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Uniqueness results on certain polynomials of meromorphic functions sharing a small function

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Abstract

This study explores specific polynomials that share a small function by applying the notions of weakly weighted and relaxed weighted sharing of meromorphic functions. In particular, the research delves into the uniqueness of two polynomial types related to the meromorphic function f: the homogeneous differential polynomial $\Phi[z]$ and the non-constant differential-difference polynomial $\Psi[z,f]$ and examines the value distribution of these polynomial functions within the context of weakly weighted and relaxed weighted sharing, resulting in the equation $\Phi[z] \equiv \Psi[z,f]$ (or) $\Phi[z].\Psi[z,f] \equiv a^2$. The findings of this investigation enhance and extend the previous work of Harina P. Waghamore and Vijayalakshmi S. B. [27].

Keywords: Uniqueness, meromorphic function, homogeneous differential polynomial, Differential-difference polynomial, Weakly weighted sharing, Relaxed weighted sharing
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1 Introduction and main results

It is well known that, the value distribution theory is a fundamental area of complex analysis that studies the behavior of meromorphic functions and their value-sharing properties. This involves examining the conditions under which two meromorphic functions sharing certain values must be identical or have a specific relationship. It mainly analyzes the distribution of solutions to the equation $f(z) = a$, where $f(z)$ is an entire or meromorphic function in $\mathbb{C}, z \in \mathbb{C}$, and $a \in \mathbb{C} \cup \{\infty\}$. Hayman [9], Yi and Yang [34], and Yang [37] provide standard definitions and notations for Nevanlinna theory. The Nevanlinna characteristic function is denoted by $T(r,f)$, and $S(r,f)$ is a small quantity defined by $o(T(r,f)) = S(r,f)$ as $r \to \infty$ and $r \notin E$, where $E \subseteq \mathbb{R}^+$ and the measure of E is finite. For $a \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)},$$

and

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

are respectively called the “deficiency” and “ramification index” of a for the function f.

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In order to ensure our paper is comprehensive, we present the following fundamental definitions from Nevanlinna theory.

Definition 1.1. Let f and g be two meromorphic functions in the complex plane $\mathbb{C}$ and let a is a small function with respect to f and g. We write $E(a, f) = \{z \in \mathbb{C} : f(z) - a(z) = 0\}$, where zeros of $f - a$ are counted according to their multiplicities. Also by $\overline{E}(a, f)$, we denote the zeros of $f - a$, where a zero is counted only once. We say that f and g share the function a CM (counting multiplicity) if $E(a, f) = E(a, g)$. Further, if $\overline{E}(a, f) = \overline{E}(a, g)$ we say that f and g share the function a IM (ignoring multiplicity).

Definition 1.2. [11] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_k(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the function a with weight k. We write f and g share (a, k) to mean that f and g share the function a with weight k. Since $E_k(a, f) = E_k(a, g)$ implies that $E_l(a, f) = E_l(a, g)$, for any integer $l(0 \leq l < k)$, if f, g share (a, k), then f, g share $(a, l), (0 \leq l < k)$. Moreover, we note that f and g share the function $a(z)$ IM (ignoring multilicity) or CM (counting multiplicity) if and only if f and g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.3. [15] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a(z) \in \mathbb{C}, t \in \mathbb{Z}^+ \cup \{\infty\}$. If

$$\overline{N}_{t)}(r, a; f) + \overline{N}_{t)}(r, a; g) - 2\overline{N}^E_{t)}(r, a) = S(r, f) + S(r, g),$$

$$\overline{N}_{(t+1}(r, a; f) + \overline{N}_{(t+1}(r, a; g) - 2\overline{N}^0_{(t+1}(r, a) = S(r, f) + S(r, g),$$

or, if $t = 0$ and

$$\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_0(r, a) = S(r, f) + S(r, g),$$

then we say that f and g weakly share a with weight t and the notion will be denoted by $\varkappa(a, t)$. Here, we write f, g share $\varkappa(a, t)$ to mean that f, g weakly share a with weight t.

Obviously, if f and g share function $\varkappa(a, t)$, then for any $p(0 \leq p \leq t)$, f and g share $\varkappa(a, p)$. In addition, f and g share a IM or CM if and only if they share $\varkappa(a, 0)$ or $\varkappa(a, \infty)$, respectively.

In 2007, Banerjee and Mukherjee [3] introduced a new type of sharing known as relaxed weighted sharing which is weaker than weakly weighted sharing as follows.

Definition 1.4. [3] We denote by $N(r, a; f| = p; g| = q)$ the reduced counting function of common a-points of f and g with multiplicities p and q, respectively.

Definition 1.5. [3] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. Suppose that f and g share the value a IM. If for $p \neq q$,

$$\sum_{p,q \leq k} N(r, a; f| = p; g| = q) = S(r),$$

then we say that f and g share the value a with weight k in a relaxed manner and we write f and g share $\varkappa(a, k)^*$

The study of value sharing of meromorphic and entire functions has been a central theme in complex analyses for decades. Rooted in Nevanlinna theory, which provides a framework for understanding the value distribution of meromorphic functions, the concept of value sharing has led to advancements in function theory and uniqueness problems. Research has focused on exploring conditions under which meromorphic functions sharing values exhibit identical or closely related behaviors.

