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Stability for class of starlike functions related to a shell-like curve connected with Fibonacci numbers

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Abstract

In this paper, the problem of stability for the subclass $\mathcal{SL}$ of analytic univalent functions in T_δ-neighborhoods is studied. Also, the lower and upper bounds of the radius of stability are obtained.

Keywords: Stability of Hadamard product, Integral convoiution, Analytic functions, starlike and convex functions, fibonacci numbers, Hypergeometric function
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1 Introduction

Assume that $\mathcal{A}$ is the class of all holomorphic functions f in the open unit disc $\mathbb{U}$ with the normalization $f(0)=0, f'(0)=1$. We say that f is subordinate to F in $\mathbb{U}$, written as $f\prec F$ if and only if $f(z)=F(\omega(z))$ for some holomorphic function ω such that $\omega(0)=0$ and $|\omega(z)|<1$, for all $z\in\mathbb{U}$. The class $\mathcal{SL}$ of shell-like functions is the set of functions $f\in\mathcal{A}$ satisfying the condition

$$\frac{zf'(z)}{f'(z)}\prec\tilde{p}(z) \tag{1.1}$$

where

$$\tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},\qquad \tau=\frac{1-\sqrt{5}}{2}\approx-0.618.\ (z\in\mathbb{U}) \tag{1.2}$$

was investigated by J. Dziok, R.K. Raina and J. Sokół in [8]. The function (1.2) has some nice properties. The name attributed to the class $\mathcal{SL}$ is motivated by the shape of the curve

$$W=\left\{\tilde{p}(e^{it}), t\in[0,2\pi)\setminus\{\pi\}\right\},$$

which is a shell-like curve and a simple transformation converts it into a curve called the conchoid of de Sluze (René Francois Baron de Sluze 1622-1685). Moreover, the coefficients of (1.2) are connected with the Fibonacci numbers as explained in Lemma 1.2 and in the next section. A geometric description of the conchoid of de Sluze is given here as follows:

A ray OB is drawn from the point $O\,(0,0)$ and it cuts the directrix $x=a$, where $a>0$ at the point $B(a,b)$. From the point $B(a,b)$, segments BM and BN are laid off in either direction along the ray such that $|OB||BM|=k^2$ and

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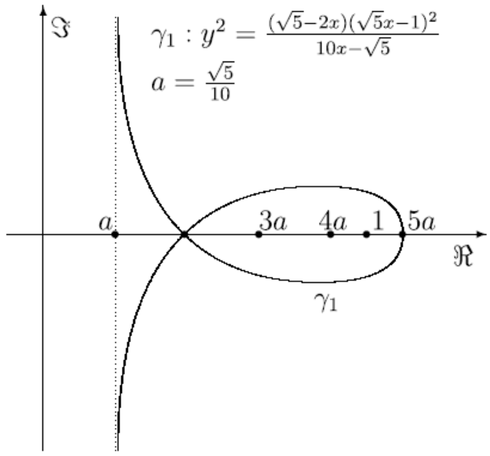

Figure 1:

$|OB||BN| = k^2$, where $k > 0$ is given. As B changes, the ray revolves and the point M describes a curve (called the conchoid of de Sluze) given by

$$a(x-a)(x^2+y^2) + k^2x^2 = 0, \tag{1.3}$$

while the point N describes a curve (called the conjugate of the conchoid of de Sluze) given by

$$a(x-a)(x^2-y^2) - k^2x^2 = 0. \tag{1.4}$$

Definition 1.1. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it is satisfies the condition following

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)\ (z \in \mathbb{U}), \tag{1.5}$$

where the function $\tilde{p}$ is defined in (1.2).

The radius of univalence and several other properties of the function $\tilde{p}$ were found in [10]. Let us recall some of them.

Lemma 1.2. [3] Let the function $\tilde{p}$ be given by (1.2), then it satisfies the following:

1. $\tilde{p}$ is univalent in the disc $|z| < \frac{3-\sqrt{5}}{2} \approx 0.38$, any increase in the greater side makes the assertion false,
2. $\tilde{p}(z) = 1 + \sum_{n=1}^{\infty}(u_{n-1} + u_{n+1})\tau^n z^n = 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + ...$, where u_n is the sequence of Fibonacci numbers $u_0 = 0, u_1 = 1$,
$u_{n+2} = u_n + u_{n+1} (n = 0, 1, 2, 3, ...)$,
3. $\lim_{\varphi\to\pi^-} \Im m\left[\tilde{p}(e^{i\varphi})\right] = -\infty$ and $\lim_{\varphi\to\pi^+} \Im m\left[\tilde{p}(e^{i\varphi})\right] = \infty$,
4. $\Re e\tilde{p}(e^{i\varphi}) = \frac{\sqrt{5}}{2(3-2\cos\varphi)} \geq \frac{\sqrt{5}}{10} = a$ for all $\varphi \in [0, 2\pi)$.

