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Maximal area integral problem for a family of multivalent functions of starlike type

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Abstract

For a normalized analytic multivalent function f defined in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, let $\Delta(r, f)$ and $L(r, f, p)$ denote the Dirichlet integral and the integral mean of f respectively. In this research article, we consider a subclass $S_p^*(A, B, \lambda)$ of normalized analytic multivalent functions in the open unit disk and solve the Yamashitha's conjecture for this subclass. Also, we determine the extremal function for which the integral mean $L(r, f, p)$ is maximum.

Keywords: multivalent analytic function, subordination, integral means, area integral, Gaussian hypergeometric functions.
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1 Introduction

In Geometric Function Theory, the branch of Complex Analysis which deals with the interplay of analytic functions and the geometry of the images of certain domains in the complex plane under these functions, an interesting problem is to determine the maximum value of the area of the image of a subdisk inside the unit disk and that of a length of an arc under these analytic functions. The conjecture of Yamashita [20] regarding the maximum of area and length for univalent functions has been a driving force for the researchers working in univalent function theory to solve the extremal problems of this type for various subclasses of univalent functions defined in the open unit disk. For more details in this regard, one may refer to [10, 11, 12, 13, 18].

Theory of univalent functions had been a central object of study in Geometric funcion theory for more than a century since the appearance of the Bieberbach conjecture in the year 1916. Many tools to study various aspects of several subclasses of univalent functions have been developed in the past [4]. Denote by $\mathcal{A}$ the class of all normalized analytic functions f in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = 0 = f'(0) - 1$. These functions have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions [4]. A nice way to compare the ranges of two members of the class $\mathcal{S}$ is through subordination, which we recall now:

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Definition 1.1. [4] Given two functions f and g, analytic in Δ, we say that f is subordinate to g if there exists a function ω analytic in Δ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ for $z \in \Delta$. We denote this by $f \prec g$. In particular, if the function g is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

It is worth recalling a remarkable result proved by Hallenbeck and Ruscheweyh proved in 1975 regarding subordination of analytic functions.

Theorem 1.2. [8] Suppose that $f(z) = zp + a_{p+1}z^{p+1} + \cdots$ and $F(z) = z^p + A_{p+1}z^{p+1} + \cdots$ are analytic in Δ and that $g(z) = zf'(z)/f(z) \prec zF'(z)/F(z) = G(z)$. If $G(z)$ is univalent and starlike with respect to $w = p$, then $f(z)/z^p \prec g(z)/z^p$.

The following result of Rogosinski established in the year 1943 will be used in proving our results.

Theorem 1.3. [15] Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in Δ and suppose $g \prec f$. Then,

$$\sum_{k=1}^{n} |b_k|^2 \leq \sum_{k=1}^{n} |a_k|^2, \ n = 1, 2, \cdots .$$

We now recall the concept of Dirichlet integral for analytic functions defined in the open unit disk.

Definition 1.4. [20] For each analytic function g defined in the open unit disk and for each $0 < r \leq 1$, the area of the image of the disk Δ_r under g, denoted by $\Delta(r, g)$ is defined as

$$\Delta(r, g) = \int\int_{|z|<r} |g'(z)|^2 dxdy, \ z = x + iy.$$

The above integral is called the **Dirichlet Integral** of g. A function g is said to be Dirichlet finite if $\Delta(1, g)$, the area of the image of the open unit disk under f is finite.

If $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then, by the Parseval-Gutzmer formula,

$$\Delta(r, g) = \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}.$$

An univalent function $f : \Delta \to \mathbb{C}$ is a function which assumes each of its values exactly once. A generalization of the univalent function is the concept of multivalent function. A function f defined on the open unit disk and analytic is said to be multivalent if it assumes each of its values more than once. This is equivalent to saying that the number of roots of the equation $f(z) = w$ in Δ for any w does not exceed p.

Let $\mathcal{A}_p$ denote the class of all multivalent functions having Taylor's series

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, p \in \mathbb{N} \tag{1.2}$$

Various subclasses of multivalent functions were introduced and studied in the past by several researchers. A detailed theory about multivalent functions can be found in [2, 3, 7, 17].

For a function $f \in \mathcal{A}_p$ the integral mean is given by,

$$L_1(r, f, p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r^{2p}}{|f(re^{i\theta})|^2} d\theta$$

and the Dirichlet integral is given by,

$$\Delta(r, f) = \int\int_{|z|<r} |f'(z)|^2 dxdy = \pi p r^{2p} + \pi \sum_{n=1}^{\infty} (n + p)|a_{n+p}|^2 r^{2(n+p)}, \ z = x + iy.$$

where $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$. Computing this area for various subclasses of analytic functions is known as the area problem for the functions of type g. All polynomials are Dirichlet-finite. In general, all functions $g \in \mathcal{A}$ for which g' is bounded on Δ are Dirichlet-finite.

Special functions has good connections with Geometric Function Theory. Among special functions, the hypergeometric functions play a very important role in the univalent function theory since its surprising appearance in the proof of Bieberbach conjecture by Loius de Branges in the year 1985 [6]. We now recall hypergeometric functions. For complex numbers a, b and c with c neither zero nor a negative integer, the Gaussian hypergeometric function ${}_2F_1(a, b; c, z)$ is defined by

$$ {}_2F_1(a, b; c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \ |z| < 1. $$

Observe that the function ${}_2F_1(a, b; c, z)$ is an analytic function in Δ. The notation $(a)_n$ is the Pochammer symbol defined by

$$ (a)_n = a(a+1)(a+2)\cdots(a+n-1), \ n \geq 1 $$

with $(a)_0 = 1$ if $a \neq 0$. Further the hypergeometric function ${}_0F_1(a; z)$ is defined by

$$ {}_0F_1(a; z) = \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!}, \ |z| < 1. $$

For more details about special functions, one may refer to [1, 14, 19]. In the year 1973, Janowski generalized the family of starlike functions using subordination by introducing the subclass defined below:

Definition 1.5. [9] For $-1 \leq B < A \leq 1$, the subclass of functions $f \in \mathcal{A}$ satisfying the condition

$$ \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}. $$

Sahoo et.al. in [16] verified the Yamashita conjecture for the above subclass in the year 2015. In 2017, N.L. Sharma [18] solved the problem of maximal area integral for a subclass of multivalent Janowski type starlike functions which is defined as :
For $-1 \leq B \leq 0$ and $A \in \mathbb{C}, \ A \neq B$ Sharma considered the subclass of all functions in the class $\mathcal{A}$ satisfying

$$ \frac{zf'(z)}{pf(z)} \prec \frac{1+Az}{1+Bz}. $$

He showed that for functions f in this subclass,

$$ max_{f \in \mathcal{S}_p^*(A,B)} \Delta(r, \frac{z}{f}) = E_{A,B}(r, p), $$

where

$$ E_{A,B}(r, p) = \pi |\overline{A} - B|^2 p^2 r^2 \, {}_2F_1(p\alpha + 1, p\overline{\alpha} + 1; 2, B^2 r^2). $$

Motivated by the work of N.L. Sharma, in this research paper, we consider a more general subclass of multivalent Janowski type starlike functions defined in the open unit disk and for functions in this subclass, we solve the maximal area integral problem and find an estimate of the length of image of an arc inside the unit disk.