R. Nevanlinna's pioneering work laid the foundation for value distribution theory, introducing key concepts such as the First and Second Main Theorems. These results open the door for further investigation of uniqueness theorems based on shared values. Mathematicians such as Hayman, Gundersen, Yi, and others have extended these ideas by developing new uniqueness criteria and refining the classification of value-sharing into types, such as ignoring multiplicities (IM) and counting multiplicities (CM), to be more general. Later, the study included weighted sharing, difference sharing of polynomials, etc., and their role in uniqueness theory. Researchers have examined how value-sharing properties of functions extend to their derivatives and algebraic combinations, leading to differential and functional equations; one can refer to [1, 12, 17, 24, 32, 30, 31, 29, 39, 40].

Subsequently, a number of authors have shown interest in finding the uniqueness of entire and meromorphic functions whose differential polynomials share certain values or fixed points and obtained some remarkable results (see [2, 8, 14, 20, 26]). In recent years, some mathematicians worldwide have begun to study the uniqueness questions of meromorphic functions sharing values with their shifts, differences, and difference polynomials, and have produced many works; for example, see ([5, 6, 10, 18, 19, 21, 22, 23, 36, 38]).

Since then, a number of improvements and generalizations have been made by mathematicians regarding the uniqueness of $f^{(k)}$ and $P[f]$. In 2014, Banerjee and Majumder [4] explored the weighted sharing of f^n and $(f^m)^{(k)}$ and demonstrated the following:

Theorem A. [4] Let f be a non-constant meromorphic function, $k, n, m \in N$ and l be a non-negative integer. Suppose $a(\not\equiv 0, \infty)$ is a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$ such that f^n and $(f^m)^{(k)}$ share (a, l). If

(i) $l \geq 2$ and

$$(k+3)\Theta(\infty, f) + (k+4)\Theta(0, f) > 2k + 7 - n, \quad \text{or}$$

(ii) $l = 1$ and

$$\left(k + \frac{7}{2}\right)\Theta(\infty, f) + \left(k + \frac{9}{2}\right)\Theta(0, f) > 2k + 8 - n, \quad \text{or}$$

(iii) $l = 0$ and

$$(2k+6)\,\Theta(\infty, f) + (2k+7)\,\Theta(0, f) > 4k + 13 - n,$$

then $f^n = (f^m)^{(k)}$.

In 2016, regarding the uniqueness of $p(f)$ and $P[f]$, the authors Kuldeep Singh Charak and Banarsi Lal [7] demonstrated the following result:

Theorem B. [7] Let f be a non-constant meromorphic function, $a(\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$, and $p(z)$ be a polynomial of degree $n \geq 1$ with $p(0) = 0$. Let $P[f]$ be a non-constant differential polynomial of f. suppose $p(f)$ and $P[f]$ share (a, l) with one of the following conditions:

(i) $l \geq 2$ and

$$(Q+3)\Theta(\infty, f) + 2n\Theta(0, p(f)) + \overline{d}(P)\delta(0, f) > Q + 3 + 2\overline{d}(P) - \underline{d}(P) + n,$$

(ii) $l = 1$ and

$$\left(Q + \frac{7}{2}\right)\Theta(\infty, f) + \frac{5n}{2}\Theta(0, p(f)) + \overline{d}(P)\delta(0, f)) > Q + \frac{7}{2} + 2\overline{d}(P) - \underline{d}(P) + \frac{3n}{2},$$

(iii) $l = 0$ and

$$(2Q+6)\,\Theta(\infty, f) + 4n\Theta(0, p(f)) + 2\overline{d}(P)\delta(0, f)) > 2Q + 6 + 4\overline{d}(P) - 2\underline{d}(P) + 3n,$$

then $p(f) \equiv P[f]$.

In 2018, Harina P. Waghamore and Naveenkumar S.H. [25] established the uniqueness of meromorphic functions of the form $P(f) = f_1^p P(f_1) - a$ and $H[f] - a$ and obtained the following result:

Theorem C. [25] Let $k(\geq 1), n(\geq 1), p(\geq 1)$ and $m(\geq 0)$ be integers, f and $f_1 = f - \omega_p$ be two non-constant meromorphic functions and $H[f]$ be a non-constant differential polynomial generated by f. Let $p(z) = a_{m+n}z^{m+n} + \cdots + a_n z^n + \cdots + a_0, a_{m+n} \neq 0$ be a polynomial in z of degree $m + n$ such that $p(f) = f_1^p P(f_1)$. Also, let $a(z)(\not\equiv 0)$ be a small function with respect to f. Suppose $p(f) - a$ and $H[f] - a$ share $(0, l)$ with one of the following conditions.

