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Journey from UPM to SUPMWS

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Abstract

In this article, we systematically develop and extend the concepts and criteria associated with the uniqueness polynomial of meromorphic functions (UPM). Starting from the foundational UPM, we extend the concept to encompass the (reduced) strong uniqueness polynomial for meromorphic functions in a wider sense ((R)SUPMWS), completing this progression. This extension marks a significant advancement in this area of research. The article consolidates and organizes all related definitions of these polynomials in a concise and systematic manner. Furthermore, we classify and analyze various categories of polynomials for degree n, where $4 \leq n \leq 6$.

Keywords: Uniqueness polynomial, SUPM, wider sense, meromorphic functions, shared sets
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1 Introduction and examples

The characterization of uniqueness properties of meromorphic functions, particularly in relation to understanding and categorizing different classes of polynomials, represents a challenging yet significant area of research. The foundational investigations in this domain were initiated three decades ago by Li and Yang [8]. Their pioneering work laid the foundation for subsequent research in this field, providing insights into the uniqueness properties of meromorphic functions by analyzing the intricate structure of polynomials. Li-Yang [8] first introduced the following novel definition.

Definition 1.1. *[8]* A polynomial P in $\mathbb{C}$, is called a uniqueness polynomial for meromorphic *(*entire*)* functions, if for any two non-constant meromorphic *(*entire*)* functions f and g, $P(f) \equiv P(g)$ implies $f \equiv g$. We say P is a UPM *(*UPE*)* in brief.

Clearly $P(z) = a_1 z + a_0$, $(a_1 \neq 0)$ is a UPM. Next it is natural to ask about the existence of UPM (UPE) of degree 2 and 3. In this regard, we will consider the following examples:

Example 1.1. Consider the following polynomial of degree 2, $P_2(z) = a_2 z^2 + a_1 z + a_0$, where a_i, $i = 0, 1, 2$ are complex numbers such that $P_2(z)$ has no multiple zeros.

<u>Case</u> **1:** $a_1 \neq 0$. Consider $f \equiv e^z g$, where $g(z) = -\frac{a_1}{a_2(1+e^z)}$. As $(a_2 f^2 + a_1 f) = (a_2 g^2 + a_1 g)$, we have $P_2(f) = P_2(g)$, but $f \not\equiv g$.

<u>Case</u> **2:** $a_1 = 0$. Suppose $f(z) = \frac{e^z}{(e^z+1)}$, $g = -f$. Clearly, $P_2(f) = P_2(g)$ but $f \not\equiv g$.

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Example 1.2. Consider the following polynomial of degree 3, $P_3(z) = a_3z^3 + a_2z^2 + a_1z + a_0$, where a_i, $i = 0, 1, 2, 3$ are complex numbers such that $P_3(z)$ has multiple zeros.

Case 1: $a_1 \neq 0$, $a_2 = 0$. Take a pair of entire and a pair of meromorphic functions as follows:

$$f \equiv \frac{e^z}{1-w^2} + \frac{a_1e^{-z}}{a_3(w^2-w)}, \quad g \equiv \frac{e^z}{w^2-w} + \frac{a_1e^{-z}}{a_3(w^2-w)};$$

$$f \equiv \frac{\sqrt{a_1}\,(-sinz + i\sqrt{3})}{\sqrt{3a_3}\,cosz}, \qquad g \equiv \frac{2\sqrt{a_1}\,sinz}{\sqrt{3a_3}\,cosz}.$$

For both the pairs $P_3(f) \equiv P_3(g)$ but $f \not\equiv g$.

Case 2: $a_1 = 0$, $a_2 \neq 0$. Next we assume a pair of entire and a pair of meromorphic function as follows:

$$f \equiv \frac{sinz + cosz - 1}{2a_3}, \quad g \equiv -\frac{cosz}{2a_3} \text{ with } a_2 = \frac{2 - cosz}{2};$$

$$f \equiv -\frac{a_2}{a_3}\left(\frac{e^z(e^z+1)}{e^{2z}+e^z+1}\right) \text{ where, } g \equiv e^{-z}f.$$

Here, $P_3(f) \equiv P_3(g)$ but $f \not\equiv g$.

The above example shows that, any polynomial of degree 3 can not be a UPM or UPE. Considering $P(z) = z^n + a_{n-1}z^{n-1} + a_0$, $a_{n-1} \neq 0$ and $n \geq 4$, it is easy to verify that $P(z)$ is a UPE but not a UPM, so UPE is a larger class.

The following theorem elucidate about the existence of 4-th degree UPE.

Theorem 1.1. *[8]* Let $P_4(z) = z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$. Then

(a) $P_4(z)$ is not a UPM.

(b) $P_4(z)$ is a UPE if and only if $\frac{{a_3}^3}{8} - \frac{a_2a_3}{2} + a_1 \neq 0$.