We first define our subclass:

Definition 1.6. For $-1 \leq B \leq 0, \ A \in \mathbb{C}$ and $0 \leq \lambda \leq 1$, a function $\mathcal{A}_p$ is said to be in the subclass $S_p^*(A, B, \lambda)$ if

$$ \frac{zf'(z)}{(1-\lambda)pf(z) + \lambda zf'(z)} \prec \frac{1+Az}{1+Bz} $$

If $$ k_{A,B,\lambda}(z) = \begin{cases} z^p \left(1 + \frac{B-\lambda A}{1-\lambda} z\right)^{(\frac{A-B}{B-\lambda A})p}, if B \neq 0. \\ z^p \left(1 - \frac{\lambda A z}{1-\lambda}\right)^{-\frac{p}{\lambda}}, if B = 0. \end{cases} $$

then $k_{A,B,\lambda}$ plays the role of extremal function for the subclass $S_p^*(A, B, \lambda)$.

Remark 1.7. When $\lambda = 0$, the subclass $S_p^*(A, B, \lambda)$ reduces to the class $S_p^*(A, B)$ considered by Sharma [18].

Remark 1.8. If $\mathcal{A}_p$ and $\frac{z^p}{f} \neq 0$ in Δ, then

$$\frac{z^p}{f} = 1 + \sum_{n=1}^{\infty} b_{n+p-1} z^n, \ z \in \Delta. \tag{1.3}$$

Note that the non-vanishing condition above is ensured whenever $f \in \mathcal{S}$.

2 Results on Maximal Area Integrals for $f \in S_p^*(A, B, \lambda)$

Lemma 2.1. Let $f \in S_p^*(A, B, \lambda)$ for $A \in \mathbb{C}$, $-1 \leq B \leq 0$, $A \neq B$, $p \in \mathbb{N}$ and f be of the form (3). Then

$$\sum_{k=1}^{\infty} [(1-\lambda)^2 k^2 - |(B - \lambda A)k + (\overline{A} - B)p|^2] |b_{k+p-1}|^2 \leq |\overline{A} - B|^2 p^2.$$

Proof . If $f \in S_p^*(A, B, \lambda)$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{p(1 + Az)}{1 + \frac{B - \lambda A}{1-\lambda} z}. \tag{2.1}$$

Let $g(z) = \frac{z^p}{f}$, Then (4) becomes

$$\frac{zg'(z)}{pg(z)} = \frac{(B-A)zw(z)}{1 - \lambda + (B - \lambda A)zw(z)}$$

where $w : \Delta \to \overline{\Delta}$ with $w(0) = 1$. Using the series representation (3) for f we obtain

$$\sum_{k=1}^{n} (1-\lambda) k b_{k+p-1} z^{k-1} + \sum_{k=n+1}^{\infty} c_{k+p-1} z^{k-1} = \left((A-B)p + \sum_{k=1}^{n-1} ((B - \lambda A)k + (A-B)p) b_{k+p-1} z^k \right) w(z).$$

Since $|w(z)| < 1$, for $|z| = r$, by Clunie's method [5]+ we find

$$\sum_{k=1}^{n} (1-\lambda)^2 k^2 |b_{k+p-1}|^2 r^{2k-2} \leq |A-B|^2 p^2 + \sum_{k=1}^{n-1} |(B - \lambda A)k + (A-B)p|^2 |b_{k+p-1}|^2 r^{2k}$$

which is equivalent to

$$\sum_{k=1}^{n} (1-\lambda)^2 k^2 |b_{k+p-1}|^2 r^{2k-2} - \sum_{k=1}^{n-1} |(B - \lambda A)k + (A-B)p|^2 |b_{k+p-1}|^2 r^{2k} \leq |A-B|^2 p^2. \tag{2.2}$$

Taking $r = 1$ and allowing $n \to \infty$, we obtain

$$\sum_{k=1}^{\infty} [(1-\lambda)^2 k^2 - |(B - \lambda A)k + (A-B)p|^2] |b_{k+p-1}|^2 \leq |A-B|^2 p^2.$$

□

Lemma 2.2. Let $f \in S_p^*(A, B, \lambda)$ with $A \in \mathbb{C}, -1 \leq B < 0$, $A \neq \frac{B}{\lambda}$ and $0 < \lambda < 1$. Suppose that for $|z| < r$, $0 < r < 1$,

$$\frac{z^p}{f} = 1 + \sum_{k=1}^{\infty} b_{k+p-1} z^k$$

and

$$\left(1 + \frac{B - \lambda A}{1 - \lambda} z\right)^{-(\frac{A-B}{B-\lambda A})p} = 1 + \sum_{k=1}^{\infty} d_{k+p-1} z^k, \ 0 < r \leq 1.$$

Then,

$$\sum_{k=1}^{N} k |b_{k+p-1}|^2 r^{2k} \leq \sum_{k=1}^{N} k |d_{k+p-1}|^2 r^{2k}. \tag{2.3}$$

Proof . By Lemma 1.1, we have

$$\sum_{k=1}^{n}(1-\lambda)^2k^2|b_{k+p-1}|^2r^{2k}+(1-\lambda)^2n^2|b_{n+p-1}|^2r^{2n}$$
$$-\sum_{k=1}^{n-1}|(B-\lambda A)k+(A-B)p|^2|b_{k+p-1}|^2r^{2k+2}\le|A-B|^2p^2$$

which can again be rewritten as

$$\sum_{k=1}^{n-1}[k^2-|k+p\alpha|^2\beta^2r^2]|b_{k+p-1}|^2r^{2k}+n^2|b_{n+p-1}|^2r^{2n}\le\alpha^2p^2\beta^2r^2 \tag{2.4}$$

where $\alpha=\frac{A-B}{B-\lambda A}$ and $\beta=\frac{B-\lambda A}{1-\lambda}$. It is clear that equality in above is attained for the function $\left(1+\frac{B-\lambda A}{1-\lambda}z\right)^{-(\frac{A-B}{B-\lambda A})p}$ as $n\to\infty$ with $b_{k+p-1}=c_{k+p-1}$.

The rest of the proof is divided into the following steps:

Step-1: Crammer's Rule.