(i) $l \geq 2$ and

$$(Q+3)\Theta(\infty, f)+\mu_2\delta_{\mu_2^*}(w_p, f)+\overline{d}(H)\delta_{k+2}(0, f) > Q+3+\mu_2+\overline{d}(H)-p, \quad \text{or}$$

(ii) $l = 1$ and

$$\left(Q+\frac{7}{2}\right)\Theta(\infty, f)+\mu_2\delta_{\mu_2^*}(w_p, f)+\overline{d}(H)\delta_{k+2}(0, f)+\frac{1}{2}\Theta(w_p, f) > Q+4+\mu_2+\overline{d}(H)+\frac{m+n-3p}{2}, \quad \text{or}$$

(iii) $l = 0$ and

$$(2Q+6)\Theta(\infty, f)+2\Theta(w_p, f)+\mu_2\delta_{\mu_2^*}(w_p, f)+\overline{d}(H)\delta_{k+2}(0, f) + \overline{d}(H)\delta_{k+1}(0, f) > 2Q+8+\mu_2+2\overline{d}(H)+2(m+n)-3p,$$

then $p(f) \equiv H[f]$.

In 2019, Harina P. Waghamore and Vijayalakshmi S. B. [27] proved the following result concerning the uniqueness of meromorphic functions sharing a small function:

Theorem D. [27] Let f be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and $k(\geq 1)$ is the highest derivative in $M[f]$. Let $P[f]$ be a non-constant differential polynomial of f. Also, let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $M[f]-a$ and $P[f]-a$ share $(0, l)$ with one of the following conditions:

(i) $l \geq 2$ and

$$(Q+3\lambda+3)\Theta(\infty, f)+\lambda\Theta(0, f)+2d_M\delta_{1+k}(0, f)+\overline{d}(P)\delta(0, f) > Q+4\lambda+3+d_M+2\overline{d}(P)-\underline{d}(P), \quad \text{or}$$

(ii) $l = 1$ and

$$\left(\frac{2Q+7\lambda+7}{2}\right)\Theta(\infty, f)+\lambda\Theta(0, f)+\frac{5d_M}{2}\delta_{1+k}(0, f)+\overline{d}(P)\delta(0, f) > \frac{2Q+9\lambda+7}{2}+\frac{3d_M}{2}+2\overline{d}(P)-\underline{d}(P), \quad \text{or}$$

(iii) $l = 0$ and

$$(2Q+5\lambda+6)\Theta(\infty, f)+\lambda\Theta(0, f)+4d_M\delta_{1+k}(0, f)+2\overline{d}(P)\delta(0, f) > 2Q+6\lambda+6+3d_M+4\overline{d}(P)-2\underline{d}(P),$$

then $M[f] \equiv P[f]$.

The results presented above naturally lead to the question of whether further generalizations are possible. Inspired by numerous investigations into various forms of Bruck's conjecture, we try to broaden the scope of these results. Our focus extends the results to include both homogeneous differential polynomials and non-homogeneous differential-difference polynomials of meromorphic functions, which we have defined as follows:

Definition 1.6. [13] We define a homogenous differential polynomial as

$$\Phi[z] = \sum_{r=1}^{n} a_r \prod_{s=0}^{p} \left(z^{(s)}\right)^{t_{rs}},$$

where $n(\geq 1), p(\geq 0), r, s, t \in \mathbb{Z}^+ \cup \{0\}$ and the degree of $\Phi[z]$ is d_Φ where $d_\Phi = \sum_{s=0}^{p} t_{rs}$. We define D by

$$D = \max_{1\leq r\leq n} \sum_{s=0}^{p} s t_{rs}.$$

Definition 1.7. [41] The difference polynomial in $f(z)$ and its shifts is defined as

$$\Psi_0[z,f] = \sum_{\lambda\in I} a_\lambda(z) f(z)^{i_{\lambda,0}} f(z+c_1)^{i_{\lambda_{i,1}}} \cdots f(z+c_k)^{i_{\lambda,k}},$$

where $d(\Psi_0) = \max_{\lambda\in I}\{d(\lambda)\}$, is its degree, where I is a finite set of the index $\lambda = \{i_{\lambda,0}, \cdots, i_{\lambda,k}\}$ and $a_\lambda(z)$ are meromorphic coefficients satisfying $T(r, a_\lambda(z)) = S(r,f), \lambda \in I$ and $f(z)^{i_{\lambda,0}} f(z+c_1)^{i_{\lambda_{i,1}}} \cdots f(z+c_k)^{i_{\lambda,k}}$ is monomial in $f(z)$ and its shifts $f(z+c_1), \cdots, f(z+c_k)$, where $c_1, \cdots, c_k$ are distinct non-zero complex constants and $d(\lambda) = i_{\lambda,0} + \cdots + i_{\lambda,k}$.

We now give the definition of the general differential-difference polynomial of $f(z)$ and its shifts which is given in [42] as follows.

Definition 1.8. [42] The general differential-difference polynomial of $f(z)$ and its shifts denoted by $\Psi[z,f]$, is defined as

$$\Psi[z,f] = \sum_{\lambda\in I} a_\lambda(z) \prod_{i=0}^{k} \prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}},$$

where I is a finite set of multi-indices $\lambda = (\lambda_{0,0}, \cdots, \lambda_{0,m}, \cdots, \lambda_{1,m}, \cdots, \lambda_{k,0}, \cdots, \lambda_{k,m})$, $c_0(=0)$ and $c_1, c_2, \ldots, c_k$ are distinct complex constants. We assume that the meromorphic co-efficients $a_\lambda(z), \lambda \in I$ of $\Psi[z,f]$ are of growth $S(r,f)$. We denote the degree of the monomial $\prod_{i=0}^{k}\prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}}$ by $d(\lambda) = \sum_{i=0}^{k}\sum_{j=0}^{m} \lambda_{i,j}$. Then we denote the degree and the lower degree of $\Psi[z,f]$ by $d(\Psi) = \max_{\lambda\in I}\{d(\lambda)\}$, and $d^*(\Psi) = \min_{\lambda\in I}\{d(\lambda)\}$ respectively.

