Int. J. Nonlinear Anal. Appl. In Press, 1–16
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2025.33690.5033

A new discrete distribution: The exponentiated discrete inverse Weibull distribution

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(Communicated by Javd Damirchi)

Abstract

Discrete distributions are used to model count data; however, at the same time, they can play an important role in lifetime phenomena. In this paper, a new discrete distribution is proposed called the exponentiated discrete inverse Weibull distribution. The new distribution includes several older distributions as special cases. Several distributional properties of the new distribution such as probability mass function, hazard rate function, moments, skewness, kurtosis, and order statistics are studied. The new distribution can be a good candidate for modeling platykurtic data sets. The problem of point estimation of the parameters based on the maximum likelihood method is discussed. The asymptotic confidence intervals for the unknown parameters are obtained as well. A simulation study is carried out to evaluate the maximum likelihood estimators and asymptotic confidence intervals. A real data set is analyzed in order to show the applicability of the new distribution. We end the paper with some remarks.

Keywords: Discrete inverse Weibull distribution, Maximum likelihood estimation, Moments, Order statistics, Platykurtic data
2020 MSC: Primary 60E05; Secondary 62F10, 62E99

1 Introduction

Discrete distributions are often used to model count data. Sometimes in a lifetime phenomenon, we encounter count data in terms of the number of years, number of days, number of hours, and so on. Such data are inherently discrete. In some cases, the lifetime of a component can be expressed in terms of a discrete random variable that does not have a temporal nature, for example, the lifetime of a rifle may be measured in the number of rounds fired until failure, see [40] and [35] for more details. Therefore, discrete distributions may possess a key role in reliability problems.

With the advancement of science and the development of sciences in various fields of engineering, etc., the need for new distributions was felt. Therefore, statisticians tried to generalize older distributions and introduced new discrete distributions. One of the ways of generalizing statistical distributions is the exponentiation method, which has been applied to generalizing discrete and continuous distributions. For example, [25] introduced the exponentiated Poisson and exponentiated geometric distributions. [15] and [34] applied the exponentiated geometric distribution to positive data and [31] applied the exponentiated geometric distribution to nonnegative data. [35] introduced the

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Received: April 2024 *Accepted:* May 2025

exponentiated discrete Weibull distribution and discussed its properties. A special case of the exponentiated discrete Weibull distribution, namely the exponentiated Rayleigh distribution was studied by [6]. In addition, the exponentiated generalized geometric distribution was proposed by [11]. [22] introduced the exponentiated discrete inverse Rayleigh distribution. Recently, [27] proposed the exponentiated discrete hypo exponential distribution. For examples of applying the exponentiation method to continuous distributions, see [2, 24, 30, 32, 38].

The discrete inverse Weibull (DIW) distribution has been introduced by [3]. The probability mass function (PMF) of the DIW is given by

$$p_X(x) = q^{x^{-\beta}} - q^{(x-1)^{-\beta}}, \quad x = 1, 2, \cdots,$$

where $q \in (0,1)$ and $\beta > 0$. We note that $q^{\infty} \equiv 0$. The cumulative distribution function (CDF) of the DIW is also given by

$$F(x) = q^{[x]^{-\beta}}, \quad x \geq 1,$$

where $[x]$ denotes the integer part of x.

The DIW is the discrete analogue of the continuous inverse Weibull (or Fréchet) distribution. The Fréchet distribution has a key role in reliability and possesses a rich literature in statistical and engineering studies. In this paper, we intend to generalize the DIW distribution using the exponentiation method, aiming at finding a new discrete distribution, called the exponentiated discrete inverse Weibull (EDIW) distribution with higher flexibility. The additional parameter of the new distribution affects the skewness and kurtosis considerably, see Table 2 in Section 3. [36] underlined that there appears to be no universal agreement about the meaning and interpretation of the term kurtosis. In symmetrical distributions, kurtosis may be interpreted as the peakedness of the distribution around the symmetry point, however, this interpretation has been criticized by numerous authors, see [36]. Generally, kurtosis may be interpreted as a statistical measure that describes the shape of a distribution's tails in relation to its overall shape, particularly focusing on the presence of outliers. [10] hinted that kurtosis offers insights into the behavior of the tails and the presence of outliers within a data set. According to [10] a platykurtic distribution has a kurtosis of less than 3, indicating lighter tails and a flatter peak compared to a normal distribution. It suggests that the data set has fewer extreme values and is less prone to outliers [10]. In other words, as highlighted by [33], we expect fewer outliers for a platykurtic data set than normal distributed data sets. On the other side, many authors believe that a platykurtic data set is one whose frequency curve is more flat-topped than the normal curve, see for example [37, 43]. It seems that the DIW might not be suitable for platykurtic data sets and this problem becomes fixed with the help of the newly introduced generalized distribution, namely the EDIW distribution. Moreover, as we will see later, the new model can fit a real data set better than its sub-models, see Section 6 for more details. In what follows, the new distribution is proposed in Section 2. Some distributional properties of the new model are discussed in Section 3. Section 4 is devoted to parameter estimation and confidence intervals. A simulation study is given in Section 5 to evaluate the point and interval estimators of the unknown parameters proposed in this paper. A real data set is analyzed in order to show the applicability of the new distribution in Section 6. We end the paper with some remarks in Section 7.