Further advancement in the direction of characterization of polynomials of higher degree was made by Yang-Hua [11] in the following manner:

Theorem 1.2. *[11]* Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ $(n \geq 4)$ be a polynomial. If there exists an integer t with $1 \leq t < n-2$ and $\gcd(n,t) = 1$ such that $a_{n-1} = \cdots = a_{t+1} = 0$, but $a_t \neq 0$, then P is a UPE.

The following examples show that in the above theorem, the conditions $1 \leq t$ and $\gcd(n,t) = 1$ are needed.

Example 1.3. Let $t = 0$ and $P(z) = z^n + a_0$. Then for any function f and n-th root of unity u, we have $P(f) = P(uf)$ and thus P is not a UPE.

Example 1.4. Let $P(z) = z^6 + z^2 + 1$. Then $t = 2 < n - 2 = 4$ but $\gcd(t,n) \neq 1$ and $P(f) = P(-f)$ for any f. So P is not a UPE.

Example 1.5. Let $P(z) = z^6 + z^3 + 1$. Then $t = 3 < n - 2 = 4$ but $\gcd(t,n) \neq 1$ and $P(f) = P(\omega f)$ for any f, where ω is the non real cube root of unity. So P is not a UPE.

The following theorem characterize UPM.

Theorem 1.3. *[11]* Let $P(z) = z^n + a_mz^m + a_0$ be a monic polynomial such that gcd $(n,m) = 1$ and $a_m \neq 0$. If $n \geq 5$ and $1 \leq m < n-1$, then P is a UPM.

The next definition is the refinement of UPM due to Fujimoto [3].

Definition 1.2. *[3]* Let P be a polynomial in $\mathbb{C}$. If for an non-constant meromorphic *(*entire*)* functions f and g, $P(f) \equiv cP(g)$ implies $f \equiv g$, where c is any arbitrary nonzero constant, then P is called *SUPM (SUPE).*

It is clear from the above definitions that a SUPM(SUPE) is a UPM(UPE) but a UPM(UPE) may not be a SUPM(SUPE). The following example shows that one degree polynomials are UPM(UPE) but may not be SUPM(SUPE).

Example 1.6. Let $P(z) = a_1 z + a_0$ $(a \neq 0)$. Clearly $P(z)$ is a UPM(UPE). We note that for any non-constant meromorphic (entire) function g, if we take $f := cg - \frac{a_0}{a_1}(1-c)$ $(c \neq 0, 1)$, then $P(f) = cP(g)$ but $f \neq g$.

The next examples show that for higher degree polynomial $P(z)$, the same conclusion can be derived. We know for degree $n = 2$ or 3, $P(z)$ is not a UPE, so there is no question of existence of UPM, SUPE or SUPM for these two integers. Hence the case $n \geq 4$ deserves attention.

Example 1.7. Let $P(z) = z^4 + az$ where a is a non-zero complex number, then from **Theorem 1.1** or **Theorem 1.2** we know $P(z)$ is a UPE. But for the imaginary cube root of unity ω, choosing $g = \omega f$, $(g = \omega^2 f)$ it is easy to verify $P(f) = \omega^2 P(g)$ $(P(f) = \omega P(g))$. So $P(z)$ is not a SUPE.

Example 1.8. Consider $P(z) = z^{n-r}(z^r + a)$ where a is a non-zero complex number and $\gcd(n, r) = 1$, $r \geq 2$ and $n \geq 5$. Then from **Theorem 1.3**, P is a uniqueness polynomial but for any non-constant meromorphic (entire) function g if we take $f = \omega g$, where ω is non-real $r - th$ root of unity, then $P(f) = \omega^{n-r} P(g)$. So $P(z)$ is not an SUPM.

The following definition known as critical injection property plays a crucial role for representing UPM and SUPM.

Example 1.9. *[2]* Let $P(z)$ be a polynomial such that $P'(z)$ has k mutually distinct zeros given by $d_1,\ d_2,\ ...,\ d_k$ with multiplicities $q_1,\ q_2,\ ...,\ q_k$, respectively. Then $P(z)$ is said to satisfy the critical injection property if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, ..., k\}$.

Next to recall the definition of Counting Multiplicities (CM) in the literature under the notion of multiplicity. Let $f(z)$ be a non-constant meromorphic function and $a \in \mathbb{C}$. Define the function $\nu_f^a : \mathbb{C} \to \mathbb{N}$ by

$$\nu_f^a = \begin{cases} 0 & \text{if } f(z) \neq a \\ d & \text{if } f(z) = a \text{ with multiplicity d} \end{cases}$$

and define $\nu_f^\infty = \nu_{\frac{1}{f}}^0$. Further, we define $E(a, f)$ as $E(a, f) = \{(z, \nu_f^a(z)) : z \in \mathbb{C}\}$. If $E(a, f) = E(a, g)$ we say that f and g share the value a CM.

2 Definitions

In this section, we introduce some relevant definitions that are fundamental to the subject matter of this article. To this end, we begin by slightly modifying the definition presented in [1], as this adaptation is essential for the subsequent stages of our discussion.