In particular, $\Psi[z,f]$ is called a homogeneous differential-difference polynomial if $d(\Psi) = d^*(\Psi)$. Otherwise, $\Psi[z,f]$ is a non-homogeneous differential-difference polynomial.

The following are the main results of our paper which extend and generalize the earlier results.

Theorem 1.9. Let $f(z)$ be a non-constant meromorphic function of finite order and $a(z)$ be a small function such that $a(z)(\neq 0,\infty)$. Let $\Phi[z]$ be a homogenous differential polynomial and $\Psi[z,f]$ be a non-constant differential-difference polynomial. Suppose $\Phi[z]$ and $\Psi[z,f]$ share $\varkappa(a,t)$ and if one of the following holds:

(i) $t \geq 2$ and

$$(Q^*+4)\Theta(\infty,f) + (d^*(\Psi)+d_\Phi)\delta(0,f) > Q^* + d(\Psi) + 4, \tag{1.1}$$

(ii) $t = 1$ and

$$(Q^*+4)\Theta(\infty,f) + (d^*(\Psi)+d_\Phi)\delta(0,f) > Q^* + d(\Psi) + 4, \tag{1.2}$$

(iii) $t = 0$ and

$$(2Q^*+2D+7)\Theta(\infty,f) + (2d^*(\Psi)+d_\Phi)\delta(0,f) > 2Q^* + 2d(\Psi) + 2D + 7, \tag{1.3}$$

then, either $\Phi[z].\Psi[z,f] \equiv a^2$ (or) $\Phi[z] \equiv \Psi[z,f]$.

Theorem 1.10. Let $f(z)$ be a non-constant meromorphic function of finite order and $a(z)$ be a small function such that $a(z)(\neq 0,\infty)$. Let $\Phi[z]$ be a homogenous differential polynomial and $\Psi[z,f]$ be a non-constant differential-difference polynomial. Suppose $\Phi[z]$ and $\Psi[z,f]$ share $\varkappa(a,2)^*$ and $(2d_\Phi + d^*(\Psi))\delta(0,f) + (D+Q^*+5)\Theta(\infty,f) > Q^* + D + d_\Phi + d(\Psi) + 5$, then one of the conclusions of Theorem 1.9 holds.

2 Auxiliary Lemmas

In this section, we will present some lemmas that will be used to prove the main results.

Lemma 2.1. [15] Let F and G be two non-constant meromorphic functions sharing $\varkappa(1,t)$ where $t \in \mathbb{Z}^+ \cup \{0\} \cup \{\infty\}$ and let,

$$\Omega = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

If $\Omega \not\equiv 0$, then,

(i) $T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,F) + S(r,G)$, when $2 \le t \le \infty$;

(ii) $T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \overline{N}_L(r,1;F) + S(r,F) + S(r,G)$, when $t = 1$;

(iii) $T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G)$, when $t = 0$;
and the same inequality holds for $T(r,G)$.

Lemma 2.2. [33] Let F and G be non-constant meromorphic functions sharing $\varkappa(1,1)$. Then,

$$\overline{N}_L(r,1;F) \le \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F).$$

Lemma 2.3. [33] Let F and G be non-constant meromorphic functions sharing $\varkappa(1,0)$. Then,

$$\overline{N}_L(r,1;F) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + S(r,F).$$

Lemma 2.4. [3] Let F and G be two non-constant meromorphic functions that share $\varkappa(1,2)^*$ and $\Omega \not\equiv 0$. Then

$$\begin{aligned} T(r,F) \le{}& N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \overline{N}(r,0;F) + \overline{N}(r,\infty;F) \\ &- m(r,1;G) + S(r,F) + S(r,G), \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.5. [13] Let f be a non-constant meromorphic function and $\Phi[z]$ is defined as above. Then,

(i) $T(r,\Phi[z]) = d_\Phi T(r,f) + D\overline{N}(r,\infty;f) + S(r,f)$;

(ii) $N(r,0;\Phi[z]) \le T(r,\Phi[z]) - d_\Phi T(r,f) + d_\Phi N(r,0;f) + S(r,f) \le D\overline{N}(r,\infty;f) + d_\Phi N(r,0;f) + S(r,f)$.