For each $n=1,2,\cdots,N$, the $n-th$ coefficient of (7) is multiplied by a factor $\lambda_{n,N}$ which are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (7) giving raise to

$$\sum_{k=1}^{N}k|b_{k+p-1}|^2r^{2k}\le\beta^2p^2r^2|\alpha|^2\lambda_{n,N}.$$

From (6) and above inequality, we have

$$\beta^2p^2r^2|\alpha|^2\lambda_{n,N}\le\sum_{k=1}^{N}k|d_{k+p-1}|^2r^{2k},\ n=1,2,\cdots,N$$

and hence $d_{k+p-1}=\frac{\beta^k(p\alpha)_k}{k!}$. In the computation of the factors $\lambda_{n,N}$ we obtain the following system of linear equations

$$k=k^2\lambda_{k,N}+\sum_{n=k+1}^{N}\lambda_{n,N}(k^2-|k+p\alpha|^2\beta^2r^2),k=1,2,\cdots,N. \tag{2.5}$$

The matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Using Crammer's rule, the solution of the system (8) is given by

$$\lambda_{n,N}=\frac{((n-1)!)^2}{(N!)^2}DetA_{n,N},$$

where $A_{n,N}$ is the $(N-n+1)\times(N-n+1)$ matrix

$$A_{n,N}=\begin{pmatrix} n & n^2-|n+p\alpha|^2\beta^2r^2 & . & . & . & n^2-|n+p\alpha|^2\beta^2r^2\\ n+1 & (n+1)^2 & . & . & . & (n+1)^2-|n+1+p\alpha|^2\beta^2r^2\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ N & 0 & . & . & . & N^2\end{pmatrix}$$

We obtain the determinant of the above matrix by expanding using the Laplace's rule with respect to the last row wherein the first coefficient is N, the remaining $N-n-1$ entries are zeros and the last entry is 0. Expanding the determinant of the above matrix and usage of Mathematical Induction yields the following :

$$\lambda_{k,N}=\lambda_{k,N-1}-\frac{1}{N}\left(1-\left|1+\frac{p\alpha}{k}\right|^2\beta^2r^2\right)\prod_{\gamma=k+1}^{N-1}\left|1+\frac{p\alpha}{\gamma}\right|^2\beta^2r^2.$$

Call $E_{k,p} = 1 - \left|1 + \frac{p\alpha}{k}\right|^2 \beta^2 r^2$. Then, we obtain

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N} E_{k,p} \prod_{\gamma=k+1}^{N-1} \left(\left|1 + \frac{p\alpha}{\gamma}\right|^2 \beta^2 r^2 \right). \tag{2.6}$$

Observe that $E_{k,p}$ in (9) may be positive as well as negative for all $k \in \mathbb{N}$.

Case (i): Suppose $U_{k,p}$ is negative. From (9), we see that for fixed $k \in \mathbb{N}$, $k \leq N-1$, the sequence $\{\lambda_{k,N}\}$ is strictly non-decreasing, i.e. $\lambda_{k,N} - \lambda_{k,N-1} > 0$ so that

$$\lambda_{k,N} > \lambda_{k,N-1} > \cdots > \lambda_{k,k} = \frac{1}{k} > 0$$

and thus $\lambda_{k,N} \geq 0$ when $N \to \infty$ as required.

Case (ii): Suppose $U_{k,p}$ is positive.

For fixed $k \in \mathbb{N}$, $N \geq k$, the sequence $\{\lambda_{k,N}\}$ is strictly non-increasing, i.e. $\lambda_{k,N} - \lambda_{k,N-1} < 0$ with

$$\lambda_k = lim_{N\to\infty} \lambda_{k,N} = \frac{1}{k} - E_{k,p} \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{\gamma=k+1}^{\infty} \left(\left|1 + \frac{p\alpha}{\gamma}\right|^2 \beta^2 r^2 \right). \tag{2.7}$$

For all $N \in \mathbb{N}$, $1 \leq k \leq N$, to show that $\lambda_{k,N} > 0$, it is enough to prove $\lambda_k \geq 0$ for $k \in \mathbb{N}$. This will be proved in the following step:

Step-II: Positivity of Multipliers.

Let,

$$L_k = \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{\gamma=k+1}^{n-1} \left(\left|1 + \frac{p\alpha}{m}\right|^2 \beta^2 r^2 \right), \ k \in \mathbb{N}.$$

We show that

$$L_k \leq \frac{1}{kE_{k,p}}.$$

From (10) we have

$$\lambda_k = \frac{1}{k} - L_k + \left(\left|1 + \frac{p\alpha}{k}\right|^2 \beta^2 r^2 \right) L_k, \ k \in \mathbb{N}$$

Set

$$M_k = \frac{1}{k} + \left(\left|1 + \frac{p\alpha}{k}\right|^2 \beta^2 r^2 \right) L_k.$$

It is enough to show that

$$M_k \leq \frac{1}{kE_{k,p}}. \tag{2.8}$$

To show this, we use the inequality

$$\frac{1}{nE_{n,p}} > \frac{1}{(n+1)E_{n+1,p}} \tag{2.9}$$

and the identity

$$\frac{1}{nE_{n,p}} = \frac{1}{n} + \frac{\left(\left|1 + \frac{p\alpha}{n}\right|^2 \beta^2 r^2 \right)}{nE_{n,p}} \tag{2.10}$$

which are valid for each $n \in \mathbb{N}$. Repeated applications of (12) and (13) for $n = k, k+1, \cdots, R$ results in the inequality

$$\frac{1}{kE_{k,p}} > \sum_{n=k}^{R} \frac{1}{n} \prod_{\gamma=k}^{n-1} \left(\left|1 + \frac{p\alpha}{\gamma}\right|^2 \beta^2 r^2 \right) + \frac{\prod_{\gamma=k}^{R} \left(\left|1 + \frac{p\alpha}{\gamma}\right|^2 \beta^2 r^2 \right)}{RE_{R,p}} = L_{k,R} + U_{k,R}, \text{ for } k \leq R.$$

Since $U_{k,R} > 0$, taking the limit as $R \to \infty$, we have

$$\frac{1}{kE_{k,p}} \geq \lim_{R\to\infty} L_{k,R} = \sum_{n=k}^{\infty} \frac{1}{n} \prod_{\gamma=k}^{n-1} \left(\left|1 + \frac{p\alpha}{\gamma}\right|^2 \beta^2 r^2 \right) = R_k$$

and we complete the proof of the inequality (11).