2 Preliminary lemmas

We note that if $f(z) \in \mathcal{SL}$ then $\frac{zf'(z)}{f(z)}$ lies in the region G, see Fig .1. For some in-depth understanding of the class $\mathcal{SL}$ it would be worthwhile here to find the shape of the curve $W = \left\{\tilde{p}(e^{it}), t \in [0, 2\pi) \setminus \{\pi\}\right\}$. We begin our study by noting that $\tilde{p}(0) = \tilde{p}(-\frac{1}{2\tau}) = 1$ and $\tilde{p}(\tau^4) = \frac{\sqrt{5}}{2}$. Moreover because $\tilde{p}(e^{\pm i \arccos(1/4)}) = \frac{\sqrt{5}}{5} = 2a$, so the curve W

intersect itself on the real axis in the point $2a$. If we denote $\Re e\tilde{p}(e^{i\varphi}) = x$ and $\Im m(e^{i\varphi}) = y$, $\varphi \in [0, 2\pi) \setminus \{\pi\}$, then after simple calculations, we get

$$x = \frac{1-2\tau}{2(3-2\cos\varphi)}, \quad y = \frac{\sin\varphi(4\cos\varphi - 1)}{2(3-2\cos\varphi)(1+\cos\varphi)}, \ \varphi \in [0, 2\pi) \setminus \{\pi\}. \tag{2.1}$$

Note that the boundery of $\tilde{p}(z)$ is $W = x + iy$ where x and y are in (2.1). It is useful here to use (2.1) to find the corresponding Cartesian equation of the curve W. This curve is described by the equation

$$(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2. \tag{2.2}$$

It is worthy to point out that for $k = 2a$, the conchoid of de Sluze (1.3) becomes the trisectrix of Maclaurin (Colin Maclaurin 1698 – 1746):

$$x^3 - 3ax^2 + (x - a)y^2 = 0. \tag{2.3}$$

while the conjugate of the conchoid (1.4) becomes the conjugate of the trisectrix of Maclaurin given by

$$x^3 - 5ax^2 + (x - a)y^2 = 0. \tag{2.4}$$

If we rewrite (2.2) in the following form

$$(\frac{\sqrt{5}}{5} - x)^3 + \frac{3\sqrt{5}}{10}(\frac{\sqrt{5}}{5} - x)^2 + \left[(\frac{\sqrt{5}}{5} - x) - \frac{\sqrt{5}}{10}\right] y^2 = 0.$$

then the image of the unit circle under the function $\tilde{p}$ is translated into a trisectrix of Maclaurin (2.3) (with $a = \frac{1-2\tau}{10} = \frac{\sqrt{5}}{10}$). Therefore the curve W has a shell-like shape, see Fig .1.

Let the Hadamard product (or convolution) of two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, be given by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ and the integral convolution be given by

$$(f \otimes g)(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n$$

Also note that if I denotes $I(z) = z$ then $f * I = I$ and $f \otimes I = I$.

The convolution has the algebraic properties of ordinary multiplication. In convolution theory, the concept of duality is important. Many authors have used the powerful method of duality for study properties of analytic function (for example, see [7, 11, 20]. The concept of duality in geometric function theory was stated by Ruscheweyh in the book[19]. Denote the dual set of $\nu \subset \mathcal{A}$ by ν^* then according to definition in [18] we have

$$\nu^* = \{g \in \mathcal{A} : \frac{(f * g)(z)}{z} \neq 0\,, f \in \nu, z \in \mathbb{U}\}.$$

Let $D \subset \mathcal{A}$ be given such that $D^* = \mathcal{SL}$. Then it is easy to see that

$$f \in \mathcal{SL} \qquad iff \qquad \frac{(f * g)(z)}{z} \neq 0, \quad (g \in D, z \in \mathbb{U})\}.$$

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then T_δ - neighborhood of the function f is defined as

$$TN_\delta(f) = \{g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \delta\},$$

where $\delta > 0$ and $T = \{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers.
In[20, 6] authors investigated T_δ - neighborhood for various subclasses of analytic functions.

We also define $TN_\delta(A) = \bigcup_{f \in A} TN_\delta(f), (A \subset \mathcal{A})$. St. Ruscheweyh in [17] considered $T = \{T_n\}_{n=2}^{\infty}$ and showed that if $f \in \mathcal{K}$, then $TN_{1/4}(f) \subset S^*$.