Definition 2.1. **Initial term non-gap and gap polynomials:** A polynomial

$$P(z) = a_n z^n + a_{n-1} z^{z-1} + \ldots + a_1 z + a_0$$

of degree n is called an initial term non-gap polynomial *(ITNGP)* if there exists at least one consecutive non-zero term after the first term, i.e., if $a_{n-i} \neq 0$ for some $1 \leq i < n$, then $a_{n-1} \neq 0$, $a_{n-2} \neq 0$, $\ldots$, $a_{n-i+1} \neq 0$. Otherwise $P(z)$ is called an initial term gap polynomial *(ITGP).*

Definition 2.2. **Degeneracy of a polynomial:** A polynomial $P(z)$ is said to be a non-degenerate polynomial if $0 \notin \{z \mid P(z) = 0\}$, otherwise it is called a degenerate polynomial.

We are now introducing analogues definition of *Definition* 2.1, vis-a-vis terminal term non gap and gap polynomials.

Definition 2.3. **Terminal term non-gap and gap polynomials:** A non-degenerate polynomial

$$P(z) = a_n z^n + a_{n-1} z^{z-1} + \ldots + a_1 z + a_0$$

is called a terminal term non-gap polynomial *(TTNGP)* if before a_0, there exists at least one non-zero consecutive term, i.e., if $a_t \neq 0$, then $a_{t-1} \neq 0$, $a_{t-2} \neq 0, \ldots$, $a_1 \neq 0$ for $t = 1, 2, \ldots, (n-1)$. Otherwise, the polynomial is said to be a terminal term gap polynomials *(TTGP)*.

Next, in view of *Definitions* 2.1 and 2.3, we want to propose the definitions of Index and Reserve index of a polynomial in the following manner.

Definition 2.4. **Index of a polynomial:** Let us consider the polynomial: $P^{[s]}(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$.

A. If $P^{[s]}(z)$ is an *ITNGP*, then it is said to be initial term non-gap polynomial of index s *(ITNGP*$_s$ in short*)* if one of the followings is satisfied*:*

i*)* $a_{n-s+1} \neq 0$ but $a_0 = 0$ *(*$1 \leq s \leq n$*);*

ii*)* $a_i \neq 0$, $i = 0, 1, 2, \ldots, n$; then $s = n + 1$.

B. If $P^{[s]}(z)$ is an *ITGP*, then it is of index 1.

Note 2.1. Any polynomial of degree n is of index $s \geq 1$.

Definition 2.5. **Reverse index of a polynomial:** Suppose $P_{[\hat{s}]}(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0$, $b_0 \neq 0$.

A. If $P_{[\hat{s}]}(z)$ is a *TTNGP*, then it is said to be terminal term non-gap polynomial of reverse index $\hat{s}$ *(TTNGP*$_{\hat{s}}$ in short*)* if one of the followings is satisfied*:*

i*)* $b_{\hat{s}-1} \neq 0$; $(1 < \hat{s} < n)$

ii*)* $b_i \neq 0$ for $i = 1, \ldots, n-1$; then $\hat{s} = n + 1$.

B. If $P_{[\hat{s}]}(z)$ is a *TTGP*, then reverse index is 1.

Note 2.2. For a polynomial of degree n, n cannot be reverse index of the polynomial.

Definition 2.6. **Order of a polynomial:** For a polynomial $P(z)$ with index s and reverse index $\hat{s}$, we will call it polynomial of order $(s, \hat{s})$.

Definition 2.7. **Intermediate polynomial:** Any polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_z + a_0$ of order ≥ 4 and order $(1, 1)$ is said to be an intermediate polynomial if there exists some i such that $a_i \neq 0$, $0 < i < (n-1)$.

Generalizing the definition of SUPM we present the following definition.

Let us first generalize *Definition 1.2* for weighted set sharing in the wider sense as follows:

Definition 2.8. **SUPMWS:** Let $P(z)$ and $Q(z)$ be two polynomials in $\mathbb{C}$. If for any non-constant meromorphic *(*entire*)* functions f and g, $P(f) \equiv cQ(g)$ implies $f \equiv g$, where c is any arbitrary nonzero constant, then $P(z)$ and $Q(z)$ are called strong uniqueness polynomial for meromorphic *(*entire*)* functions in the wider sense, *SUPMWS (SUPEWS*) in brief.

It is worth mentioning that in the direction of SUPMWS, some allied investigations were carried out by researchers, as documented in [6], [7], [9] and [10]. Notably, in 2004, Khoai and Yang [7] established a significant result concerning the absence of solutions to the functional equation $P(f) = Q(g)$, where $P(x)$ and $Q(x)$ are polynomials of different degrees. The result is stated as follows:

Theorem 2.1. *[7]* Let $P(f) = Q(g)$ be a functional equation, where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m. The equation has no pair of non-constant meromorphic function solutions f and g if n, m satisfy one of the following conditions:

1) $(n, m) = 1, n > m \geq 2, n \geq 5$,

2) $(n, m) \geq 2, n \geq 6$,

3) $n = m \geq 4$.