Step-III:

We now prove that

$$\beta^2 p^2 r^2 |\alpha|^2 \lambda_{n,N} \leq \sum_{k=1}^{N} k|d_{k+p-1}|^2 r^{2k}$$

for N=2, n=1 and using the inequality (9), we obtain,

$$\begin{aligned}
\beta^2 p^2 r^2 |\alpha|^2 \lambda_{1,2} &= \beta^2 p^2 r^2 |\alpha|^2 \Big(\lambda_{1,1} - 1/2(1 - |1 + p\alpha|^2 \beta^2 r^2) \Big) \\
&= \frac{\beta^2 p^2 r^2 |\alpha|^2}{2} + \frac{\beta^4 p^2 r^4 |\alpha|^2 |1 + p\alpha|^2}{2}, \text{ since } \lambda_{1,1} = 1 \\
&\leq \beta^2 p^2 r^2 |\alpha|^2 + \frac{\beta^4 p^2 r^4 |\alpha|^2 |1 + p\alpha|^2}{2} \\
&= \sum_{k=1}^{2} \frac{k|(p\alpha)_k|^2}{(k!)^2} (\beta r)^{2k}.
\end{aligned}$$

Since $d_{k+p-1} = \frac{\beta^k (p\alpha)_k}{k!}$ then the inequality

$$\beta^2 p^2 r^2 |\alpha|^2 \lambda_{n,N} \leq \sum_{k=1}^{N} k|d_{k+p-1}|^2 r^{2k} \tag{2.11}$$

holds for N=2, n=1. Now we can complete the proof by induction method. Assume that the inequality (12) is true for N=γ, (i.e)

$$\beta^2 p^2 r^2 |\alpha|^2 \lambda_{n,\gamma} \leq \sum_{k=1}^{\gamma} k|d_{k+p-1}|^2 r^{2k}. \tag{2.12}$$

Then for $N = \gamma + 1$ using (9) we deduce that

$$\begin{aligned}
\beta^2 p^2 r^2 |\alpha|^2 \lambda_{n,\gamma+1} &= \beta^2 p^2 r^2 |\alpha|^2 \left(\lambda_{n,\gamma} - \frac{1}{\gamma+1} \left(1 - \left|1 + \frac{p\alpha}{n}\right|^2 \beta^2 r^2 \right) \prod_{h=n+1}^{\gamma} \left|1 + \frac{p\alpha}{h}\right|^2 \beta^2 r^2 \right) \Bigg) \\
&\leq \sum_{k=1}^{\gamma} k|d_{k+p-1}|^2 r^{2k} - \frac{1}{\gamma+1} \left(1 - \left|1 + \frac{p\alpha}{n}\right|^2 \beta^2 r^2 \right) \prod_{h=n+1}^{\gamma} \left|1 + \frac{p\alpha}{h}\right|^2 \beta^2 r^2 \Big) \beta^2 p^2 r^2 |\alpha|^2 \\
&= \sum_{k=1}^{\gamma} k|d_{k+p-1}|^2 r^{2k} - \frac{1}{\gamma+1} \prod_{h=n+1}^{\gamma} \left(\left|1 + \frac{p\alpha}{h}\right|^2 \beta^2 r^2 \right) \beta^2 p^2 r^2 |\alpha|^2 + \\
&\frac{1}{\gamma+1} \prod_{h=n}^{\gamma} \left(\left|1 + \frac{p\alpha}{h}\right|^2 \beta^2 r^2 \right) \beta^2 r^2 \Big) \beta^2 p^2 r^2 |\alpha|^2 \\
&\leq \sum_{k=1}^{\gamma} \frac{k|(p\alpha)_k|^2}{(k!)^2} (\beta r)^{2k} + \frac{1}{\gamma+1} \prod_{h=n}^{\gamma} \left(\left|1 + \frac{p\alpha}{h}\right|^2 \beta^2 r^2 \right) \beta^2 p^2 r^2 |\alpha|^2
\end{aligned}$$

since $d_{k+p-1} = \frac{\beta^k (p\alpha)_k}{k!}$. The last inequality implies that,

$$\beta^2 p^2 r^2 |\alpha|^2 \lambda_{n,\gamma+1} \leq \sum_{k=1}^{\gamma} \frac{k|(p\alpha)_k|^2}{(k!)^2} (\beta r)^{2k} + \frac{1}{\gamma+1} \prod_{h=1}^{\gamma} \left(\left|1 + \frac{p\alpha}{h}\right|^2 \beta^2 r^2 \right) \beta^2 p^2 r^2 |\alpha|^2$$

or equivalently,

$$\begin{aligned}
\beta^2p^2r^2|\alpha|^2\lambda_{n,\gamma+1} &\leq \sum_{k=1}^{\gamma}\frac{k|(p\alpha)_k|^2}{(k!)^2}(\beta r)^{2k}+\frac{(\gamma+1)(\beta^2r^2)^{\gamma+1}}{(1)^2_{\gamma+1}}\prod_{h=1}^{\gamma}\left(\left|1+\frac{p\alpha}{h}\right|^2\right)(1)^2_{\gamma}p^2|\alpha|^2\\
&=\sum_{k=1}^{\gamma}\frac{k|(p\alpha)_k|^2}{(k!)^2}(\beta r)^{2k}+\frac{(\gamma+1)(\beta^2r^2)^{\gamma+1}}{(1)^2_{\gamma+1}}\prod_{h=1}^{\gamma}(|h+p\alpha|^2)p^2|\alpha|^2\\
&=\sum_{k=1}^{\gamma}\frac{k|(p\alpha)_k|^2}{(k!)^2}(\beta r)^{2k}+\frac{(\gamma+1)(\beta^2r^2)^{\gamma+1}}{(1)^2_{\gamma+1}}|(p\alpha)_{\gamma+1}|^2\\
&=\sum_{k=1}^{\gamma+1}\frac{k|(p\alpha)_k|^2}{(k!)^2}(\beta r)^{2k}
\end{aligned}$$

Hence we have the obtained the desired inequality. □

Theorem 2.3. Let $f\in S_p^*(A,B,\lambda)$ for $A\in\mathbb{C}$ with $-1\leq B<0,\ A\neq\frac{B}{\lambda}$, $p\in\mathbb{N}$ and $0<\lambda<1$ and $\frac{z^p}{f(z)}$ be a non-vanishing analytic function in Δ. Then, for $0<r\leq 1$,

$$\max_{f\in\mathcal{S}_p^*(A,B,\lambda)}\Delta\left(r,\frac{z^p}{f}\right)=E_{A,\frac{B-\lambda A}{1-\lambda},\lambda}(r,p)$$

where $E_{A,\frac{B-\lambda A}{1-\lambda},\lambda}(r,p)$

$$=\pi\frac{|\overline{A}-B|^2}{(1-\lambda)^2}p^2r^2\ {}_2F_1\left(\left(\frac{A-B}{B-\lambda A}\right)p+1,\left(\frac{\overline{A}-B}{B-\lambda\overline{A}}\right)p+1;2;\left(\frac{B-\lambda A}{1-\lambda}\right)^2r^2\right).$$