2 The new distribution

The DIW distribution was introduced by [3]. Now, using the method of exponentiation, we generalize the DIW model to arrive at a more flexible discrete distribution, called the exponentiated discrete inverse Weibull (EDIW) distribution. The CDF and PMF of the new model are given by

$$F(x; q, \beta, \gamma) = 1 - \left(1 - q^{([x]+1)^{-\beta}}\right)^{\gamma}, \qquad x \geq 0, \tag{2.1}$$

and

$$p_X(x) = \begin{cases} 1 - (1-q)^{\gamma}, & x = 0, \\ \left(1 - q^{x^{-\beta}}\right)^{\gamma} - \left(1 - q^{(x+1)^{-\beta}}\right)^{\gamma}, & x = 1, 2, 3, \cdots, \end{cases} \tag{2.2}$$

respectively, where $q \in (0,1), \beta > 0$ and $\gamma > 0$ are the model parameters. The EDIW distribution is the discrete analogue of the continuous exponentiated inverse Weibull (or exponentiated Fréchet) distribution, introduced by [32]. If a discrete random variable possesses CDF (2.1) with parameters q, β and γ, then we write $X \sim EDIW(q, \beta, \gamma)$.

Several special cases of the EDIW distribution are as follows:

- For $\gamma = 1$, we obtain discrete inverse Weibull distribution (starting from zero), introduced by [3].
- For $\beta = 2$, we obtain the exponentiated discrete inverse Rayleigh distribution, introduced by [22].
- For $\beta = 1$, we obtain the discrete generalized inverted exponential distribution, proposed by [1].
- For $\beta = 2$ and $\gamma = 1$, we obtain the discrete inverse Rayleigh distribution, see [23].
- For $\beta = \gamma = 1$, we obtain the discrete inverse exponential distribution.

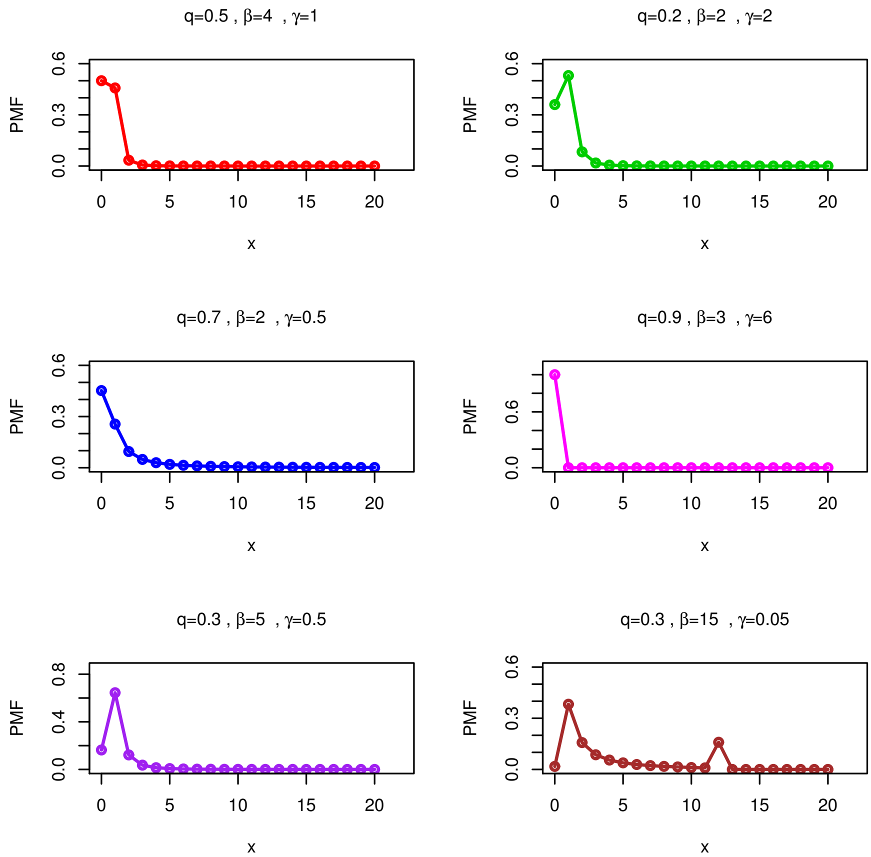

Figure 1: PMFs of the EDIW distribution for selected values of the parameters.

Figure 1 shows the plots of the PMF of the EDIW distribution for selected values of parameters. Figure 1 reveals that the PMF can be decreasing, unimodal or even bimodal depending on the values of the parameters. For example, for a large value of q (near 1), a small value of β (near zero) and a large value of γ, we expect to observe that the PMF is decreasing. Besides, as q approaches zero, and β moves further from zero, we expect to observe a unimodal PMF if γ is not a small or large number (a moderate value). We may state that q Influences the tail behavior of the distribution. As q approaches 1, the distribution tail behavior may become more pronounced. The parameter β may control how quickly probabilities decrease in value. However, as we will mention later, γ possesses a great effect on the values of skewness and kurtosis, which makes the role of this additional parameter significant.