Lemma 2.6. [28] Let f be a transcendental meromorphic function. Let $\Psi[z,f]$ be defined as in Definition 1.8. If $\Psi[z,f] \not\equiv 0$, then we have

$$N(r,0;\Psi[z,f]) \le T(r,\Psi[z,f]) - T(r,f^{d(\Psi)}) + (d(\Psi) - d^*(\Psi))m\left(r,\frac{1}{f}\right) + N(r,0;f^{d(\Psi)}) + S(r,f).$$

$$N(r,0;\Psi[z,f]) \le Q^*\overline{N}(r,\infty;f) + (d(\Psi) - d^*(\Psi))m\left(r,\frac{1}{f}\right) + N(r,0;f^{d(\Psi)}) + S(r,f),$$

where $Q^* = \max_{0 \le i \le k, \lambda \in I}\{\lambda_{i,1} + 2\lambda_{i,2} + \cdots + m\lambda_{i,m}\}$.

3 Proof of Main Theorems

3.1 Proof of Theorem 1.7.

Let us consider $F = \frac{\Phi[z]}{a}$ and $G = \frac{\Psi[z,f]}{a}$. Since F and G share $\varkappa(a,t)$, then it immediately follows that F and G share $\varkappa(1,t)$, except at the poles and zeros of a. We prove the result through following three cases.
Case 1. Assume that $\Omega \not\equiv 0$. Now, we consider the following three subcases:
Subcase 1.1. Suppose that $2 \le t \le \infty$. We deduce from Lemma 2.1,

$$\begin{aligned} T(r,F) &\le N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,F) + S(r,G) \\ &\le 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty:f) + N_2(r,0;F) + N_2(r,0;G) + S(r,F) + S(r,G). \end{aligned}$$

Now, with the help of Lemmas 2.5 and 2.6,

$$\begin{aligned}T(r,F) \leq\ & 4\overline{N}(r,\infty;f) + T(r,F) - d_\Phi T(r,f) + d_\Phi N(r,0;f) + Q^*\overline{N}(r,\infty:f)\\ &+ N(r,0;f^{d(\Psi)}) + (d(\Psi)-d^*(\Psi))m\left(r,\frac{1}{f}\right) + S(r,f),\end{aligned}$$

hence

$$\begin{aligned}d_\Phi T(r,f) &\leq (Q^*+4)\overline{N}(r,\infty;f) + (d(\Psi)+d_\Phi)N(r,0;f) + (d(\Psi)-d^*(\Psi))m\left(r,\frac{1}{f}\right) + S(r,f)\\ &\leq (Q^*+4)\overline{N}(r,\infty;f) + d_\Phi N(r,0;f) + d(\Psi)T(r,f) - d^*(\Psi)m\left(r,\frac{1}{f}\right) + S(r,f)\\ &\leq (Q^*+4)\overline{N}(r,\infty;f) + d_\Phi N(r,0;f) + d(\Psi)T(r,f) + d^*(\Psi)N(r,0;f) - d^*(\Psi)T(r,f) + S(r,f),\end{aligned}$$

which implies that

$$(d_\Phi - d(\Psi) + d^*(\Psi))T(r,f) \leq (Q^*+4)\overline{N}(r,\infty;f) + (d^*(\Psi)+d_\Phi)N(r,0;f) + S(r,f).$$

Therefore, we have

$$[(Q^*+4)\Theta(\infty,f) + (d^*(\Psi)+d_\Phi)\delta(0,f) - Q^* - d(\Psi) - 4]T(r,f) \leq S(r,f),$$

which contradicts (1.1).
Subcase 1.2. When $t=1$. We deduce from Lemmas 2.1 and 2.2,

$$\begin{aligned}T(r,F) &\leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;G) + N_2(r,\infty;F) + \overline{N}_L(r,1;F) + S(r,F) + S(r,G)\\ &\leq 4\overline{N}(r,\infty;f) + N(r,0;F) + N(r,0;G) + \overline{N}_L(r,1;F) + S(r,f).\end{aligned}$$

Now, with the help of Lemmas 2.5 and 2.6, we get

$$\begin{aligned}T(r,F) \leq\ & 4\overline{N}(r,\infty;f) + T(r,F) - d_\Phi T(r,f) + d_\Phi N(r,0;f) + Q^*N(r,\infty;f) + N(r,0;f^{d(\Psi)})\\ &+ (d(\Psi)-d^*(\Psi))m\left(r,\frac{1}{f}\right) + \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,f),\end{aligned}$$

hence

$$\begin{aligned}d_\Phi T(r,f) \leq\ & \left(Q^*+\frac{9}{2}\right)\overline{N}(r,\infty;f) + d_\Phi N(r,0;f) + d(\Psi)T(r,f) - d^*(\Psi)m\left(r,\frac{1}{f}\right)\\ &+ \frac{D}{2}\overline{N}(r,0;f) + \frac{d_\Phi}{2}\overline{N}(r,0;f) + S(r,f)\\ \leq\ & \left(Q^*+\frac{9}{2}+\frac{D}{2}\right)\overline{N}(r,\infty;f) + \frac{3}{2}d_\Phi N(r,0;f) + d(\Psi)T(r,f) + d^*(\Psi)N(r,0;f) - d^*(\Psi)T(r,f) + S(r,f),\end{aligned}$$

which implies that

$$(d_\Phi - d(\Psi) + d^*(\Psi))T(r,f) \leq \left(Q^*+\frac{9}{2}+\frac{D}{2}\right)\overline{N}(r,\infty;f) + \left(\frac{3}{2}d_\Phi + d^*(\Psi)\right)N(r,0;f) + S(r,f).$$