The maximum is attained for the function

$$k_{A,B,\lambda}(z)=\left(1+\frac{B-\lambda A}{1-\lambda}z\right)^{(\frac{A-B}{B-\lambda A})p}.$$

Proof . Suppose $f\in S_p^*(A,B,\lambda)$ with $-1\leq B<0$ and $A\neq\frac{B}{\lambda}$ with $0<\lambda<1$. Set $g(z):=z^p/f(z)$. By the definition of $S_p^*(A,B,\lambda)$, we obtain

$$\frac{zf'(z)}{pf(z)}\prec\frac{1+Az}{1+(\frac{B-\lambda A}{1-\lambda})z}=\frac{zk'_{A,B,\lambda}(z)}{pk_{A,B,\lambda}(z)}$$

By Hallenbeck and Ruscheweyh's result [8] and subordinate principle, we find that

$$g(z)=\frac{z^p}{f(z)}\prec\left(1+\frac{B-\lambda A}{1-\lambda}z\right)^{-(\frac{A-B}{B-\lambda A})p}=\frac{z^p}{k_{A,B,\lambda}(z)},$$

$$\frac{z^p}{f(z)}=1+\sum_{k=1}^{\infty}b_{k+p-1}z^k\text{ and }\frac{z^p}{k_{A,B,\lambda}(z)}=1+\sum_{k=1}^{\infty}d_{k+p-1}z^k,\ |z|<r$$

then from Lemma 1.2 for each $N\in\mathbb{N}$ it follows

$$\sum_{k=1}^{N}k|b_{k+p-1}|^2r^{2k}\leq\sum_{k=1}^{N}k|d_{k+p-1}|^2r^{2k},0<r\leq 1,$$

which implies that

$$\pi\sum_{k=1}^{\infty}k|b_{k+p-1}|^2r^{2k}\leq\pi\sum_{k=1}^{\infty}k|d_{k+p-1}|^2r^{2k}$$

i.e.,

$$\Delta\left(r,\frac{z^p}{f}\right)\leq\Delta\left(r,\frac{z^p}{k_{A,B,\lambda}}\right)$$

Note that

$$d_{k+p-1} = \frac{(-1)^k \beta^k (p\alpha)_k}{k!}$$

where $\alpha = \frac{A-B}{B-\lambda A}$ and $\beta = \frac{B-\lambda A}{1-\lambda}$. Applying the area's formula for $\frac{z^p}{k_{A,B,\lambda}}$, we have

$$\begin{aligned}
\pi^{-1}\Delta\left(r, \frac{z^p}{k_{A,B,\lambda}}\right) &= \sum_{k=1}^{\infty} k|d_{k+p-1}|^2 r^{2k}, \ |z| < r \\
&= \sum_{k=1}^{\infty} k \frac{\left((\frac{A-B}{B-\lambda A})p\right)_k \left((\frac{\overline{A}-B}{B-\lambda \overline{A}})p\right)_k}{(1)_k (1)_k} \left(\frac{B-\lambda A}{1-\lambda}\right)^{2k} r^{2k} \\
&= \left|\frac{\overline{A}-B}{B-\lambda\overline{A}}\right|^2 p^2 \left(\frac{B-\lambda\overline{A}}{1-\lambda}\right)^2 r^2 \sum_{k=0}^{\infty} \frac{\left((\frac{A-B}{B-\lambda A})p\right)_k \left((\frac{\overline{A}-B}{B-\lambda \overline{A}})p\right)_k}{(2)_k (1)_k} \\
&= \frac{|\overline{A}-B|^2}{(1-\lambda)^2} p^2 r^2 \ {}_2F_1\left(\left(\frac{A-B}{B-\lambda A}\right)p+1, \left(\frac{\overline{A}-B}{B-\lambda\overline{A}}\right)p+1; 2; \left(\frac{B-\lambda A}{1-\lambda}\right)^2 r^2\right)
\end{aligned}$$

from which the required result follows. □

Remark 2.4. When $\lambda = 0$, the above lemmas and theorems reduces to those obtained by Sharma et.al. [18].

Lemma 2.5. Let $f \in S_p^*(A, 0, \lambda)$, $0 < |A| \leq 1$. For $|z| < r$, Suppose that

$$\frac{z^p}{f(z)} = 1 + \sum_{k=1}^{\infty} b_{k+p-1} z^k$$

and

$$\left(1 - \frac{\lambda A z}{1-\lambda}\right)^{\frac{p}{\lambda}} = 1 + \sum_{k=1}^{\infty} c_{k+p-1} z^k, r \in (0, 1].$$

Then for all $N \in \mathbb{N}$

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq \sum_{k=1}^{N} k|c_{k+p-1}|^2 r^{2k}. \tag{2.13}$$

Proof . It is enough to prove the result for $0 < A \leq 1$. From Lemma 1.1 for B=0, we get

$$\sum_{k=1}^{n-1} \left[(1-\lambda)^2 k^2 - \left(A(p-\lambda k)\right)^2 r^2\right] |b_{k+p-1}|^2 r^{2k-2} + n^2 |b_{n+p-1}|^2 r^{2n-2} \leq A^2 p^2 \tag{2.14}$$

Multiplying (17) throughout by r^2 and dividing by $(1-\lambda)^2$, we get

$$\sum_{k=1}^{n-1} \left[k^2 - \left(\frac{p-\lambda k}{1-\lambda}\right)^2 A^2 r^2\right] |b_{k+p-1}|^2 r^{2k} + \left(\frac{n}{1-\lambda}\right)^2 |b_{n+p-1}|^2 r^{2n} \leq \left(\frac{A}{1-\lambda}\right)^2 p^2 r^2 \tag{2.15}$$

Equality in (18) is attained with $b_{k+p-1} = c_{k+p-1}$ for the function $\left(1 - \lambda - \lambda A z\right)^{\frac{p}{\lambda}}$. The remaining part of the proof is split into three parts:

Step 1: Crammer's Rule

For each $n = 1, 2, \cdots, N$, the $n-th$ coefficient is multiplied by a factor $\lambda_{n,N}$ which are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (16) and hence from the modified inequalities we get

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq A^2 \left(\frac{p-\lambda k}{1-\lambda}\right)^2 r^2 \lambda_{n,N} \tag{2.16}$$