Assume that A, B are subclasses of the class $\mathcal{A}$. Then the set of all functions $f * g$ and $f \otimes g$, where $f \in A$ and $g \in B$, will be denoted by $A * B$ and $A \otimes B$, respectively. Let $A * B \subset C$, the Hadamard product is called T-C-stable on the pair of classes (A, B) if there exists $\delta > 0$ such that $TN_\delta(A) * TN_\delta(B) \subset C$. Stability of the integral convolution is defined in a similar way. The constant δ_T which characterizes the stability of Hadamard or integral convolution is called the radius of stability and it is defined as follows.

Definition 2.1. Let A, B, C be the subclasses of the class $\mathcal{A}$ and $A*B \subset C$. Then the constant $\delta_T(A*B, C)$, defined by

$$\delta_T(A*B, C) = sup\{\delta : TN_\delta(A) * TN_\delta(B) \subset C\},$$

is called the radius of stability of the convolution on the pair (A, B). The constant $\delta_T(A \otimes B, C)$, defined by

$$\delta_T(A \otimes B, C) = sup\left\{\delta : TN_\delta(A) \otimes TN_\delta(B) \subset C\right\},$$

is called the radius of stability of the integral convolution on the pair (A, B).

Bednarz in [3] studied T-C-stability for certain classes of analytic functions. Also, Bednarz, Kanas, Sokół, Aghalary and Shams et al. [1, 4, 5, 20] recently investigated the problem of stability for various subclasses of analytic functions. In this paper we investigate the problem of stability for the class $\mathcal{SL}$ in T_δ - neighborhoods and we find the lower and upper bounds for radius of stability.

3 Preliminaries

For obtaining our results we need the following definitions and lemmas to prove our main results.

Lemma 3.1. Let $H_\varphi(z)$ be given by

$$H_\varphi(z) = \left[\frac{z}{(1-z)^2} - \frac{Wz}{1-z}\right]/(1-W)$$

where $W = x + iy$ and x , y are introduced in (2.1), then a function $f \in \mathcal{A}$ is in $\mathcal{SL}$ if and only if for all z in $\mathbb{U}$, $\dfrac{(f*H_\varphi)(z)}{z} \neq 0$.

Proof . Let us assume that for $f \in \mathcal{A}$, $\dfrac{(f*H_\varphi)(z)}{z} \neq 0$, $(z \in \mathbb{U})$, then we have

$$\begin{aligned}\frac{(f*H_\varphi)(z)}{z} &= \frac{1}{1-W}\{f(z)*[\frac{z}{(1-z)^2} - \frac{Wz}{1-z}]\} \\ &= [zf'(z) - Wf(z)]/(1-W) \\ &\neq 0\end{aligned}$$

or

$$\frac{zf'(z)}{f(z)} \neq W$$

since boundry of region G can be taken as W of the form (2.1), this means that $\dfrac{zf'(z)}{f(z)}$ lies completely either inside G or complement of G for all $z \in \mathbb{U}$. At $z = 0$, $\dfrac{zf'(z)}{f(z)} = 1 \in G$ so that $\dfrac{zf'(z)}{f(z)} \in G$ for all $z \in \mathbb{U}$, which shows that $f \in \mathcal{SL}$. The converse part follows easily since all the steps can be retraced back. This completes the proof of Lemma 3.1 □

Using the definition of dual set and Lemma 3.1 we can easily obtain the following result.

Corollary 3.2. Let $D = \left\{h \in \mathcal{A} : h(z) = \dfrac{1}{1-W}\left[\frac{z}{(1-z)^2} - \frac{Wz}{1-z}\right]\right\}$ where W of the form (2.1) then $D^* = \mathcal{SL}$.

Lemma 3.3. If $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in D$. Then

$$|c_n| \leq \frac{20n + 4\tau + 3}{[1 - 2\tau - 2(3 - 2\cos\varphi)]\,(1 + \cos\varphi)}, \quad \cos\varphi \neq -1, \frac{5}{4} + \frac{\tau}{2}$$

Proof . From the power series expansion of the function $h(z) \in D$ in Corollary 3.2 also from (2.1) we obtain