In addition, the hazard rate function (HRF) of the new distribution is given by

$$r(x) = \frac{p_X(x)}{P(X \geq x)} = \begin{cases} 1 - (1-q)^{\gamma} & x = 0, \\ 1 - \left(\dfrac{1 - q^{(x+1)^{-\beta}}}{1 - q^{x^{-\beta}}} \right)^{\gamma} & x = 1, 2, 3, \cdots . \end{cases}$$

Figure 2 shows the plots of the HRF of the EDIW distribution for selected values of parameters. We can observe both decreasing and increasing-decreasing HRFs as the values of the parameters vary in Figure 2.

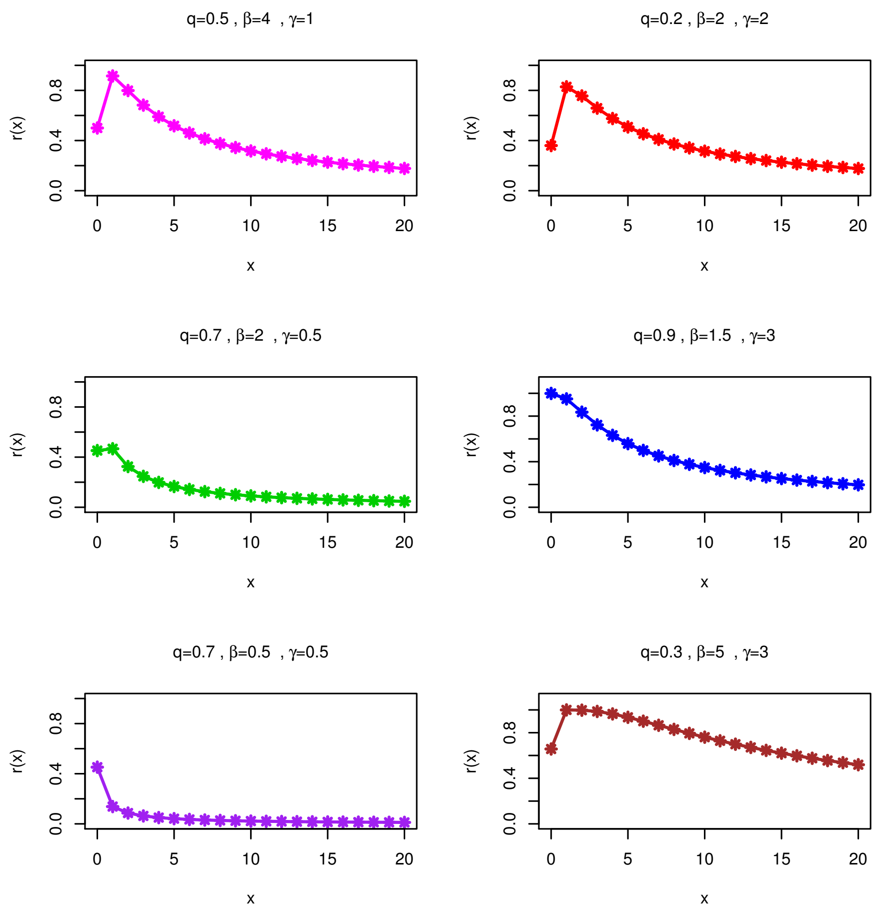

Figure 2: HRFs of the EDIW distribution for selected values of the parameters.

3 Some distributional properties

In this section, we present several distributional properties of the new model such as moments, skewness, kurtosis, data generation, and order statistics.

3.1 Moments and related measures

The k-th moment of $X \sim EDIW(q, \beta, \gamma)$ is given by (provided that it is convergent)

$$E(X^k) = \sum_{x=1}^{\infty} x^k \left\{ \left(1 - q^{x^{-\beta}}\right)^{\gamma} - \left(1 - q^{(x+1)^{-\beta}}\right)^{\gamma} \right\}.$$

Therefore, the mean and variance of the new model are given by (provided that they are convergent)

$$\mu = E(X) = \sum_{x=1}^{\infty} x \left\{ \left(1 - q^{x^{-\beta}}\right)^{\gamma} - \left(1 - q^{(x+1)^{-\beta}}\right)^{\gamma} \right\},$$

and

$$Var(X) = \sum_{x=1}^{\infty} x^2 \left\{ \left(1 - q^{x^{-\beta}}\right)^{\gamma} - \left(1 - q^{(x+1)^{-\beta}}\right)^{\gamma} \right\} - (E(X))^2,$$

respectively. Besides, the skewness and kurtosis of the new model will be obtained through the following relations (provided that they are convergent)

$$S = \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{[Var(X)]^{\frac{3}{2}}},$$

and

$$K = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{[Var(X)]^2},$$

respectively. Table 1 includes the computed values of the mean and variance for selected parameter values and the computed values of skewness and kurtosis for selected values of parameters are also given in Table 2 (the related sums are considered up to 10^9). We notice the great effect of γ (the additional parameter) on the values of skewness and kurtosis. Thus, the additional parameter may help the new model to fit right-skewed and leptokurtic/platykurtic data sets with various positive values of skewness or kurtosis adequately. Note that for the case $\gamma = 1$ which corresponds to the DIW distribution, the kurtosis values are greater than 3 for all the considered cases. So it seems that the DIW may not be suitable for platykurtic data sets and this problem may be resolved by the new distribution, namely the EDIW distribution.