First we shall evaluate the suitable multipliers $\lambda_{n,N}$ by Cramer's rule. Secondly, in step-II, we will prove that these multipliers are all positive. Finally from (16) and (19), we will prove the inequality

$$A^2\left(\frac{p-\lambda k}{1-\lambda}\right)^2 r^2\lambda_{n,N} \leq \sum_{k=1}^{N} k|c_{k+p-1}|^2 r^{2k} \tag{2.17}$$

in step III. Here $c_{k+p-1} = \frac{A^k\left(\frac{p-\lambda k}{1-\lambda}\right)^k}{k!}$
For the calculation of the factors $\lambda_{n,N}$, we get the following system of linear equations,

$$k = k^2\lambda_{k,N} + \sum_{n=k+1}^{N} \lambda_{n,N}\left(k^2 - A^2\left(\frac{p-\lambda k}{1-\lambda}\right)^2 r^2\right), k = 1, 2, ...N \tag{2.18}$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (21) in the form

$$\lambda_{n,N} = \left(\frac{(n-1)!}{N!}\right)^2 DetA_{n,N},$$

where $A_{n,N}$ is the $(N-n+1)\times(N-n+1)$ matrix constructed as follows

$$A_{n,N} = \begin{pmatrix} n & n^2 - \left(\frac{p-\lambda k}{1-\lambda}\right)^2 A^2r^2 & . \;.\;. & n^2 - \left(\frac{p-\lambda k}{1-\lambda}\right)^2 A^2r^2 \\ n+1 & (n+1)^2 & . \;.\;. & (n+1)^2 - \left(\frac{p-\lambda k}{1-\lambda}\right)^2 A^2r^2 \\ . & . & . \;.\;. & . \\ . & . & . \;.\;. & . \\ . & . & . \;.\;. & . \\ N & 0 & . \;.\;. & N^2 \end{pmatrix}$$

Determinants of these matrices can be obtained by expanding, according to Laplace's rule with respect to the last row, wherein the first coefficient is N, and the last entry is N^2. The rest of the entries are zeros. This expansion and a mathematical induction lead to the following formula: if $k \leq N-1$

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1 - \left(\frac{p}{k} - \lambda\right)^2 \frac{A^2r^2}{(1-\lambda)^2}\right) \prod_{\gamma=k+1}^{N-1}\left(\frac{(\frac{p-\lambda k}{1-\lambda})^2 A^2r^2}{\gamma^2}\right).$$

We see that the sequence $\lambda_{k,N}$ is strictly decreasing in N when $k \in \mathbb{N}$ is fixed and $k \leq N$, i.e., $\lambda_{k,N} < \lambda_{k,N-1}$ with

$$\lambda_k := lim_{N\to\infty}\lambda_{k,N} = \frac{1}{k} - \left(1 - \left(\frac{p}{k} - \lambda\right)^2 \frac{A^2r^2}{(1-\lambda)^2}\right) \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{\gamma=k+1}^{N-1}\left(\frac{(\frac{p-\lambda k}{1-\lambda})^2 A^2r^2}{\gamma^2}\right) \tag{2.19}$$

To prove that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}$, $1 \leq k \leq \mathbb{N}$ it is adequate to show that $\lambda_k \geq 0$ for $k \in \mathbb{N}$. This will be completed in step II. But before that we want the remark that the proof of the said inequality is sufficient for the proof of the theorem, since, as we remarked for (18), equality holds for $b_{k+p-1} = c_{k+p-1}$.

Step-II: Positivity of Multipliers. In this step, we show that

$$\sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{\gamma=k+1}^{n-1}\left(\frac{(\frac{p-\lambda k}{1-\lambda})^2 A^2r^2}{\gamma^2}\right) \leq \frac{1}{k\left(1 - \left(\frac{p}{k} - \lambda\right)^2 \frac{A^2r^2}{(1-\lambda)^2}\right)}$$

$$= \frac{1}{k}\sum_{n=k+1}^{\infty}\left(\left(\frac{p}{k} - \lambda\right)^2 \frac{A^2r^2}{(1-\lambda)^2}\right)^n$$

which is indeed easy to prove i.e from (22) $\lambda_k \geq 0$.

Step-III Since the sequence $\lambda_{n,N}$ is strictly decreasing in N for each fixed n, $n \leq N$, i.e., $\lambda_{n,N} < \lambda_{n,n}$, so that

$$\left(\left(\frac{p-\lambda k}{1-\lambda}\right)^2 A^2 r^2\right)\lambda_{n,N} < \left(\left(\frac{p-\lambda k}{1-\lambda}\right)^2 A^2 r^2\right)\lambda_{n,n} = \left(\frac{(\frac{p-\lambda k}{1-\lambda})^2 A^2 r^2}{n}\right)$$

$$< \left(\left(\frac{p-\lambda k}{1-\lambda}\right)^2 A^2 r^2\right) \leq \sum_{k=1}^{N} k \frac{\left(\frac{p-\lambda k}{1-\lambda}\right)^{2k}}{(k!)^2} A^{2k} r^{2k}.$$

Thus the inequality (20) holds. □

Theorem 2.6. Let $f \in \mathcal{S}_p^*(A, 0, \lambda)$ for $0 < A \leq 1$ and $p \in \mathbb{N}$, be of the form $\frac{z^p}{f(z)}$. Then for $0 < r \leq 1$, we have

$$\max_{f \in \mathcal{S}_p^*(A,0,\lambda)} \Delta\left(r, \frac{z^p}{f}\right) = E_{A,0,\lambda}(r,p)$$

where

$$E_{A,0,\lambda}(r,p) = \pi\left(\frac{A}{1-\lambda}\right)^2 \left(p - \lambda(k-1)\right)^2 r_0^2 F_1\left(2, (\frac{A}{1-\lambda})^2 \left(p - \lambda(k-1)\right)^2 r^2\right).$$

The maximum is attained by the rotations of the function

$$k_{A,0,\lambda}(z) = z^p \left(1 - \frac{\lambda A z}{1-\lambda}\right)^{-\frac{p}{\lambda}}.$$

Proof . Let $f \in \mathcal{S}_p^*(A, 0, \lambda)$ for $0 < A \leq 1$ and $p \in \mathbb{N}$. It is enough to prove the theorem for $0 < A \leq 1$. By the definition of $\mathcal{S}_p^*(A, 0, \lambda)$, we get

$$\frac{zf'(z)}{pf(z)} \prec \frac{1+Az}{1-\frac{\lambda A z}{1-\lambda}} = \frac{zk'_{A,0,\lambda}(z)}{k_{A,0,\lambda}(z)}.$$

Let $g(z) = \frac{z^p}{f(z)}$. Then using Theorem 1.1 (see [8]) we get

$$g(z) \prec \left(1 - \frac{\lambda A z}{1-\lambda}\right)^{\frac{p}{\lambda}} = \frac{z^p}{k_{A,B,\lambda}(z)} \tag{2.20}$$