$$c_n = \frac{n-W}{1-W}$$

and therefore

$$\begin{aligned}
|c_n|^2 &= \frac{|n-W|^2}{|1-W|^2} \\
&= \frac{\left(n - \frac{1-2\tau}{2(3-2\cos\varphi)}\right)^2 + \left(\frac{\sin\varphi(4\cos\varphi-1)}{2(3-2\cos\varphi)(1+\cos\varphi)}\right)}{\left(1 - \frac{1-2\tau}{2(3-2\cos\varphi)}\right)^2 + \left(\frac{\sin\varphi(4\cos\varphi-1)}{2(3-2\cos\varphi)(1+\cos\varphi)}\right)^2} \\
&\leq \frac{\left(n - \frac{1-2\tau}{2(3-2\cos\varphi)}\right)^2 + \left(\frac{\sin\varphi(4\cos\varphi-1)}{2(3-2\cos\varphi)(1+\cos\varphi)}\right)^2}{\left(1 - \frac{1-2\tau}{2(3-2\cos\varphi)}\right)^2} \\
&= \frac{(1+\cos\varphi)^2\left[2n(3-2\cos\varphi)+2\tau-1\right]^2 + \sin^2\varphi(4\cos\varphi+1)^2}{\left[2(3-2\cos\varphi)+2\tau-1\right]^2(1+\cos\varphi)^2} \\
&\leq \frac{4(10n-1+2\tau)^2+25}{\left[2(3-2\cos\varphi)+2\tau-1\right]^2(1+\cos\varphi)^2} \\
&\leq \frac{(20n+4\tau+3)^2}{\left[2(3-2\cos\varphi)+2\tau-1\right]^2(1+\cos\varphi)^2} \qquad (3.1)
\end{aligned}$$

and so

$$|c_n| \leq \frac{20n+4\tau+3}{\left[1-2\tau-2(3-2\cos\varphi)\right](1+\cos\varphi)}, \quad \cos\varphi \neq -1, \frac{5}{4}+\frac{\tau}{2}$$

□

Corollary 3.4. $g(z) = z + Az^n \in \mathcal{SL}$ if and only if

$$|A| \leq \frac{\left[1-2\tau-2(3-\cos\varphi)\right](1+\cos\varphi)}{20n+4\tau+3} \qquad (3.2)$$

Proof . First we prove the sufficient condition. Since for every $h \in D$ we have

$$\begin{aligned}
\left|\frac{(g*h)(z)}{z}\right| &= |1 + c_n A z^{n-1}| \\
&\geq 1 - |c_n A z| \\
&\geq 1 - |z| \\
&> 0 \qquad (3.3)
\end{aligned}$$

so Corollary 3.2 gives $g \in D^* = \mathcal{SL}$. For necessity suppose that $g \in \mathcal{SL}$, and

$$h(z) = z + \sum_{n=2}^{\infty} \frac{20n+4\tau+3}{\left[1-2\tau-2(3-2\cos\varphi)\right](1+\cos\varphi)} z^n \in D.$$

Then

$$\frac{(g*h)(z)}{z} = 1 + A\frac{20n+4\tau+3}{\left[1-2\tau-2(3-2\cos\varphi)\right](1+\cos\varphi)} z^{n-1}.$$

Then for $|A| > \frac{[1-2\tau-2(3-\cos\varphi)](1+\cos\varphi)}{20n+4\tau+3}$ there exist a point $\xi \in \mathbb{U}$ such that $\frac{(g*h)(\xi)}{\xi} = 0$, this shows that the inequality 3.2 must hold. □

Corollary 3.5. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{A}$. If

$$\sum_{n=2}^{\infty}\frac{20n+4\tau+3}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}|a_n|\leq 1,$$

then $f\in\mathcal{SL}$.

Proof . Let $h(z)=z+\sum_{n=2}^{\infty}c_nz^n\in D$, since for all $n\geq 2$,

$$|c_n|\leq\frac{20n+4\tau+3}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)},$$

then we have

$$\begin{aligned}|\frac{(f*h)(z)}{z}| &= |1+\sum_{n=2}^{\infty}a_nc_nz^{n-1}|\\ &\geq 1-\sum_{n=2}^{\infty}|a_n||c_n||z|\\ &> 1-\sum_{n=2}^{\infty}|a_n||c_n|\\ &\geq 0.\end{aligned}\tag{3.4}$$

Then $\frac{(f*h)(z)}{z}\neq 0$ and from Corollary 3.2 we have $f\in D^*=\mathcal{SL}$. □

Lemma 3.6. if for $f\in\mathcal{A}$ and every $\epsilon\in\mathbb{C}$ such that $|\epsilon|<\delta$, $F_\epsilon(z)=\frac{f(z)+\epsilon z}{1+\epsilon}\in\mathcal{SL}$, then for every $h\in D$, $|\frac{(f*h)(z)}{z}|\geq\delta$, $(z\in\mathbb{U})$.

Proof . Let $F_\epsilon\in\mathcal{SL}$, then by Corollary 3.2 for every $h\in D$ we have $\frac{(F_\epsilon*h)(z)}{z}\neq 0\quad z\in\mathbb{U}$. Equivalently, $\frac{(f*h)(z)+\epsilon z}{(1+\epsilon)z}\neq 0$ in $\mathbb{U}$ or $\frac{(f*h)(z)}{z}\neq-\epsilon$ which shows that $|\frac{(f*h)(z)}{z}|\geq\delta$. □

Lemma 3.7. [19] Let f(z) and g(z) be in the class $\mathcal{K}$ and $\mathcal{S}^*$ respectively. Then for every function $F(z)$ analytic in $\mathbb{U}$, we have

$$\frac{f(z)*F(z)g(z)}{f(z)*g(z)}\in\overline{Co}(F(\mathbb{U}))\ ,\ z\in\mathbb{U},$$

where $\overline{Co}$ denote the closed convex hull.