Table 1: The computed values of the mean (and variance in parentheses) for $X \sim EDIW(q, \beta, \gamma)$ for selected values of the parameters.

	$\gamma = 1$		
β \ q	0.25	0.5	0.75
3	1.01430 (1.17753)	0.63600 (0.83627)	0.30742 (0.46612)
5	0.80025 (0.28379)	0.52536 (0.31180)	0.26058 (0.21871)
	$\gamma = 3$		
β \ q	0.25	0.5	0.75
3	0.42604 (0.25316)	0.12559 (0.11103)	0.01567 (0.01552)
5	0.42195 (0.24406)	0.12501 (0.10940)	0.01563 (0.01538)
	$\gamma = 5$		
β \ q	0.25	0.5	0.75
3	0.23741 (0.18125)	0.03135 (0.03029)	0.00098 (0.00098)
5	0.23730 (0.18099)	0.03125 (0.03027)	0.00098 (0.00098)

Table 2: The computed values of the skewness (and kurtosis in parentheses) for $X \sim EDIW(q, \beta, \gamma)$ for selected values of the parameters.

	$\gamma = 1$		
β \ q	0.25	0.5	0.75
3	34.85498 (1268856)	29.30100 (998365.8)	29.56449 (994897.1)
5	0.65175 (15.11181)	0.78358 (7.87722)	1.71077 (9.40562)
	$\gamma = 3$		
β \ q	0.25	0.5	0.75
3	0.40851 (1.49240)	2.31244 (6.58775)	7.87678 (63.98628)
5	0.31798 (1.10633)	2.26847 (6.14933)	7.81222 (62.04306)
	$\gamma = 5$		
β \ q	0.25	0.5	0.75
3	1.23831 (2.54613)	5.39005 (30.07019)	31.95767 (1022.528)
5	1.23497 (2.52516)	5.38816 (30.03230)	31.95311 (1022.002)

3.2 Data generation

Here, we present an approach to generate data from the new model. To this end, suppose that W is a continuous random variable distributed as the exponentiated inverse Weibull (or exponentiated Fréchet) distribution with the following CDF [32]

$$F_W(w) = 1 - \left(1 - \mathrm{e}^{-\alpha w^{-\beta}}\right)^{\gamma}, \qquad w > 0, \alpha > 0, \beta > 0, \gamma > 0.$$

Set $q = \mathrm{e}^{-\alpha}$, then, for a positive integer x, we have

$$\begin{aligned} P([W] = x) &= P(x \leq W < x+1) = F_W(x+1) - F_W(x) \\ &= \left(1 - q^{x^{-\beta}}\right)^{\gamma} - \left(1 - q^{(x+1)^{-\beta}}\right)^{\gamma}, \end{aligned}$$

which is the PMF of the EDIW distribution with parameter q, β, and γ. For $x = 0$, we have $P([W] = 0) = F_W(1) = 1-(1-q)^{\gamma}$. Therefore, the EDIW distribution is the discrete analogue of the continuous exponentiated inverse Weibull distribution with parameters $\alpha = -\ln(q)$, β and γ. In other words, if W possesses the continuous exponentiated inverse Weibull distribution with parameters α, β and γ, then $X = [W]$ is distributed as the EDIW distribution with parameters $q = \mathrm{e}^{-\alpha}$, β, and γ. Now, if we intend to generate an observation from the new discrete model, we first generate an observation from the continuous exponentiated inverse Weibull distribution with parameters $\alpha = -\ln(q)$, β and γ, say W. Then, the integer value of W will be the desired generated random observation from the EDIW distribution with parameters q, β, and γ.

3.3 Order statistics

Order statistics play a key role in statistical inference and experiments with censored data. Let $X_1, \cdots, X_n$ be a random sample of size n from $EDIW(q, \beta, \gamma)$. Then, the CDF of the k-th order statistic for a nonnegative value x is given by (see [7])

$$F_{k:n}(x; q, \beta, \gamma) = \sum_{i=k}^{n} \binom{n}{i} [F(x; q, \beta, \gamma)]^i [1 - F(x; q, \beta, \gamma)]^{n-i}.$$

Using the binomial expansion, we have

$$\begin{aligned} F_{k:n}(x; q, \beta, \gamma) &= \sum_{i=k}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^j [1 - F(x; q, \beta, \gamma)]^{n-i+j} \\ &= \sum_{i=k}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^j \left(1 - q^{([x]+1)^{-\beta}}\right)^{\gamma(n-i+j)}. \end{aligned}$$

Now, consider the following generalized binomial expansion (see [20])

$$(1-z)^{\lambda}=\sum_{m=0}^{\infty}\binom{\lambda}{m}(-1)^{m}z^{m},\qquad |z|<1.$$