By rewriting (23) in power series form, we have

$$1 + \sum_{k=1}^{\infty} b_{k+p-1} z^k \prec \left(1 - \frac{\lambda A z}{1-\lambda}\right)^{\frac{p}{\lambda}} = 1 + \sum_{k=1}^{\infty} c_{k+p-1} z^k$$

where $c_{k+p-1} = (-1)^k \frac{\left(\frac{p}{\lambda} - (k-1)\right)_k}{k!} \left(\frac{\lambda A}{1-\lambda}\right)^k$. Now, by lemma (1.3) for $r \in (0, 1]$, we have

$$\sum_{k=1}^{N} k|b_{k+p-1}|^{2k} \leq \sum_{k=1}^{N} k|c_{k+p-1}|^{2k}, N \in \mathbb{N}.$$

If we assume $N \to \infty$, then it follows that,

$$\pi \sum_{k=1}^{N} k|b_{k+p-1}|^{2k} \leq \pi \sum_{k=1}^{N} k|c_{k+p-1}|^{2k}, N \in \mathbb{N}$$

i.e.,

$$\Delta\left(r, \frac{z^p}{f}\right) = \Delta\left(r, \frac{z^p}{k_{A,0,\lambda}}\right).$$

We claim that

$$\pi\sum_{k=1}^{N}k|c_{k+p-1}|^{2k}=E_{A,0,\lambda}(r,p).$$

To prove the claim, we have

$$\pi^{-1}\Delta\left(r,\frac{z^p}{k_{A,0,\lambda}}\right)=\sum_{k=1}^{N}k|c_{k+p-1}|^{2k}$$

$$=\sum_{k=1}^{\infty}k\frac{\left(\frac{p}{\lambda}-(k-1)\right)_k\left(\frac{p}{\lambda}-(k-1)\right)_k}{(1)_k^2}\left(\frac{\lambda A}{1-\lambda}\right)^{2k}r^{2k}$$

$$=\left(\frac{\lambda A}{1-\lambda}\right)^2 r^2\left(\frac{p-\lambda(k-1)}{\lambda}\right)^2\sum_{k=0}^{\infty}\frac{\left(\left(\frac{p}{\lambda}-(k-1)\right)+1\right)_k\left(\left(\frac{p}{\lambda}-(k-1)\right)+1\right)_k}{(2)_k(1)_k}\left(\frac{\lambda A}{1-\lambda}\right)^{2k}r^{2k}$$

$$=\left(\frac{A}{1-\lambda}\right)^2(p-\lambda(k-1))^2r^2\sum_{k=0}^{\infty}\frac{\left(\left(\frac{p}{\lambda}-(k-1)\right)+1\right)_k\left(\left(\frac{p}{\lambda}-(k-1)\right)+1\right)_k}{(2)_k(1)_k}\left(\frac{\lambda A}{1-\lambda}\right)^{2k}r^{2k}$$

$$=\left(\frac{A}{1-\lambda}\right)^2(p-\lambda(k-1))^2r_0^2F_1\left(2,\left(\frac{A}{1-\lambda}\right)^2(p-\lambda(k-1))^2r^2\right).$$

Hence

$$\Delta\left(r,\frac{z^p}{k_{A,0,\lambda}}\right)=\pi\left(\frac{A}{1-\lambda}\right)^2(p-\lambda(k-1))^2r_0^2F_1\left(2,\left(\frac{A}{1-\lambda}\right)^2(p-\lambda(k-1))^2r^2\right)=E_{A,0,\lambda}(r,p)$$

□

Corollary 2.7. Let $f\in S_p^*(1,-1,\lambda)$ with $0\le\lambda<1$ and $p\in\mathbb{N}$. Then for $0<r\le1$,

$$\max_{f\in S_p^*(1,-1,\lambda)}\Delta(r,\frac{z}{f})=\frac{4\pi}{(1-\lambda)^2}p^2r_2^2F_1\left(\frac{-2p}{1+\lambda}+1,\frac{-2p}{1+\lambda}+1;2;\left(\frac{1+\lambda}{1-\lambda}\right)^2r^2\right)$$

and the maximum is attained by the rotation of the function

$$k_{1,-1,\lambda}(z)=z^p\left(1-\frac{1+\lambda}{1-\lambda}z\right)^{\frac{-2p}{1+\lambda}}.$$

Corollary 2.8. Let $f\in S_p^*((1-\frac{2\beta}{p}),-1,\lambda)$ with $0\le\lambda<1$ and $p\in\mathbb{N}$. Then for $0<r\le1$, $\max_{f\in S_p^*((1-\frac{2\beta}{p}),-1,\lambda)}\Delta(r,\frac{z}{f})$

$$=\frac{4\pi}{(1-\lambda)^2}(p-\beta)^2r_2^2F_1\left(\frac{2(\beta-p)}{1+\frac{\lambda(p-2\beta)}{p}}+1,\frac{2(\beta-p)}{1+\frac{\lambda(p-2\beta)}{p}}+1;2;\left(\frac{1+\frac{\lambda(p-2\beta)}{p}}{1-\lambda}\right)^2r^2\right)$$

and the maximum is attained by the rotation of the function

$$k_{(1-\frac{2\beta}{p}),-1,\lambda}(z)=z^p\left(1-\left(\frac{1+\frac{\lambda(p-2\beta)}{p}}{1-\lambda}\right)\right)^{\frac{2(p-\beta)}{1+\frac{\lambda(p-2\beta)}{p}}}.$$

Corollary 2.9. Let $f\in S_p^*(1,0,\lambda)$ with $0\le\lambda<1$ and $p\in\mathbb{N}$. Then for $0<r\le1$,

$$\max_{f\in S_p^*(1,0,\lambda)}\Delta(r,\frac{z}{f})=\frac{\pi}{(1-\lambda)}^2p^2r_2^2F_1\left(-\frac{p}{\lambda}+1,-\frac{p}{\lambda}+1;2;\left(\frac{\lambda}{1-\lambda}\right)^2r^2\right)$$

and the maximum is attained by the rotation of the function

$$k_{1,0,\lambda}(z)=z^p\left(1-\frac{\lambda}{1-\lambda}z\right)^{-\frac{p}{\lambda}}.$$