Lemma 3.8. If $f\in\mathcal{SL}$ and $g\in\mathcal{K}$. Then $f*g\in\mathcal{SL}$.

Proof . It is sufficient to prove that

$$\frac{z(g*f)'(z)}{(g*f)(z)}\in G.$$

We have

$$\begin{aligned}\frac{z(g*f)'(z)}{(g*f)(z)} &= \frac{g(z)*zf'(z)}{g(z)*f(z)}\\ &= \frac{g(z)*\frac{zf'(z)}{f(z)}f(z)}{g(z)*f(z)},\end{aligned}\tag{3.5}$$

where $F(z) = \frac{zf'(z)}{f(z)}$. Since

$$\mathcal{R}e\frac{zf'(z)}{f(z)} > 0,$$

so using Lemma3.7 we obtain

$$\frac{z(g*f)'(z)}{(g*f)(z)} \in \bar{C}o(F(\mathbb{U})) \subset G.$$

Then $f*g \in \mathcal{SL}$. $\square$

Definition 3.9. A function $f \in \mathcal{A}$ is said to be in the class N if for all $z \in \mathbb{U}$,

$$zf'(z) \in \mathcal{SL}.$$

Lemma 3.10. If $f \in \mathcal{N}$, then for ϵ with $|\epsilon| < \dfrac{1}{4}$,

$$F_\epsilon(z) = \frac{f(z) + \epsilon z}{1+\epsilon} \in \mathcal{SL}.$$

Proof . Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$\begin{aligned} F_\epsilon(z) &= \frac{f(z)+\epsilon z}{1+\epsilon} \\ &= \frac{z(1+\epsilon) + \sum_{n=2}^{\infty} a_n z^n}{1+\epsilon} \\ &= \frac{f(z) * [z(1+\epsilon) + \sum_{n=2}^{\infty} z^n]}{1+\epsilon} \\ &= f(z) * \frac{z - \dfrac{\epsilon}{1+\epsilon} z^2}{1-z} = f(z) * h(z) \end{aligned} \tag{3.6}$$

where $h(z) = \dfrac{z - \dfrac{\epsilon}{1+\epsilon}z^2}{1-z}$.Differentiating logarithmically we obtain,

$$\begin{aligned} \frac{zh'(z)}{h(z)} &= \frac{z - \frac{2\epsilon}{1+\epsilon}z^2}{z - \frac{\epsilon}{1+\epsilon}z^2} + \frac{z}{1-z} \\ &= \frac{-\rho z}{1-\rho z} + \frac{1}{1-z} \end{aligned}$$

where $\rho = \dfrac{\epsilon}{1+\epsilon}$. Hence $\mathcal{R}e\dfrac{zh'(z)}{h(z)} > 0$ if $|\rho| < \dfrac{1}{3}$ then $|\epsilon| < \dfrac{1}{4}$. Therefore h is starlike in $\mathbb{U}$ if $|\epsilon| < \dfrac{1}{4}$and so

$$\int_0^z \frac{h(t)}{t} dt = z + \sum_{n=2}^{\infty} \frac{h_n z^n}{n} = h(z) * \log(\frac{1}{1-z}),$$

is convex for $|\epsilon| < \dfrac{1}{4}$. Also we have,

$$F_\epsilon(z) = (f*h)(z) = zf'(z) * (h(z) * \log\frac{1}{1-z}) \in \mathcal{SL}.$$

$\square$

Lemma 3.11. If $f \in \mathcal{N}$ and $h \in D$, then $|\dfrac{(f*h)(z)}{z}| \geq \dfrac{1}{4}$.

Proof . Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{N}$ and $h\in D$, then from Lemma 3.10 for $|\epsilon|<\frac{1}{4}$ we have $F_\epsilon(z)=\dfrac{f(z)+\epsilon z}{1+\epsilon}\in\mathcal{SL}$. Thus

$$\frac{1}{z}[h(z)*F_\epsilon(z)]\neq 0\ ,\ |\epsilon|<\frac{1}{4}.$$

Now from the properties of Hadamard product we obtain

$$\begin{aligned}\frac{1+\epsilon}{z}[h(z)*\frac{f(z)+\epsilon z}{1+\epsilon}] &= \frac{1}{z}[h(z)*(f(z)+\epsilon z)]\\ &= \frac{1}{z}[h(z)*f(z)]+\epsilon\neq 0.\end{aligned}\tag{3.7}$$

Hence for $|\epsilon|<\frac{1}{4}$, $\frac{1}{z}[h(z)*f(z)]\neq-\epsilon$, and so $|\frac{(f*h)(z)}{z}|\geq\frac{1}{4}$. □

Theorem 3.12. [9] If $f\in\mathcal{SL}$ and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ then

$$|a_n|\leq|\tau|^{n-1}u_n,$$

where $u_n=\frac{(1-\tau)^n-\tau^n}{\sqrt{5}}$, $\tau=\frac{1-\sqrt{5}}{2}$.