If λ is a positive integer value, then the above expansion will be finite. Now, using the above expansion, we get

$$\begin{aligned}F_{k:n}(x;q,\beta,\gamma) &= \sum_{i=k}^{n}\sum_{j=0}^{i}\sum_{m=0}^{\infty}\binom{n}{i}\binom{i}{j}\binom{\gamma(n-i+j)}{m}(-1)^{m+j}q^{m([x]+1)^{-\beta}}\\ &= \sum_{i=k}^{n}\sum_{j=0}^{i}\sum_{m=0}^{\infty}\binom{n}{i}\binom{i}{j}\binom{\gamma(n-i+j)}{m}(-1)^{m+j}F_{DIW}(x;q^{m},\beta),\end{aligned}$$

where $F_{DIW}(\cdot;q^m,\beta)$ is CDF of the discrete inverse Weibull distribution with parameters q^m and β. So, the CDF of the k-th order statistic coming from the new model can be written as an infinite linear combination of the CDFs of the discrete inverse Weibull distribution (if γ is a positive integer value, then the linear combination will be finite). Besides, we may conclude for a positive integer value x, the PMF of the k-th order statistic coming from the new model can be written as follows

$$p_{k:n}(x;q,\beta,\gamma)=\sum_{i=k}^{n}\sum_{j=0}^{i}\sum_{m=0}^{\infty}\binom{n}{i}\binom{i}{j}\binom{\gamma(n-i+j)}{m}(-1)^{m+j}p_{DIW}(x;q^{m},\beta),$$

where $p_{DIW}(\cdot;q^m,\beta)$ is the PMF of the discrete inverse Weibull distribution with parameters q^m and β. Based on the results of this subsection, we might obtain some distributional properties of the order statistics such as moments using the distributional properties of the discrete inverse Weibull distribution.

4 Point and interval estimation

The maximum likelihood (ML) estimation is one of the most popular methods of estimation. Let $\boldsymbol{X}=(X_1,\cdots,X_n)$ be a random sample of size n from $EDIW(q,\beta,\gamma)$ and $\boldsymbol{x}=(x_1,\cdots,x_n)$ be the observed set of $\boldsymbol{X}$. Then, the likelihood function of the parameters given $\boldsymbol{x}$ is given by

$$\mathcal{L}(q,\beta,\gamma|\boldsymbol{x})=\Big[1-(1-q)^{\gamma}\Big]^{n-\sum_{i=1}^{n}u(x_i)}\prod_{i=1}^{n}\Big[\Big(1-q^{x_i^{-\beta}}\Big)^{\gamma}-\Big(1-q^{(x_i+1)^{-\beta}}\Big)^{\gamma}\Big]^{u(x_i)},$$

where

$$u(x)=\begin{cases}0, & x=0,\\ 1, & x=1,2,3,\cdots.\end{cases}$$

The log-likelihood function is then given by

$$\ell(q,\beta,\gamma|\boldsymbol{x})=\big(n-\sum_{i=1}^{n}u(x_i)\big)\ln\Big[1-(1-q)^{\gamma}\Big]+\sum_{i=1}^{n}u(x_i)\ln\Big[\Big(1-q^{x_i^{-\beta}}\Big)^{\gamma}-\Big(1-q^{(x_i+1)^{-\beta}}\Big)^{\gamma}\Big].$$

The ML estimates of the parameters are obtained by maximizing the above log-likelihood function with respect to (w.r.t.) the parameters. Taking the first derivatives w.r.t. the parameters and equating them equal to zero, we get

$$\begin{aligned}
\frac{\partial \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial q} =& \frac{\left(n-\sum_{i=1}^{n} u(x_i)\right)\gamma(1-q)^{\gamma-1}}{1-(1-q)^{\gamma}} + \gamma \sum_{i=1}^{n} \frac{u(x_i)\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma-1}(x_i+1)^{-\beta} q^{(x_i+1)^{-\beta}-1}}{\left(1-q^{x_i^{-\beta}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma}} \\
&-\gamma \sum_{i=1}^{n} \frac{u(x_i)\left(1-q^{x_i^{-\beta}}\right)^{\gamma-1} x_i^{-\beta} q^{x_i^{-\beta}-1}}{\left(1-q^{x_i^{-\beta}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma}} = 0, \\
\frac{\partial \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial \beta} =& \gamma \ln(q) \sum_{i=1}^{n} \frac{u(x_i)(1-q^{x_i^{-\beta}})^{\gamma-1} q^{x_i^{-\beta}} x_i^{-\beta} \ln(x_i)}{\left(1-q^{x_i^{-\beta}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma}} \\
&-\gamma \ln(q) \sum_{i=1}^{n} \frac{u(x_i)(1-q^{(x_i+1)^{-\beta}})^{\gamma-1} q^{(x_i+1)^{-\beta}} (x_i+1)^{-\beta} \ln(x_i+1)}{\left(1-q^{x_i^{-\beta}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma}} = 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial \gamma} =& \sum_{i=1}^{n} \frac{u(x_i)\left(1-q^{x_i^{-\beta}}\right)^{\gamma} \ln\left(1-q^{x_i^{-\beta}}\right)}{\left(1-q^{x_i^{-\beta}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma}} - \sum_{i=1}^{n} \frac{u(x_i)\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma} \ln\left(1-q^{(x_i+1)^{-\beta}}\right)}{\left(1-q^{x_i^{-\beta}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma}} \\
&-\frac{\left(n-\sum_{i=1}^{n} u(x_i)\right)(1-q)^{\gamma}\ln(1-q)}{1-(1-q)^{\gamma}} = 0.
\end{aligned}$$

One may solve the above nonlinear equations with the help of numerical techniques, such as Newton-Raphson and Broyden-Fletcher-Goldfarb-Shanno (BFGS) methods. The BFGS method, that is used in this paper, is a quasi-Newton method, see [12, 13, 18, 19, 41].