Therefore, we have

$$\left[\left(Q^*+\frac{D}{2}+\frac{9}{2}\right)\Theta(\infty,f) + \left(\frac{3}{2}d_\Phi + d^*(\Psi)\right)\delta(0,f) - Q^* - d(\Psi) - \frac{d_\Phi + D + 9}{2}\right]T(r,f) \leq S(r,f),$$

which contradicts (1.2).
Subcase 1.3. When $t=0$. We deduce from Lemmas 2.1 and 2.3,

$$\begin{aligned}T(r,F) &\leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}_2(r,\infty;F) + \overline{N}_2(r,\infty;G) + 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G)\\ &\leq 4\overline{N}(r,\infty;f) + N(r,0;F) + N(r,0;G) + 2\overline{N}(r,0;F) + 2\overline{N}(r,\infty;F)\\ &\quad + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,f).\end{aligned}$$

Now, with the help of Lemmas 2.5and 2.6, we get

$$\begin{aligned} T(r,F) \le\ & 7\overline{N}(r,\infty;f) + T(r,F) - d_\Phi T(r,f) + d_\Phi N(r,0;f) + Q^* N(r,\infty;f) \\ & + (d(\Psi) - d^*(\Psi))m\left(r,\frac{1}{f}\right) + N(r,0;f^{d(\Psi)}) + 2d_\Phi T(r,f) + 2D\overline{N}(r,\infty;f) \\ & + Q^*\overline{N}(r,\infty;f) + (d(\Psi) - d^*(\Psi))m\left(r,\frac{1}{f}\right) + N(r,0;f^{d(\Psi)}) + S(r,f), \end{aligned}$$

hence

$$d_\Phi T(r,f) \le (2D + 2Q^* + 7)\overline{N}(r,\infty;f) + d_\Phi N(r,0;f) + 2N(r,0;f^{d(\Psi)}) + 2(d(\Psi) - d^*(\Psi))m\left(r,\frac{1}{f}\right) + S(r,f),$$

which implies that

$$(d_\Phi - 2d(\Psi) + 2d^*(\Psi))T(r,f) \le (2D + 2Q^* + 7)\overline{N}(r,\infty;f) + (2d^*(\Psi) + d_\Phi)N(r,0;f) + S(r,f).$$

Therefore, we have

$$[(2Q^* + 2D + 7)\Theta(\infty,f) + (2d^*(\Psi) + d_\Phi)\delta(0,f) - 2Q^* - 2d(\Psi) - 2D - 7]T(r,f) \le S(r,f),$$

which contradicts (1.3).
Case 2. We now assume that $\Omega \equiv 0$. Now, integrating (2.1) twice, we get,

$$\frac{1}{G-1} = \frac{U}{F-1} + V,$$

where $U(\neq 0)$ and V are two complex constants. Which implies that,

$$G = \frac{(V+1)F + (U-V-1)}{VF + (U-V)} \quad \text{and} \tag{3.1}$$

$$F = \frac{(V-U)G + (U-V-1)}{VG - (V+1)}. \tag{3.2}$$

Now, we discuss the following subcases:
Subcase 2.1.

Let $V \neq 0, -1$. We obtain from (3.2), $\overline{N}\left(r,\frac{V+1}{V};G\right) = \overline{N}(r,\infty;F)$. Using Lemma 2.6 on Nevanlinna 2^{nd} fundamental theorem, we have

$$\begin{aligned} T(r,G) &\le \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}\left(r,\frac{V+1}{V};G\right) + S(r,G) \\ &\le 2\overline{N}(r,\infty;f) + T(r,G) - d(\Psi)T(r,f) + (d(\Psi) - d^*(\Psi))m\left(r,\frac{1}{f}\right) + d(\Psi)N(r,0;f) + S(r,f) \\ &\le 2\overline{N}(r,\infty;f) + d(\Psi)T(r,f) + d^*(\Psi)N(r,0;f) - d^*(\Psi)T(r,f) + S(r,f), \end{aligned}$$

which implies that

$$d^*(\Psi)T(r,f) \le 2\overline{N}(r,\infty;f) + d^*(\Psi)N(r,0;f) + S(r,f). \tag{3.3}$$

Now, we assume that $U - V - 1 \neq 0$, then it follows from (3.1) that $\overline{N}\left(r,\frac{-U+V-1}{V+1};F\right) = \overline{N}(r,0;G)$. Using Lemma 2.5 on Nevanlinna 2^{nd} fundamental theorem, we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}\left(r,\frac{-U+V-1}{V+1};F\right) + S(r,F) \\ &\le (Q^*+1)\overline{N}(r,\infty;f) + T(r,F) - d_q T(r,f) + d_q N(r,0;f) + (d(P) - d^*(P))m\left(r,\frac{1}{f}\right) \\ &\quad + d(P)N(r,0;f) + S(r,f), \end{aligned}$$

which implies that

$$d_qT(r,f) \leq (Q^*+1)\overline{N}\left(r,\infty;f\right) + d_qN(r,0;f) + d(P)T(r,f) + d^*(P)N\left(r,0;f\right) - d^*(P)T(r,f) + S(r,f). \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$[(Q^*+3)\Theta(\infty,f) + (d_q + d^*(P) + d^*(\Psi))\delta(0,f) - d(P) - Q^* + d(P) - 3]T(r,f) \leq S(r,f),$$

