

Theorem 3.1. If $A^2BC = 0, A^2BD = 0, CBC = 0, CBD = 0, CABC = 0$ and $CABD = 0$, then M has gπ-Hirano inverse.

Proof . Let $p = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$, then $M = p + q$. By Lemma 1.3, it is obvious that p, q have gπ-Hirano invertible. Now we have

$$p^3 q = \begin{pmatrix} A^2BC & A^2BD \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$qpq = \begin{pmatrix} 0 & 0 \\ CBC & CBD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$qp^2q = \begin{pmatrix} 0 & 0 \\ CABC & CABD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 2.2, M has gπ-Hirano inverse. □

Corollary 3.2. If $ABC = 0, ABD = 0, CBC = 0$ and $CBD = 0$, then M has gπ-Hirano inverse.

Proof . This is obvious by Theorem 3.1.□

Regarding a complex matrix as the operator matrix on $\mathbb{C} \times \mathbb{C}$, we now show that Corollary 3.2, is a non-trivial generalization of [7, Theorem 2].

Example 3.3. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

be complex matrices and set

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then

$$ABC = 0,\ ABD = 0,\ CBC = 0,\ CBD = 0.$$

In view of Corollary 3.2, M has gπ-Hirano inverse but $BC \neq 0$ and $BD \neq 0$.

Theorem 3.4. If $A^3B = 0, CA^2B = 0, BCB = 0, DCB = 0$, $BCAB = 0$ and $DCAB = 0$, then M has gπ-Hirano inverse.

Proof . Let $p = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}$, then $M = p + q$. It is clear that, p and q have gs-Drazin inverses. Now we have

$$p^3q = \begin{pmatrix} 0 & A^3B \\ 0 & CA^2B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$qpq = \begin{pmatrix} 0 & BCB \\ 0 & DCB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$qp^2q = \begin{pmatrix} 0 & BCAB \\ 0 & DCAB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 2.2, M has gπ-Hirano inverse.□

Corollary 3.5. If $A^2B = 0, BCB = 0, DCB = 0$ and $CAB = 0$, then M has gπ-Hirano inverse.

Proof . It is a special case of Theorem 3.4.□

Theorem 3.6. If $(A^2 + BC)BD = 0, CABD = 0, DCBD = 0$, $ABC = 0$ and $CBC = 0$, then M has gπ-Hirano inverse.

Proof . Let $p = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$, then $M = p + q$. Since $ABC = 0, CBC = 0$, it follows by Corollary 3.2, that p has gs-Drazin inverse. By Lemma 1.3, q has gs-Dazin inverse. Now we have,

$$p^3q = \begin{pmatrix} 0 & A^2BD + BCBD \\ 0 & CABD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$qpq = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$qp^2q = \begin{pmatrix} 0 & 0 \\ 0 & DCBD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 2.2, M has gπ-Hirano inverse.□

Corollary 3.7. If $BC = 0, BD = 0$, then M has gπ-Hirano inverse.

Proof . This is clear by Theorem 3.6. □

We are now ready to prove:

Theorem 3.8. If $A^3B = 0, CAB = 0, CA^2B = 0, BCB = 0$ and $DCB = 0$, then M has gπ-Hirano inverse.

Proof . Let $p = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$, then $M = p + q$. Clearly, p has gs-Drazin inverse. Since $CBC = 0$ and $CBD = 0$, it follows by Corollary 3.5, q has g-Dazin inverse. We check that

$$p^3q = \begin{pmatrix} 0 & A^3B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$qpq = \begin{pmatrix} 0 & 0 \\ 0 & CAB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$qp^2q = \begin{pmatrix} 0 & 0 \\ 0 & CA^2B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Theorem 2.2, M has gπ-Hirano inverse.□

As an immediate consequence of Theorem 3.8, we now derive

Corollary 3.9. If $AB = 0$ and $CB = 0$, then M has gπ-Hirano inverse.

Example 3.10. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, D = 0,$$

be complex matrices. Then

$$A^3B = 0, CAB = 0, CA^2B = 0, BCB = 0, DCB = 0.$$

By Theorem 3.8, M has gπ-Hirano inverse, but $AB \neq 0$ and $CB \neq 0$.

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