The Fisher information is a measure that quantifies the amount of information a data set contains about the parameter. The concept of Fisher information can be generalized to the multi-parameter case, where the Fisher information matrix is defined. This Fisher information in a one-parameter case and the Fisher information matrix in a multi-parameter case are of crucial importance in a number of aspects of statistical inference, such as the efficiency of unbiased estimators, Bayesian estimation, and asymptotic statistics. The inverse of Fisher information matrix is known as the Cramér-Rao lower bound matrix. Under certain general conditions, unbiased estimators cannot possess variances that are smaller than the corresponding diagonal elements of this matrix, see [17, 44] for more details. Moreover, in Bayesian estimation, one important tool for finding an objective prior for the parameters, namely the Jeffreys prior, is based on the Fisher information matrix, see for example [29]. Furthermore, a major result in asymptotic statistics is that, in many situations, the inverse of the Fisher information matrix can serve as the asymptotic variance-covariance matrix of the ML estimators, see [45]. Let $\widehat{q}_{ML}$, $\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ be the ML estimators of q, β and γ, respectively. Let further $\boldsymbol{J}(\theta)$ denote the Fisher information matrix about the vector of parameters $\theta = (q,\beta,\gamma)$ in $\boldsymbol{X}$. Then, we have

$$\boldsymbol{J}(\theta) = -\begin{bmatrix} E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial q^2}\Big) & E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial q \partial \beta}\Big) & E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial q \partial \gamma}\Big) \\ E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial \beta \partial q}\Big) & E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial \beta^2}\Big) & E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial \beta \partial \gamma}\Big) \\ E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial \gamma \partial q}\Big) & E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial \gamma \partial \beta}\Big) & E\Big(\frac{\partial^2 \ln p_\theta(\boldsymbol{X})}{\partial \gamma^2}\Big) \end{bmatrix},$$

where $p_\theta(\boldsymbol{x})$ is the joint PMF of $X_1, \cdots, X_n$, provided that the above expectations exist.

Under some regularity conditions (see for example [28]), the ML estimators possess an asymptotic 3-variate normal distribution. To be more exact, we have

$$\begin{bmatrix} \widehat{q}_{ML} - q \\ \widehat{\beta}_{ML} - \beta \\ \widehat{\gamma}_{ML} - \gamma \end{bmatrix} \xrightarrow{D} N_3\Big(\underline{0}, \boldsymbol{J}^{-1}(\theta)\Big),$$

where $\xrightarrow{D}$ denotes convergence in distribution, $\underline{0} = [0,0,0]^T$, and $\boldsymbol{J}^{-1}(\theta)$ is the inverse matrix of $\boldsymbol{J}(\theta)$. If we replace the unknown parameters that appear in the arrays of $\boldsymbol{J}^{-1}(\theta)$ with their corresponding ML estimators and $\hat{\boldsymbol{J}}^{-1}(\theta)$

denotes the resulted matrix, then the diagonals of $\hat{\boldsymbol{J}}^{-1}(\theta)$ are equivalent to the ML estimators of the corresponding diagonals of $\boldsymbol{J}^{-1}(\theta)$, which are denoted by $\widehat{V}ar(\widehat{q}_{ML})$, $\widehat{V}ar(\widehat{\beta}_{ML})$ and $\widehat{V}ar(\widehat{\gamma}_{ML})$. Now, the $100(1-\alpha)\%$ two-sided equi-tailed (TE) asymptotic confidence intervals (ACIs) for parameters q, β and γ are given by

$$\widehat{q}_{ML} \pm z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\widehat{q}_{ML})}, \quad \widehat{\beta}_{ML} \pm z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\widehat{\beta}_{ML})}, \quad \text{and} \quad \widehat{\gamma}_{ML} \pm z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\widehat{\gamma}_{ML})},$$

respectively, where z_α denotes the α-th upper quantile of the standard normal distribution.

However, the TE ACI for q may not be included inside $(0,1)$. To avoid this problem, we use the transformation $g(q) = \ln\left(\frac{q}{1-q}\right)$. Using the delta method (see for example, [42]), the asymptotic distribution of the ML estimator of $g(\widehat{q}_{ML})$ will be

$$g(\widehat{q}_{ML}) - g(q) \xrightarrow{D} N\Big(0, \frac{Var(\widehat{q}_{ML})}{q^2(1-q)^2}\Big).$$

Consequently, the $100(1-\alpha)\%$ TE ACI for $g(q)$ is given by

$$g(\widehat{q}_{ML}) \pm \frac{\sqrt{\widehat{V}ar(\widehat{q}_{ML})}}{\widehat{q}_{ML}(1-\widehat{q}_{ML})} z_{\frac{\alpha}{2}} \equiv (L, U). \tag{4.1}$$

Finally, the $100(1-\alpha)\%$ ACI for q is obtained to be

$$\Big(\frac{\mathrm{e}^L}{1+\mathrm{e}^L}, \frac{\mathrm{e}^U}{1+\mathrm{e}^U}\Big). \tag{4.2}$$

Besides, the $100(1-\alpha)\%$ modified ACIs for parameters β and γ are given by

$$\left(\max\left\{0, \widehat{\beta}_{ML} - z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\widehat{\beta}_{ML})}\right\}, \widehat{\beta}_{ML} + z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\widehat{\beta}_{ML})}\right),$$

and

$$\left(\max\left\{0, \widehat{\gamma}_{ML} - z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\widehat{\gamma}_{ML})}\right\}, \widehat{\gamma}_{ML} + z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\widehat{\gamma}_{ML})}\right),$$

respectively.