which contradicts (1.1), (1.2), and (1.3). Therefore, $U - V - 1 = 0$, then it follows from (3.1) that, $\overline{N}\left(r, \frac{-1}{V}; F\right) = \overline{N}(r,\infty;G)$. Using Lemma 2.5 on Nevanlinna 2^{nd} fundamental theorem, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r,0;F\right) + \overline{N}\left(r,\infty;F\right) + \overline{N}\left(r,\frac{-1}{V};F\right) + S(r,F) \\ &\leq \overline{N}\left(r,\infty;f\right) + T(r,F) - d_qT(r,f) + d_qN(r,0;f) + \overline{N}\left(r,\infty;f\right) + S(r,f), \end{aligned}$$

which implies that

$$d_qT(r,f) \leq 2\overline{N}\left(r,\infty;f\right) + d_qN(r,0;f) + S(r,f). \tag{3.5}$$

Combining (3.3) and (3.5), we get

$$[(Q^*+3)\Theta(\infty,f) + (d_q + d^*(P) + d^*(\Psi))\delta(0,f) - Q^* + d(P) + 3]T(r,f) \leq S(r,f),$$

which contradicts (1.1), (1.2), and (1.3).
Subcase 2.2. Let $V = -1$,

We obtain from (3.1) and (3.2) that, $G = \frac{U}{U+1-F}$ and $F = \frac{(U+1)G-U}{F}$. If $U+1 \neq 0$, then $\overline{N}(r,U+1;F) = \overline{N}(r,\infty;G)$ and $\overline{N}(r,\frac{U}{U+1};G) = \overline{N}(r,0;F)$. Now following the same argument as in Subcase 2.1, we arrive at a contradiction. Therefore $U+1=0$ and this implies that $FG = 1$. Hence $\Phi[z].\Psi[z,f] \equiv a^2$.
Subcase 2.3. Let $V = 0$. We obtain from (3.1) and (3.2) that, $G = \frac{F+U-1}{U}$ and $F = UG + 1 - U$. If $U - 1 \neq 0$, then $\overline{N}(r, 1-U; F) = \overline{N}(r,0;G)$ and $\overline{N}(r,\frac{U-1}{U};G) = \overline{N}(r,0;F)$. Now following the same argument as in Subcase 2.1, we arrive at a contradiction. Therefore $U - 1 = 0$ and this implies that $F = G$. Hence $\Phi[z] \equiv \Psi[z,f]$. This completes the proof of the Theorem 1.9.

3.2 Proof of Theorem 1.10.

By applying Lemmas 2.5 and 2.6 to Lemma 2.4, and proceeding similarly as in Theorem 1.9, we can prove that the conclusion of Theorem 1.10 holds.

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References

[1] A. H. Al-Khaladi, *Notes on meromorphic functions sharing small function and its derivatives*, Arab. J. Math. Sci. **21** (2015), no. 2, 194–208.

[2] A. Banerjee, *Meromorphic functions sharing one value*, Int. J. Math. Math. Sci. **22** (2005), 3587–3598.

[3] A. Banerjee and S. Mukherjee, *Uniqueness of meromorphic functions concerning differential monomials sharing the same value*, Bull. Math. Soc. Sci. **50** (2007), 191–206.

[4] A. Banerjee and S. Majumder, *Some uniqueness results related to meromorphic function that share a small function with its derivative*, Math. Rep. (Bucur.) **16** (2014), 95–111.

[5] A. Banerjee and S. Majumder, *Uniqueness of certain type of differential-difference and difference polynomials*, Tamkang J. Math. **49** (2018), no. 1, 1–24.

[6] A. Banerjee and T. Biswas, *On the value sharing of shift monomial of meromorphic functions*, Surv. Math. Appl. **15** (2020), 341–369.

[7] K.S. Charak and B. Lal, *Uniqueness of $p(f)$ and $P[f]$*, Turkish J. Math. **40** (2016), 569–581.

[8] R. G. Halburd and R. J. Korhonen, *Finite-order meromorphic solutions and the discrete Painlevé equations*, Proc. Lond. Math. Soc. **94** (2007), no. 2, 443–474.

[9] W. K. Hayman, *Meromorphic Functions*, Oxford Math. Monographs, Clarendon Press, Oxford, (1964).

[10] J. Heittokangas, R. Korhonen, I. Laine and J. Rieppo, *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ. **56** (2011), no. 1-4, 81–92.

[11] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl. **46** (2001), 241–253.

[12] I. Lahiri and A. Sarkar, *Uniqueness of a meromorphic function and its derivative*, J. Inequal. Pure Appl. Math. **5** (2004), no. 1, 9.

[13] I. Lahiri and B. Pal, *Uniqueness of meromorphic functions with their homogeneous and linear differential polynomials sharing a small function*, Bull. Korean Math. Soc. **54** (2017), 825–838.

[14] I. Laine and C. C. Yang, *Value distribution of difference polynomials*, Proc. Japan Acad. Ser. A Math. Sci. **83** (2007), no. 8, 148–151.