4 Main Results

Throughout this section $T=\{T_n\}_{n=2}^{\infty}$ will always be the sequence given by

$$T_n=\frac{20n+4\tau+3}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}u_n,$$

where $\cos\varphi\neq-1,\frac{5}{4}+\frac{\tau}{2}$, unless otherwise mentioned.

Theorem 4.1. For

$$0\leq\delta<\frac{\sqrt{(39\tau+4\tau^2-43)^2-(43+4\tau)(1+2\tau)}-(39\tau+4\tau^2-43)}{1+2\tau}.$$

we have

$$TN_\delta(\mathcal{SL})\otimes TN_\delta(\mathcal{K})\subset\mathcal{SL}.$$

Proof .. Let $f_0(z)=z+\sum_{n=2}^{\infty}a_0nz^n\in\mathcal{SL}$ and $g_0(z)=z+\sum_{n=2}^{\infty}b_0nz^n\in\mathcal{K}$. Also suppose that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in TN_\delta(f_0)$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in TN_\delta(g_0)$. We want to show that

$$\frac{(f\otimes g*h)(z)}{z}\neq 0,\quad (h\in D).$$

By the identity

$$f\otimes g*h=f_0\otimes g_0*h+f_0\otimes(g-g_0)*h+(f-f_0)\otimes g_0*h+(f-f_0)\otimes(g-g_0)*h,$$

we obtain

$$\begin{aligned}|\frac{(f\otimes g*h)(z)}{z}| &\geq |\frac{(f_0\otimes g_0*h)(z)}{z}|-|\frac{(f_0\otimes(g-g_0)*h)(z)}{z}|\\ &\quad -|\frac{((f-f_0)\otimes g_0*h)(z)}{z}|-|\frac{((f-f_0)\otimes(g-g_0)*h)(z)}{z}|.\end{aligned}\tag{4.1}$$

From Lemma 3.8 it follows that $f_0*g_0\in\mathcal{SL}$, also we have

$$z(f_0\otimes g_0)'(z)=z+\sum_{n=2}^{\infty}a_{0n}b_{0n}z^n=(f_0*g_0)(z)\in\mathcal{SL},$$

thus $f_0 \otimes g_0 \in \mathcal{N}$. Now from Lemma 3.11 for all $h \in D$ we obtain

$$\left|\frac{(f_0 \otimes g_0 * h)(z)}{z}\right| \geq \frac{1}{4}. \tag{4.2}$$

Also making use of Theorem 3.12 and Lemma 3.3 for $f_0(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$ we obtain $|a_{0n}| \leq |\tau|^{n-1} u_n$, $|c_n| \leq \frac{20n+4\tau+3}{[1-2\tau-2(3-2\cos\varphi)](1+\cos\varphi)}$. Now from the definition of $TN_\delta(f_0)$ and $TN_\delta(g_0)$ we have

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{|a_{0n}||b_n - b_{0n}||c_n|}{n} &\leq \frac{|\tau|}{2} \sum_{n=2}^{\infty} \frac{20n + 4\tau + 3}{[1 - 2\tau - 2(3 - 2\cos\varphi)]\,(1 + \cos\varphi)} u_n |b_n - b_{0n}| \\ &= \frac{|\tau|}{2} \sum_{n=2}^{\infty} T_n |b_n - b_{0n}| \\ &\leq \frac{\delta\,|\tau|}{2}.\end{aligned} \tag{4.3}$$

Similarly, we get

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{|b_{0n}||a_n - a_{0n}||c_n|}{n} &\leq \frac{1}{2} \sum_{n=2}^{\infty} \frac{20n + 4\tau + 3}{[1 - 2\tau - 2(3 - 2\cos\varphi)]\,(1 + \cos\varphi)} |a_n - a_{0n}| \\ &\leq \frac{1}{2} \sum_{n=2}^{\infty} \frac{20n + 4\tau + 3}{[1 - 2\tau - 2(3 - 2\cos\varphi)]\,(1 + \cos\varphi)} u_n |a_n - a_{0n}| \\ &\leq \frac{1}{2} \sum_{n=2}^{\infty} T_n |a_n - a_{0n}| \\ &\leq \frac{\delta}{2}.\end{aligned} \tag{4.4}$$