In 2007, Pakovich [9] addressed the problem of determining the conditions under which the pre-image sets of two polynomials corresponding to two compact sets coincide. Later, in 2010, Pakovich [10] provided a solution to the functional equation $s = P(f) = Q(g)$, where s, f, g are entire functions and P, Q are arbitrary rational functions. In 2017, in the context of functional equations, Khoai et al. [6], established the following theorem to characterize the relationships between the functions:

Theorem 2.2. *[6]* Let n, m be positive integers, $n > 2m+9$ and $c, d, e, u, v, t \in \mathbb{C}\setminus\{0\}$. Let either $m \geq 2$, $(m, n) = 1$ or $m \geq 4$; f_1, f_2, g_1, g_2 be entire functions and let $\frac{f_1}{f_2}$ and $\frac{g_1}{g_2}$ be to non-constant meromorphic functions satisfying $cf_1^n + df_1^{n-m}f_2^m + ef_2^n = ug_1^n + vg_1^{n-m}g_2^m + tg_2^n$. Then the relations holds: $g_1 = hf_1,\ \ g_2 = lf_2$, where h, l are constants satisfying the conditions: $h^n = \frac{c}{u},\ \ h^{n-m}l^m = \frac{d}{v},\ \ l^n = \frac{e}{t}$.

However, we confine our focus on the uniqueness problems that were not discussed earlier.

In the following definition we further expand the definition of SUPMWS in different manner imposing some sharing conditions over the functions f and g or multiplicity of poles and call it as restricted SUPMWS in different genre.

Definition 2.9. **<u>RSUPMWS:</u>** In addition to the definition of SUPMWS, if f, g are two non-constant meromorphic functions such that

(i) they share k number of sets, $k \in \mathbb{N} \cup \{0\}$,

(ii) they have multiple poles,

(iii) satisfying both (i) and (ii); then the pair of polynomials $P(z)$ and $Q(z)$ will be called respectively *(i)* $RSUMPMWS_{\{k\}}$, *(ii) RSUMPMWSMP*, *(iii)* $RSUMPMWS_{\{k\}}MP$. In particular, in *(i)* if $k = 0$, we get *SUPMWS* and in *(iii)* if $k = 0$, the we get definition *(ii)*.

The following charts will help the readers to understand the above definitions in a more vivid way.

CHART-1

Two polynomials $P(z)$ and $Q(z)$ are said to be **RSUPMWS{k}MP,** if $P(f) \equiv AQ(g) \Rightarrow f \equiv g$, with the restrictions that f and g sharing $k$sets and having multiple poles.

⇓

In the definition of **RSUPMWS{k}MP,** if the condition of two functions having multiple poles is omitted, then the pair $P(z)$ and $Q(z)$ are said to be **RSUPMWS{k}**.

⇓

In the definition of **RSUPMWS{k},** if $k = 0$, then $P(z)$, $Q(z)$ are called **SUPMWS.**

⇓

In the definition of **SUPMWS,** if $P(z) = Q(z)$, then the definition reduces to **SUPM.**

⇓

If $P(f) \equiv P(g) \Rightarrow f \equiv g$, then $P(z)$ is said to be **UPM.**

CHART-2

Two polynomials $P(z)$ and $Q(z)$ are **RSUPMWS$_{\{k\}}$MP,** if $P(f) \equiv AQ(g) \Rightarrow f \equiv g$, with the restrictions that f and g sharing k sets and having multiple poles.

⇓

In the definition of **RSUPMWS$_{\{k\}}$MP,** if $k = 0$ then $P(z), Q(z)$ are said to be **RSUPMWSMP.**

The goal of this article is:

i) to provide a comprehensive framework that captures various aspects of uniqueness for meromorphic functions,

ii) identifying and formalizing the criteria necessary to establish stronger uniqueness properties beyond those captured by UPM,

iii) to demonstrate the effectiveness and applicability of (R)SUPMWS in analyzing the uniqueness of meromorphic functions and

iv) to advance the understanding and application of these concepts in concerned fields.

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iv) to advance the understanding and application of these concepts in concerned fields.

3 Lemmas

The following two lemmas are needed to proceed further.

Lemma 3.1. *[4]* Let $P(z)$ be a monic polynomial without multiple zero whose first derivative has mutually k-distinct zeros, given by $d_1, d_2, .., d_k$ with multiplicities $q_1, q_2, ..., q_k$, respectively. Suppose that $P(z)$ is a critically injective polynomial. Then $P(z)$ will be a *UPM* if and only if

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{i=1}^{k} q_l .$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2$, $\min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$, then also the above inequality holds.

Lemma 3.2. *[1]* Let $\phi(z) = a^2(z^{n-m} - A)^2 - 4b(z^{n-2m} - A)(z^n - A)$, where $A, a, b \in \mathbb{C}^*$, $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$, gcd$(m, n)$= 1, $n > 2m$. If ω^l is the mth root of unity for $l = 0, 1, \ldots, m-1$, then

(i) $\phi(z)$ has no multiple zero, when $A \neq \omega^l$.