[15] S. Lin and W. Lin, *Uniqueness of meromorphic functions concerning weakly weighted sharing*, Kodai Math. J. **29** (2006), 269–280.

[16] X. Q. Lin and W. C. Lin, *Uniqueness of entire functions sharing one value*, Acta Math. Sci. Ser. B Engl. Ed. **31** (2011), 1062–1076.

[17] L. Liu and Y. Gu, *Uniqueness of meromorphic functions that share one small function with their derivatives*, Kodai Math. J. **27** (2004), no. 3, 272–279.

[18] K. Liu, X. L. Liu and T. B. Cao, *Some results on zeros and uniqueness of difference-differential polynomials*, Appl. Math. J. Chinese Univ. Ser. B **27** (2012), no. 1, 94–104.

[19] X. Luo and W. C. Lin, *Value sharing results for shifts of meromorphic functions*, J. Math. Anal. Appl. **377** (2011), no. 2, 441–449.

[20] B. N. NaveenKumar and H. P. Waghamore, *Zeros and Weighted value sharing of q-shift difference-differential polynomials of entire and meromorphic functions*, J. Fract. Calc. Appl. **15** (2024), no. 1, 1–20.

[21] B. Saha and T. Biswas, *Uniqueness of entire functions concerning the shift differential polynomials*, Electron. J. Math. Anal. Appl. **9** (2021), no. 2, 195–205.

[22] P. Sahoo and S. Seikh, *Nonlinear differential polynomials sharing a small function*, Mat. Vesnik **65** (2013), no. 252, 151–165.

[23] P. Sahoo and H. Karmakar, *Results on uniqueness of entire functions whose difference polynomials share a polynomial*, J. Contemp. Math. Anal. **52** (2017), no. 2, 102–110.

[24] H. P. Waghamore and V. Husna, *Uniqueness of a meromorphic functions that share one small function and its derivative*, Electron. J. Math. Anal. Appl. **4** (2016), no. 2, 25–37.

[25] H.P. Waghamore and S.H. Naveen kumar, *Results on uniqueness of meromorphic functions of differential polynomial*, Malaya J. Mat. **6** (2018), 14–20.

[26] H. P. Waghamore and R. Maligi, *Weighted sharing of Q-Shift difference-differential polynomials of meromorphic functions sharing a small function*, Electron. J. Math. Anal. Appl. **7** (2019), no. 2, 366–378.

[27] H. P. Waghamore and S. B. Vijayalaxmi, *Results on uniqueness of differential monomial and differential polynomial*, J. Appl. Math. Inform. **37** (2019), no. 3-4, 281–294.

[28] H. P. Waghamore and S. B. Vijayalaxmi, *Unicity result sharing one small function with the general differential-difference polynomial*, J. Anal. **29** (2021), no. 3, 781–801.

[29] H. P. Waghamore and B. N. Naveenkumar, *Some Results on Uniqueness of Meromorphic Functions Regarding Shifts and Derivatives*, Int. J. Differ. Equ. **18** (2023), no. 1, 251–265.

[30] H. P. Waghamore and B. N. Naveenkumar, *Uniqueness results on difference product of entire functions*, Electron. J. Math. Anal. Appl. **12** (2024), no. 1, 1–18.

[31] H. P. Waghamore and B. N. Naveenkumar, *Uniqueness of Differential Polynomials sharing two values*, Int. J. Open Probl. Comput. Math. **17** (2024), no. 3, 1–13.

[32] H. P. Waghamore and B. N. Naveenkumar, *Uniqueness of certain polynomials of meromorphic functions sharing a set with finite weight*, Serdica Math. J. **50** (2024), no. 3-4, 323–344.

[33] H. Y. Xu and Y. Hu, *Uniqueness of meromorphic function and its differential polynomial concerning weakly weighted sharing*, Gen. Math. **19** (2011), 101–111.

[34] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Math. Appl. **557**, Kluwer Academic Publishers, Dordrecht, (2003).

[35] C. C. Yang and I. Laine, *On analogies between nonlinear difference and differential equations*, Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), no. 1, 10–14.

[36] L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin, (1993).

[37] L. Yang, *Value distribution theory*, translated and revised from the 1982 Chinese original, Springer-Verlag, Berlin, (1993).

[38] H. X. Yi, *Meromorphic functions that share one or two values II*, Kodai Math. J. **22** (1999), no. 2, 264–272.

[39] K. Yu, *On entire and meromorphic functions that share small functions with their derivatives*, J. Inequal. Pure Appl. Math. **4** (2003), no. 1, 1–7.

[40] J. Zhang, *Meromorphic functions sharing a small function with their derivatives*, Kyungpook Math. J. **49** (2009), no. 1, 143–154.

[41] X. M. Zheng and Z. X. Chen, *On deficiencies of some difference polynomials*, Acta Math. Sin. (Chin. Ser.) **54** (2011), no. 6, 983–992.

[42] X. M. Zheng and H. Y. Xu, *On the Deficiencies of Some Differential-Difference Polynomials*, Abstr. Appl. Anal. **2014** (2014), no. 1, 378151.