3 Length Problem for the class $S_p^*(A,B,\lambda)$

Theorem 3.1. Let $f \in S_p^*(A,B,\lambda)$ for $A \in \mathbb{C}$, $-1 \le B \le 0$, $A \neq B$, $p \in \mathbb{N}$ and f be of the form (3). Then for $0 < r \le 1$, we have

$$L_1(r,f,p) = r^{2p} I_1(r,f,p) \le \ {}_2F_1\left(\frac{A-B}{B-\lambda A}p, \frac{\overline{A}-B}{B-\lambda\overline{A}}p; 1; \left(\frac{B-\lambda}{1-\lambda}\right)^2\right), \text{ if } B \neq 0$$

$$L_1(r,f,p) = r^{2p} I_1(r,f,p) \le_2 F_1\left(\left(\frac{p}{\lambda}-(n-1)\right), \left(\frac{p}{\lambda}-(n-1)\right); 1; \left(\frac{\lambda A}{1-\lambda}\right)^2\right), \text{ if } B = 0$$

where $\alpha = \frac{A-B}{B-\lambda A}$

Proof . Let $f \in S_p^*(A,B,\lambda)$. By applying the theorem of Hallenbeck and Ruscheweyh theorem [8] we get,

$$\frac{f(z)}{z^p} \prec \frac{1}{\left(1+\frac{B-\lambda A}{1-\lambda}z\right)^{-(\frac{A-B}{B-\lambda A})p}}$$

so that

$$\frac{z^p}{f(z)} \prec \left(1+\frac{B-\lambda A}{1-\lambda}z\right)^{-(\frac{A-B}{B-\lambda A})p} = \xi_{A,B,\lambda(z)} \tag{3.1}$$

where

$$\xi_{A,B,\lambda(z)} = \frac{z^p}{k_{A,B,\lambda}(z)} = \left(1+\frac{B-\lambda A}{1-\lambda}z\right)^{-(\frac{A-B}{B-\lambda A})p}$$

if $B \neq 0$ and

$$\xi_{A,B,\lambda}(z) = \left(1-\frac{\lambda A z}{1-\lambda}\right)^{\frac{p}{\lambda}} = \sum_{n=0}^{\infty} d_{n+p-1} z^n \tag{3.2}$$

if $B = 0$, with $\alpha = \frac{A-B}{B-\lambda A}$. Supposing $B \neq 0$, then

$$d_{n+p-1} = \frac{(-1)^n (p\alpha)_n (\frac{B-\lambda A}{1-\lambda})^n}{n!}$$

and if $B = 0$, then

$$d_{n+p-1} = \frac{(-1)^n \left(\frac{p}{\lambda}-(k-1)\right)_n (\frac{\lambda A}{1-\lambda})^n}{n!}.$$

From (24), $\frac{z^p}{f}$ and $\xi_{A,B,\lambda}$ are two analytic functions and have the series representation (3) and (25) with ($b_{p-1} = 1 = d_{p-1}$), respectively. Then by Theorem 1.2 (Rogosinski's Theorem [15]), we get

$$\sum_{n=0}^{k} |b_{n+p-1}|^2 r^{2n} \le \sum_{n=0}^{k} |d_{n+p-1}|^2 r^{2n}$$

for $0 < r \le 1$ and $k \in \mathbb{N}$. Thus, from (24) and (25) we obtain

$$\sum_{n=0}^{k} |b_{n+p-1}|^2 r^{2n} \le \begin{cases} \sum_{n=0}^{k} \frac{(p\alpha)_n (p\overline{\alpha})_n}{(n!)^2} \left(\frac{B-\lambda\overline{A}}{1-\lambda}\right)^{2n} r^{2n}, \text{ if } B \neq 0 \\ \sum_{n=0}^{k} \frac{\left(\frac{p}{\lambda}-(n+1)\right)_n \left(\frac{p}{\lambda}-(n-1)\right)_n}{(n!)^2} \left(\frac{\lambda\overline{A}}{1-\lambda}\right)^{2n} r^{2n}, \text{ if } B = 0. \end{cases}$$

If we take $r \to 1$ and allow $k \to \infty$, then we find the inequality

$$1 + \sum_{n=1}^{\infty} |b_{n+p-1}|^2 \le \sum_{n=0}^{\infty} \frac{(p\alpha)_n (p\overline{\alpha})_n}{(n!)^2} \left(\frac{B-\lambda\overline{A}}{1-\lambda}\right)^{2n}, \text{ if } B \neq 0$$

and

$$1+\sum_{n=1}^{\infty}|b_{n+p-1}|^2 \leq \sum_{n=0}^{\infty}\frac{\left(\frac{p}{\lambda}-(n-1)\right)_n\left(\frac{p}{\lambda}-(n-1)\right)_n}{(n!)^2}\left(\frac{\lambda\overline{A}}{1-\lambda}\right)^{2n}.$$

$$={}_2F_1\left(p\alpha, p\overline{\alpha}; 1; \left(\frac{B-\lambda\overline{A}}{1-\lambda}\right)^2\right),\ B\neq 0$$

and

$$={}_2F_1\left(\left(\frac{p}{\lambda}-(n-1)\right),\left(\frac{p}{\lambda}-(n-1)\right);1;\left(\frac{\lambda\overline{A}}{1-\lambda}\right)^2\right),\text{if } B=0.$$

Now, we evaluate the integral means for the function $\frac{z^p}{f}$ and get

$$L_1(r,f,p)=r^{2p}I_1(r,f,p)=\frac{1}{2\pi}\int_0^{2\pi}\frac{r^{2p}}{|f(re^{i\theta})|^2}d\theta=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{z^p}{f(z)}\right|^2 d\theta$$

$$=1+\sum_{n=1}^{\infty}|b_{n+p-1}|^2r^{2n}\leq 1+\sum_{n=1}^{\infty}|b_{n+p-1}|^2$$

which establishes the desired inequality. The result is sharp and it can be easily verified by considering the function $\frac{z^p}{k_{A,B,\lambda}(z)}$ □

Corollary 3.2. Let $f\in S_p^*(1,-1,\lambda)$, $p\in\mathbb{N}$ and $0\leq\lambda<1$ then we have

$$L_1(r,f,p)\leq\frac{\Gamma\left(1+\frac{4p}{1+\lambda}\right)}{\Gamma^2\left(1+\frac{2p}{1+\lambda}\right)}.$$

The inequality is sharp.

Corollary 3.3. Let $f\in S_p^*((1-\frac{2\beta}{p}),-1,\lambda)$, $p\in\mathbb{N}$, $0\leq\lambda<1$ and $0\leq\beta<p$ then we have

$$L_1(r,f,p)\leq\frac{\Gamma\left(1-\frac{4(\beta-p)}{1+\frac{\lambda(1-2\beta)}{p}}\right)}{\Gamma^2\left(1-\frac{2(\beta-p)}{1+\frac{\lambda(1-2\beta)}{p}}\right)}.$$

The inequality is sharp.

Corollary 3.4. Let $f\in S_p^*(1,0,\lambda)$, $p\in\mathbb{N}$ and $0\leq\lambda<1$ then we have

$$L_1(r,f,p)\leq\frac{\Gamma\left(1+\frac{2p}{\lambda}\right)}{\Gamma^2\left(1+\frac{p}{\lambda}\right)}.$$

The inequality is sharp.

Remark 3.5. When $\lambda=0$, each of the corollaries above reduces to the corresponding corollaires obtained by Sharma [18] in 2017.

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