Finally we have

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{|a_n - a_{0n}||b_n - b_{0n}||c_n|}{n} &\leq \frac{\delta\,[1 - 2\tau - 2(3 - 2\cos\varphi)]\,(1 + \cos\varphi)}{2(40 + 4\tau + 3)} \sum_{n=2}^{\infty} |c_n|\,|b_n - b_{0n}| \\ &\leq \frac{\delta\,[1 - 2\tau - 2(3 - 2\cos\varphi)]\,(1 + \cos\varphi)}{2(40 + 4\tau + 3)} \sum_{n=2}^{\infty} |c_n|\,u_n |b_n - b_{0n}| \\ &\leq \frac{\delta\,[1 - 2\tau - 2(3 - 2\cos\varphi)]\,(1 + \cos\varphi)}{2(40 + 4\tau + 3)} \sum_{n=2}^{\infty} T_n |b_n - b_{0n}| \\ &\leq \frac{\delta^2\,[1 - 2\tau - 2(3 - 2\cos\varphi)]\,(1 + \cos\varphi)}{2(40 + 4\tau + 3)} \\ &\leq \frac{-\delta^2(1 + 2\tau)}{43 + 4\tau}.\end{aligned} \tag{4.5}$$

By virtue of 4.2 , 4.3 , 4.4 and 4.5 , inequality 4.1 gives

$$|\frac{f \otimes g * h)(z)}{z}| \geq \frac{1}{4} - \frac{\delta\,|\tau|}{2} - \frac{\delta}{2} + \frac{\delta^2(1 + 2\tau)}{43 + 4\tau}. \tag{4.6}$$

The right side of 4.6 is positive whenever

$$0 \leq \delta < \frac{\sqrt{(39\tau + 4\tau^2 - 43)^2 - (43 + 4\tau)(1 + 2\tau)} - (39\tau + 4\tau^2 - 43)}{1 + 2\tau}.$$

□

Corollary 4.2. $\delta_T(\mathcal{SL}\otimes\mathcal{K},\mathcal{SL}) \geq \frac{\sqrt{(39\tau+4\tau^2-43)^2-(43+4\tau)(1+2\tau)}-(39\tau+4\tau^2-43)}{1+2\tau}$.

Theorem 4.3. For

$$0 \leq \delta < \frac{\sqrt{(43\tau+4\tau^2)^2-4(1+2\tau)(86+8\tau)}-(43\tau+4\tau^2)}{2(1+2\tau)},$$

we have

$$TN_\delta(\{I\})\otimes TN_\delta(\mathcal{SL}) \subset \mathcal{SL}.$$

Proof .. Let $f_0(z)=I(z)=z$ and $g_0(z)=z+\sum_{n=2}^{\infty} b_{0n}z^n \in \mathcal{SL}$. Also suppose that $f(z)=z+\sum_{n=2}^{\infty} a_n z^n \in TN_\delta(f_0)$ and $g(z)=z+\sum_{n=2}^{\infty} b_n z^n \in TN_\delta(g_0)$, then we have $\sum_{n=2}^{\infty} T_n|a_n| \leq \delta$ and $\sum_{n=2}^{\infty} T_n|b_n-b_{0n}| \leq \delta$.
We want to show

$$\frac{(f\otimes g)*h)(z)}{z} \neq 0, \qquad (h\in D).$$

We have

$$\begin{aligned}|\frac{(f\otimes g*h)(z)}{z}| \geq& |\frac{(f_0\otimes g_0*h)(z)}{z}| - |\frac{(f_0\otimes(g-g_0)*h)(z)}{z}| \\ &- |\frac{((f-f_0)\otimes g_0*h)(z)}{z}| - |\frac{((f-f_0)\otimes(g-g_0)*h)(z)}{z}|.\end{aligned}$$

observe that, $(f_0\otimes g_0*h)(z)=z$ and $(f_0\otimes(g-g_0)*h)(z)=0$. Moreover we have

$$\begin{aligned}|\frac{((f-f_0)\otimes g_0*h)(z)}{z}| &\leq \sum_{n=2}^{\infty}\frac{|a_n||b_{0n}||c_n|}{n} \\ &\leq \frac{|\tau|}{2}\sum_{n=2}^{\infty} u_n\,|c_n|\,|a_n| \\ &= \frac{|\tau|}{2}\sum_{n=2}^{\infty} T_n|a_n| \\ &\leq \delta\frac{|\tau|}{2}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\left|\frac{((f-f_0)\otimes(g-g_0)*h)(z)}{z}\right| &\leq \sum_{n=2}^{\infty}\frac{|a_n||b_n-b_{0n}||c_n|}{n} \\ &\leq \frac{1}{2}\sum_{n=2}^{\infty}|c_n|\,|a_n||b_n-b_{0n}| \\ &\leq \frac{1}{2}\sum_{n=2}^{\infty} T_n|a_n||b_n-b_{0n}| \\ &\leq \frac{\delta\,[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}{2(43+4\tau)}\sum_{n=2}^{\infty} T_n|b_n-b_{0n}| \\ &\leq \frac{\delta^2\,[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}{2(43+4\tau)}.\end{aligned}$$

□

Now, following the same techniques as in the proof of theorem 4.1 we conclude the result and we omit details.