(ii) $\phi(z)$ has exactly one multiple zero, when $A = \omega^l$ and that is of multiplicity 4. In particular, when $A = 1$, then the multiple zeros is 1.

4 Existence of various genre of SUPMWS

Further information or a detailed explanation about the conventional notations of value distribution theory are outlined in [5].

• Existence of SUPMWS:

Let us take a pair of 6 degree polynomial of order $(3,1)$;

$$\begin{cases} P_6^3(z) = \frac{z^6}{6} - \frac{2az^5}{5} + \frac{a^2z^4}{4} - c_6 = Q_6(z) - c_6, \\ \widehat{P}_6^3(z) = k_6 Q_6(z) - \widehat{c}_6, \end{cases} \tag{4.1}$$

where c_6, $\widehat{c}_6$, k_6 are three nonzero complex numbers such that both P_6^3 and $\widehat{P}_6^3$ do not have of any repeated zero and $c_6 + \frac{\widehat{c}_6}{k_6} \neq Q_6(a)$. We wish to show that, P_6^3 and $\widehat{P}_6^3$ are SUPMWS.

Proof . Let us consider two meromorphic functions f and g such that

$$P_6^3(f) \equiv A_6 \widehat{P}_6^3(g). \tag{4.2}$$

By a simple calculation we can write

$$Q_6(f) \equiv A_6 k_6 \left[Q_6(g) + \frac{1}{k_6}\left(\frac{c_6}{A_6} - \widehat{c}_6\right)\right]. \tag{4.3}$$

Case 1: Suppose $A_6 \neq \frac{c_6}{\widehat{c}_6}$. Consider the polynomial

$$\tau_1(z) = Q_6(z) + \frac{1}{k_6}\left(\frac{c_6}{A_6} - \widehat{c}_6\right).$$

Now, we wish to show that all the zeros of $\tau_1(z)$ are simple. Clearly, 0 is not a zero of $\tau_1(z)$. If a is a zero of $\tau_1(z)$, then we have

$$Q_6(a) = \frac{1}{k_6}\left(\widehat{c}_6 - \frac{c_6}{A_6}\right).$$

From (4.3), a simple calculation yields

$$Q_6(g) \equiv \frac{1}{A_6 k_6}\left[Q_6(f) - c_6 + A_6\widehat{c}_6\right].$$

Let $\tau_2(z) = Q_6(z) - c_6 + A_6\widehat{c}_6$. There are 3 possibilities.

Possibility 1: 0 is a multiple zero of $\tau_2(z)$, then $A_6 = \frac{c_6}{\widehat{c}_6}$, which is not possible.

Possibility 2: Let a be a multiple zero of $\tau_2(z)$, then we obtain

$$Q_6(a) = c_6 - A_6\widehat{c}_6.$$

Equating two expressions of $Q_6(a)$, we have

$$A_6 k_6 = -1.$$

Therefore,

$$Q_6(a) = c_6 + \frac{\widehat{c}_6}{k_6},$$

which is not possible by the hypothesis.

Possibility 3: All the zeros of $\tau_2(z)$ are simple, say, α_i, $i = 1, 2, \ldots, 6$. By the Second Fundamental Theorem we have

$$\begin{aligned} 5T(r,f) &\leq \sum_{i=1}^{6} \overline{N}(r,\alpha_i;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq \overline{N}(r,0;Q_6(g)) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq 4T(r,f) + S(r,f), \end{aligned}$$

a contradiction.

Hence, all the zeros of $\tau_1(z)$ are simple, say α'_i, $i = 1, 2, \ldots, 6$. Again by the Second Fundamental Theorem from (4.3) we have

$$\begin{aligned} 5T(r,g) &\leq \sum_{i=1}^{6} \overline{N}(r,\alpha'_i;g) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq \overline{N}(r,0;Q_6(f)) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq 4T(r,g) + S(r,g), \end{aligned}$$

a contradiction. Hence, *Case 1* does not hold.

Case 2: Assume $A_6 = \frac{c_6}{\widehat{c}_6}$. Hence, (4.3) gives

$$Q_6(f) \equiv \frac{k_6 c_6}{\widehat{c}_6} Q_6(g).$$

Now, we want to show that $f \equiv g$. On the contrary, suppose $f \not\equiv g$. Taking $h = \frac{f}{g}$ and putting it in the above equation we obtain

$$\frac{g^2}{6}\left(h^6 - \frac{k_6c_6}{\widehat{c}_6}\right) - \frac{2ag}{5}\left(h^5 - \frac{k_6c_6}{\widehat{c}_6}\right) + \frac{a^2}{4}\left(h^4 - \frac{k_6c_6}{\widehat{c}_6}\right) \equiv 0. \tag{4.4}$$

If h is a constant, we can write from (4.4) that, $h^6 = h^5 = h^4 = \frac{k_6c_6}{\widehat{c}_6}$, i.e., $h = 1$. Hence, $f \equiv g$.