In practice, after observing the random sample, we may use $\hat{\boldsymbol{I}}^{-1}(\theta)$ instead of $\hat{\boldsymbol{J}}^{-1}(\theta)$, where $\hat{\boldsymbol{I}}^{-1}(\theta)$ is the inverse of $\hat{\boldsymbol{I}}(\theta)$ and $\hat{\boldsymbol{I}}(\theta)$ is the estimate of $\boldsymbol{I}(\theta)$. The matrix $\boldsymbol{I}(\theta)$ is defined as

$$\boldsymbol{I}(\theta) = -\begin{bmatrix} \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial q^2} & \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial q \partial \beta} & \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial q \partial \gamma} \\ \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial \beta \partial q} & \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial \beta^2} & \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial \beta \partial \gamma} \\ \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial \gamma \partial q} & \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial \gamma \partial \beta} & \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial \gamma^2} \end{bmatrix}.$$

Actually, $\hat{\boldsymbol{I}}(\theta)$ is obtained by replacing the unknown parameters that appear in the elements of $\boldsymbol{I}(\theta)$ with their corresponding ML estimates. Let $t_i = \left(1 - q^{x_i^{-\beta}}\right)^\gamma - \left(1 - q^{(x_i+1)^{-\beta}}\right)^\gamma$ and $\nu = n - \sum_{i=1}^n u(x_i)$. Then, the elements of $\boldsymbol{I}(\theta)$ may be obtained through the following relations

$$\begin{aligned} \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial q^2} &= \frac{\nu\gamma(1-q)^{\gamma-2}}{1-(1-q)^\gamma}\left(1 - \frac{\gamma}{1-(1-q)^\gamma}\right) + \sum_{i=1}^n u(x_i)\frac{t_i^{(qq)}t_i - [t_i^{(q)}]^2}{t_i^2}, \\ \frac{\partial^2 \ell(q,\beta,\gamma|\boldsymbol{x})}{\partial q \partial \beta} &= \frac{\partial^2 \ell(q,\beta,\gamma)}{\partial \beta \partial q} = \sum_{i=1}^n u(x_i)\frac{t_i^{(q\beta)}t_i - t_i^{(q)}t_i^{(\beta)}}{t_i^2}, \end{aligned}$$

$$\frac{\partial^2\ell(q,\beta,\gamma|\boldsymbol{x})}{\partial q\partial\gamma}=\frac{\partial^2\ell(q,\beta,\gamma)}{\partial\gamma\partial q}=\frac{\nu(1-q)^{\gamma-1}}{1-(1-q)^\gamma}\left(1+\frac{\gamma\ln(1-q)}{1-(1-q)^\gamma}\right)+\sum_{i=1}^n u(x_i)\frac{t_i^{(q\gamma)}t_i-t^{(q)}t^{(\gamma)}}{t_i^2},$$
$$\frac{\partial^2\ell(q,\beta,\gamma|\boldsymbol{x})}{\partial\beta^2}=\sum_{i=1}^n u(x_i)\frac{t_i^{(\beta\beta)}t_i-[t_i^{(\beta)}]^2}{t_i^2},$$
$$\frac{\partial^2\ell(q,\beta,\gamma|\boldsymbol{x})}{\partial\beta\partial\gamma}=\frac{\partial^2\ell(q,\beta,\gamma)}{\partial\gamma\partial\beta}=\sum_{i=1}^n u(x_i)\frac{t_i^{(\beta\gamma)}t_i-t_i^{(\beta)}t_i^{(\gamma)}}{t_i^2},$$
$$\frac{\partial^2\ell(q,\beta,\gamma|\boldsymbol{x})}{\partial\gamma^2}=-\frac{\nu(1-q)^\gamma[\ln(1-q)]^2}{\left(1-(1-q)^\gamma\right)^2}+\sum_{i=1}^n u(x_i)\frac{t_i^{(\gamma\gamma)}t_i-[t_i^{(\gamma)}]^2}{t_i^2},$$