Corollary 4.4.

$$\delta_T((\{I\})\otimes\mathcal{SL},\mathcal{SL}) > \frac{\sqrt{(43\tau+4\tau^2)^2-4(1+2\tau)(86+8\tau)}-(43\tau+4\tau^2)}{2(1+2\tau)},$$

.

Theorem 4.5. (i) for

$$\delta_1 = \sqrt{\frac{43+4\tau}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}},$$

$$TN_{\delta_1}(\{I\}) * TN_{\delta_1}(\{I\}) \subset \mathcal{SL},$$

(ii) for

$$\delta_2 = \sqrt{\frac{2(43+4\tau)}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}},$$

$$TN_{\delta_2}(\{I\}) \otimes TN_{\delta_2}(\{I\}) \subset \mathcal{SL}.$$

The result is the best possible in each case.

Proof . (i) Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in TN_{\delta_1}(\{I\})$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in TN_{\delta_1}(\{I\}).$$

By making use of definition $TN_{\delta_1}(\{I\})$ we have

$$\sum_{n=2}^{\infty} T_n |a_n| \leq \delta_1, \tag{4.8}$$

and

$$\sum_{n=2}^{\infty} T_n |b_n| \leq \delta_1. \tag{4.9}$$

Since $T_n = \frac{20n+4\tau+3}{[1-2\tau-2(3-2\cos\varphi)](1+\cos\varphi)}$ is an increasing function of n so that from 4.8 we get

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)\delta_1}{43+4\tau},$$

which implies that

$$|a_n| \leq \frac{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)\delta_1}{43+4\tau} \qquad (n \geq 2).$$

Using the above inequality, it follows that

$$\frac{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)\delta_1}{43+4\tau} \sum_{n=2}^{\infty} |a_n||b_n| \leq \frac{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)\delta_1^2}{43+4\tau} = 1,$$

which in view of Corollary 3.5 ,$(f * g)(z) \in \mathcal{SL}$.

The proof of (ii) is similar to part (i) and we omit the details. To see that the containment relation in (i) is the best possible, we consider the function f and g defined in $\mathbb{U}$ by

$$f(z) = g(z) = z + \sqrt{\frac{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}{43+4\tau}}\, z^2,$$

clearly, $f, g \in TN_{\delta_1}(\{I\})$ and $(f * g) \in \mathcal{SL}$. Also considering the function f and g defined in $\mathbb{U}$ by

$$f(z) = g(z) = z + \sqrt{\frac{2\,[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}{43+4\tau}}\, z^2,$$

it is easily seen that the result in (ii) is the best possible. This evidently completes proof. □

Corollary 4.6. (*i*)

$$\delta_T(\{I\}*\{I\},\mathcal{SL})=\sqrt{\frac{43+4\tau}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}},$$

(*ii*)

$$\delta_T(\{I\}\otimes\{I\},\mathcal{SL})=\sqrt{\frac{2(43+4\tau)}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}}.$$

Proof . (*i*) From Theorem 4.5 we have

$$\delta_T(\{I\}*\{I\},\mathcal{SL})\geq\delta_1=\sqrt{\frac{43+4\tau}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}}. \tag{4.10}$$

Moreover, let

$$f(z)=g(z)=z+\frac{\delta(43+4\tau)}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}z^2\in TN_\delta(\{I\}).$$

Then we have

$$(f*g)(z)=z+(\frac{\delta(43+4\tau)}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)})^2z^2.$$

Let $\varphi(\delta)=(\frac{\delta(43+4\tau)}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)})^2$, then

$$\varphi(\delta)>\varphi(\delta_1)=\frac{\delta(43+4\tau)}{[1-2\tau-2(3-2\cos\varphi)]\,(1+\cos\varphi)}$$

for $\delta>\delta_1$. Therefore, by Corollary 3.4 , $(f*g)(z)\notin\mathcal{SL}$ when $\delta>\delta_1$. This means that

$$\delta_T(\{I\}*\{I\},\mathcal{SL})\leq\delta_1. \tag{4.11}$$

The relation 4.10 and 4.11 give the result. The proof of part (*ii*) is similar to part (*i*) and we omit the details. □

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