If h is non-constant, (4.4) can be written as

$$\left\{g - \frac{6a}{5}\frac{h^5 - \frac{k_6c_6}{\widehat{c}_6}}{h^6 - \frac{k_6c_6}{\widehat{c}_6}}\right\}^2 \equiv \frac{3a^2\phi(h)}{50\left(h^6 - \frac{k_6c_6}{\widehat{c}_6}\right)^2},$$

where $\phi(z) = 24\left(z^5 - \frac{k_6c_6}{\widehat{c}_6}\right)^2 - 25\left(z^6 - \frac{k_6c_6}{\widehat{c}_6}\right)\left(z^4 - \frac{k_6c_6}{\widehat{c}_6}\right)$.

Subcase 2.1: Let $\frac{k_6c_6}{\widehat{c}_6} \neq 1$. We can say from *Lemma 3.2*, $\phi(z)$ has no multiple zero and 10 distinct simple zeros, say β_i for $i = 1, 2, \ldots, 10$ and each β_i-point of h is of multiplicity 2. Then applying the Second Fundamental Theorem we can write

$$\begin{aligned} 8T(r,h) &\leq \sum_{i=1}^{10} \overline{N}(r,\beta_i;h) + S(r,h) \\ &\leq 5T(r,h) + S(r,h), \end{aligned}$$

a contradiction.

Subcase 2.2: Suppose $\frac{k_6c_6}{\widehat{c}_6} = 1$. Thus we get,

$$\begin{aligned} Q_6(f) &\equiv Q_6(g), \\ \text{i.e., } P_6^3(f) &\equiv P_6^3(g) \end{aligned}$$

and hence by Lemma 3.1, we can conclude that $f \equiv g$. □

• Existence of RSUPMWSMP:

Let us assume the following pair of 6 degree polynomial of order $(2, 1)$;

$$\begin{cases} P_6^2(z) = \frac{z^6}{6} - \frac{az^5}{5} - d_6 = R_6(z) - d_6, \\ \widehat{P}_6^2(z) = k'_6 R_6(z) - \widehat{d}_6, \end{cases} \tag{4.5}$$

where d_6, $\widehat{d}_6$, k_6' are three nonzero complex numbers such that both P_6^2 and $\widehat{P}_6^2$ do not have any repeated zero and $d_6 + \frac{\widehat{d}_6}{k_6'} \neq R_6(a)$. P_6^2 and $\widehat{P}_6^2$ are RSUPMWSMP.

Proof . Let us take two meromorphic functions f and g with multiple poles. Using the similar arguments of the previous proof, we will reach up to *Case 2* and by taking $f = gh$ we can have an analogous of equation of (4.4) such as

$$\frac{g}{6}\left(h^6 - \frac{k_6' d_6}{\widehat{d}_6}\right) - \frac{a}{5}\left(h^5 - \frac{k_6' d_6}{\widehat{d}_6}\right) \equiv 0. \tag{4.6}$$

Next, if h is constant we have $h^6 = h^5 = \frac{k_6' d_6}{\widehat{d}_6}$, i.e., $h = 1$. Hence, $f \equiv g$.

Further, if h is non-constant then we can write from (4.6),

$$g \equiv \frac{6a}{5} \frac{\left(h^5 - \frac{k_6' d_6}{\widehat{d}_6}\right)}{\left(h^6 - \frac{k_6' d_6}{\widehat{d}_6}\right)}.$$

Clearly, the two polynomials $\left(z^6 - \frac{k_6' d_6}{\widehat{d}_6}\right)$ and $\left(z^5 - \frac{k_6' d_6}{\widehat{d}_6}\right)$ do not have any common zeros. Let the zeros of the polynomial $\left(z^6 - \frac{k_6' d_6}{\widehat{d}_6}\right)$ be γ_i, $i = 1, 2, \ldots, 6$. By the Second Fundamental Theorem and noting the fact that g has multiple pole we get

$$\begin{aligned} 5T(r,h) &\leq \sum_{i=1}^{6} \overline{N}\left(r, \gamma_i; h\right) + \overline{N}(r, \infty; h) + S(r,h) \\ &\leq 4T(r,h) + S(r,h), \end{aligned}$$

a contradiction. hence, our result follows. □

- **Existence of RSUPMWS$_{\{1\}}$:**

Take

$$\begin{cases} P_5^3(z) = \frac{z^5}{5} - \frac{az^4}{2} + \frac{a^2 z^3}{3} - c_5 = Q_5(z) - c_5, \\ \widehat{P}_5^3(z) = k_5 Q_5(z) - \widehat{c}_5, \end{cases} \tag{4.7}$$

where c_5, $\widehat{c}_5$, k_5 are three nonzero complex numbers such that both P_5^3 and $\widehat{P}_5^3$ do not have any repeated zero and $c_5 + \frac{\widehat{c}_5}{k_5} \neq Q_5(a)$. Here, P_5^3 and $\widehat{P}_5^3$ are RSUPMWS$_{\{1\}}$.

Proof . Let f, g be two meromorphic functions share $\{0\}$ CM. Suppose

$$P_5^3(f) \equiv A_5 \widehat{P}_5^3(g). \tag{4.8}$$

Case 1: Suppose $A_5 \neq \frac{c_5}{\widehat{c}_5}$. Therefore, from (4.8) Hence, 0 is an e.v.P. of f and g. By a simple calculation, from (4.8) we get

$$Q_5(f) \equiv A_5 k_5 \left[Q_5(g) + \frac{1}{k_5}\left(\frac{c_5}{A_5} - \widehat{c}_5\right)\right]. \tag{4.9}$$

Consider the polynomial

$$\Gamma_1(z) = Q_5(z) + \frac{1}{k_5}\left(\frac{c_5}{A_5} - \widehat{c}_5\right).$$

Using the similar argument that was used in the case of the polynomials $P_6^3(z)$ and $\widehat{P}_6^3(z)$ and by the hypothesis of the statement, we can show that all the zeros of $\Gamma_1(z)$ are simple, say δ_i, $i = 1, 2 \ldots, 5$. By the Second Fundamental

Theorem and (4.9) we obtain,

$$\begin{aligned} 5T(r,g) &\leq \overline{N}(r,0;g)+\sum_{i=1}^{5}\overline{N}(r,\delta_i;g)+\overline{N}(r,\infty;g)+S(r,g)\\ &\leq 3T(r,g)+S(r,g), \end{aligned}$$

a contradiction.

Case 2: Let $A_5=\frac{c_5}{\widehat{c}_5}$. Hence, from (4.8) we get

$$\frac{g^2}{5}\left(h^5-\frac{k_5c_5}{\widehat{c}_5}\right)-\frac{ag}{2}\left(h^4-\frac{k_5c_5}{\widehat{c}_5}\right)+\frac{a^2}{3}\left(h^3-\frac{k_5c_5}{\widehat{c}_5}\right)\equiv 0.$$

Now, using the similar argument that was used after (4.4) we will reach up to

$$Q_5(f)\equiv Q_5(g),$$

i.e.,

$$\frac{f^5}{5}-\frac{af^4}{2}+\frac{f^3}{3}\equiv\frac{g^5}{5}-\frac{ag^4}{2}+\frac{g^3}{3}.$$

We now wish to show that $f\equiv g$. On the contrary, let $f=gh$. As f and g share $\{0\}$ and $\{\infty\}$ CM, it is evident that h does not have any zeros or poles. We have

$$\frac{g^2}{5}\left(h^5-1\right)-\frac{ag}{2}\left(h^4-1\right)+\frac{a^2}{3}\left(h^3-1\right)\equiv 0. \tag{4.10}$$

If h is constant, we have from (4.10), $h^5=h^4=h^3=1$, i.e., $h=1$.

Next, if h is non-constant we get from (4.10),

$$\left\{g-\frac{5a}{4}\frac{h^4-1}{h^5-1}\right\}^2\equiv\frac{5a^2\phi_1(h)}{48(h^5-1)^2},$$

where $\phi_1(z)=15(z^4-1)^2-16(z^5-1)(z^3-1)$. Now, by *Lemma 3.2*, $\phi_1(z)$ has exactly one multiple zero at 1 with multiplicity exactly 4 and remaining 4 simple zeros are say θ_i, $i=1,2,3,4$. By the Second Fundamental Theorem we have,

$$4T(r,h)\leq\sum_{i=1}^{4}\overline{N}(r,\theta_i;h)+S(r,h)\leq 2T(r,h)+S(r,h),$$

a contradiction. Hence, $f\equiv g$. □

• Existence of RSUPMWS$_{\{1\}}$MP:

Suppose pair of polynomial of degree 5 and of order $(2,1)$ is given by;

$$\begin{cases} P_5^2(z)=\frac{z^5}{5}-\frac{az^4}{4}-d_5=R_5(z)-d_5,\\ \widehat{P}_5^2(z)=k_5'R_5(z)-\widehat{d}_5, \end{cases} \tag{4.11}$$

where d_5, $\widehat{d}_5$, k_5' are three nonzero complex numbers such that both P_5^2 and $\widehat{P}_5^2$ do not have any repeated zero and $d_5+\frac{\widehat{d}_5}{k_5'}\neq R_5(a)$. Here, P_5^2 and $\widehat{P}_5^2$ are RSUPMWS$_{\{1\}}$MP.

Proof . Let us take two meromorphic functions f and g with multiple poles share $\{0\}$ CM. Using the similar arguments of the case of *Existence of RSUPMWSMP* for 6-th degree polynomial we get an analogous equation of (4.6) as follows:

$$\frac{g}{5}\left(h^5-\frac{k_5'd_5}{\widehat{d}_5}\right)-\frac{a}{4}\left(h^4-\frac{k_5'd_5}{\widehat{d}_5}\right)\equiv 0. \tag{4.12}$$

Next, if h is constant we have $h^5 = h^4 = \frac{k_5' d_5}{\widehat{d}_5}$, i.e., $h = 1$. Hence, $f \equiv g$.

Further, if h is non-constant then we can write from the above equation we get,

$$g \equiv \frac{5a}{4} \frac{\left(h^4 - \frac{k_5' d_5}{\widehat{d}_5}\right)}{\left(h^5 - \frac{k_5' d_5}{\widehat{d}_5}\right)}.$$

Clearly, the two polynomials $\left(z^5 - \frac{k_5' d_5}{\widehat{d}_5}\right)$ and $\left(z^4 - \frac{k_5' d_5}{\widehat{d}_5}\right)$ do not have any common zeros. Let the zeros of the polynomial $\left(z^5 - \frac{k_5' d_5}{\widehat{d}_5}\right)$ be η_i, $i = 1, 2, \ldots, 5$. By the Second Fundamental Theorem and the fact that g has multiple pole we get

$$\begin{aligned} 4T(r,h) &\leq \sum_{i=1}^{5} \overline{N}\left(r, \eta_i; h\right) + \overline{N}(r, \infty; h) + S(r,h) \\ &\leq \frac{5}{2} T(r,h) + S(r,h), \end{aligned}$$

a contradiction. hence, our result follows. □

- **Existence of RSUPMWS$_{\{2\}}$MP:**

Consider two polynomials

$$\begin{cases} P_4^3(z) = \frac{z^4}{4} - \frac{2az^3}{3} + \frac{a^2z^2}{2} - c_4 = Q_4(z) - c_4, \\ \widehat{P}_4^3(z) = k_4 Q_(z) - \widehat{c}_4, \end{cases} \tag{4.13}$$

where c_4, $\widehat{c}_4$, k_4 are three nonzero complex numbers such that both P_4^3 and $\widehat{P}_4^3$ do not have any repeated zero. Here, P_4^3 and $\widehat{P}_4^3$ are RSUPMWS$_{\{2\}}$MP.

Proof . Suppose f and g be two meromorphic functions consisting multiple poles share $\{0\}$ and $\{a\}$ CM. Consider

$$P_4^3(f) \equiv A_4 \widehat{P}_4^3(g). \tag{4.14}$$

Case 1: Suppose $A_4 \neq \frac{c_4}{\widehat{c}_4}$. If 0 is not an e.v.P. of f and g, then from (4.14) we have $A_4 = \frac{c_4}{\widehat{c}_4}$, not possible. Hence, 0 is an e.v.P. of f and g. By a simple calculation we can write

$$Q_4(f) \equiv A_4 k_4 \left[Q_4(g) + \frac{1}{k_4}\left(\frac{c_4}{A_4} - \widehat{c}_4\right)\right]. \tag{4.15}$$

Now let

$$\Pi(z) = Q_4(z) + \frac{1}{k_4}\left(\frac{c_4}{A_4} - \widehat{c}_4\right).$$

Now, we want to show that all the zeros of $\Pi(z)$ are simple. Clearly, 0 is not a zero of $\Pi(z)$. If possible, Let a is zero of $\Pi(z)$ of multiplicity 3 and other simple zero is π. As f, g share $\{a\}$ CM, a is an e.v.P. of both the functions. By the Second Fundamental Theorem we have,

$$\begin{aligned} 2T(r,f) &\leq \overline{N}(r, 0; Q_4(f)) + \overline{N}(r, \infty; f) + S(r,f) \\ &\leq \overline{N}(r, \pi; g) + \overline{N}(r, \infty; g) + S(r,f) \leq \frac{3}{2} T(r,f) + S(r,f), \end{aligned}$$

which is a contradictions. Hence, all the zeros of $\Pi(z)$ are simple. Now, using the similar arguments that was used for the case of *Existence of* RSUPMWS$_{\{1\}}$, we can show that $f \equiv g$. □

5 Some relevant discussion and an open question

Minutely observing the structures of the (R)SUPMWS of degree n for $4 \leq n \leq 6$, we have noticed that as the degree n decreases, additional criteria need to be imposed on the meromorphic functions. This observation leads to a conjecture that $n = 6$ might be the lowest degree for which SUPMWS actually exists. However, there could be a chance of further reduction of the degree of SUPMWS. In fact, the definition of intermediate polynomials in *Definition 2.7* may present an exciting opportunity for further research and analysis in this area. It is really interesting to consider how the degree of SUPMWS may vary depending on the criteria imposed on meromorphic functions under the consideration of intermediate polynomials. Further exploration and analysis in this direction could provide valuable insights into the nature of these polynomials and their existence conditions. In view of all the discussions we pose the following question for further investigations:

Question 5.1. What is the minimum degree of *SUPMWS*?

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