where

$$t_i^{(q)}=\frac{\partial t_i}{\partial q}=\gamma\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma-1}(x_i+1)^{-\beta}q^{(x_i+1)^{-\beta}-1}-\gamma\left(1-q^{x_i^{-\beta}}\right)^{\gamma-1}x_i^{-\beta}q^{x_i^{-\beta}-1},$$
$$t_i^{(\beta)}=\frac{\partial t_i}{\partial\beta}=\gamma\ln(q)\Big((1-q^{x_i^{-\beta}})^{\gamma-1}q^{x_i^{-\beta}}x_i^{-\beta}\ln(x_i)-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma-1}q^{(x_i+1)^{-\beta}}(x_i+1)^{-\beta}\ln(x_i+1)\Big),$$
$$t_i^{(\gamma)}=\frac{\partial t_i}{\partial\gamma}=\left(1-q^{x_i^{-\beta}}\right)^\gamma\ln\left(1-q^{x_i^{-\beta}}\right)-\left(1-q^{(x_i+1)^{-\beta}}\right)^\gamma\ln\left(1-q^{(x_i+1)^{-\beta}}\right),$$

$$\begin{aligned}
t_i^{(qq)}=\frac{\partial^2 t_i}{\partial q^2}=&\gamma\left(1-q^{x_i^{-\beta}}\right)^{\gamma-2}x_i^{-\beta}q^{x_i^{-\beta}-2}\left(q^{x_i^{-\beta}}\left(\gamma x_i^{-\beta}-1\right)+1-x_i^{-\beta}\right)\\
&-\gamma\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma-2}(x_i+1)^{-\beta}q^{(x_i+1)^{-\beta}-2}\left(q^{(x_i+1)^{-\beta}}\left(\gamma(x_i+1)^{-\beta}-1\right)+1-(x_i+1)^{-\beta}\right),\\
t_i^{(q\beta)}=\frac{\partial^2 t_i}{\partial q\partial\beta}=&\gamma\left(1-q^{x_i^{-\beta}}\right)^{\gamma-2}x_i^{-\beta}q^{x_i^{-\beta}-1}\ln(x_i)\left(x_i^{-\beta}\ln(q)\left(1-\gamma q^{x_i^{-\beta}}\right)+1-q^{x_i^{-\beta}}\right)\\
&-\gamma\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma-2}(x_i+1)^{-\beta}q^{(x_i+1)^{-\beta}-1}\ln(x_i+1)\\
&\times\left((x_i+1)^{-\beta}\ln(q)\left(1-\gamma q^{(x_i+1)^{-\beta}}\right)+1-q^{(x_i+1)^{-\beta}}\right),\\
t_i^{(q\gamma)}=\frac{\partial^2 t_i}{\partial q\partial\gamma}=&-\left(1-q^{x_i^{-\beta}}\right)^{\gamma-1}x_i^{-\beta}q^{x_i^{-\beta}-1}\left(1+\gamma\ln\left(1-q^{x_i^{-\beta}}\right)\right)\\
&+\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma-1}(x_i+1)^{-\beta}q^{(x_i+1)^{-\beta}-1}\left(1+\gamma\ln\left(1-q^{(x_i+1)^{-\beta}}\right)\right),\\
t_i^{(\beta\beta)}=\frac{\partial^2 t_i}{\partial\beta^2}=&\gamma\ln(q)\Big\{\left(1-q^{x_i^{-\beta}}\right)^{\gamma-2}x_i^{-\beta}q^{x_i^{-\beta}}(\ln(x_i))^2\left(x_i^{-\beta}\ln(q)\left(\gamma q^{x_i^{-\beta}}-1\right)-1+q^{x_i^{-\beta}}\right)\\
&-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma-2}(x_i+1)^{-\beta}q^{(x_i+1)^{-\beta}}(\ln(x_i+1))^2\\
&\times\left((x_i+1)^{-\beta}\ln(q)\left(\gamma q^{(x_i+1)^{-\beta}}-1\right)-1+q^{(x_i+1)^{-\beta}}\right)\Big\},
\end{aligned}$$

$$\begin{aligned}
t_i^{(\beta\gamma)}=\frac{\partial^2 t_i}{\partial\beta\partial\gamma}=&\left(1-q^{x_i^{-\beta}}\right)^{\gamma-1}x_i^{-\beta}q^{x_i^{-\beta}}\ln(q)\ln(x_i)\left(1+\gamma\ln\left(1-q^{x_i^{-\beta}}\right)\right)\\
&-\left(1-q^{(x_i+1)^{-\beta}}\right)^{\gamma-1}(x_i+1)^{-\beta}q^{(x_i+1)^{-\beta}}\ln(q)\ln(x_i+1)\left(1+\gamma\ln\left(1-q^{(x_i+1)^{-\beta}}\right)\right),\\
t_i^{(\gamma\gamma)}=\frac{\partial^2 t_i}{\partial\gamma^2}=&\left(1-q^{x_i^{-\beta}}\right)^\gamma\left(\ln\left(1-q^{x_i^{-\beta}}\right)\right)^2-\left(1-q^{(x_i+1)^{-\beta}}\right)^\gamma\left(\ln\left(1-q^{(x_i+1)^{-\beta}}\right)\right)^2.
\end{aligned}$$

5 A simulation study

In this section, a simulation study is given to evaluate the point and interval estimators, discussed in the previous section. To this end, we consider 12 different parameter combinations for q, β and γ. The number of replications of the simulation is $N = 5000$ and the sample sizes are taken to $n = 100$ and 200. In each replication, the ML estimates of the q, β, and γ, as well as the 95% ACIs for q and 95% modified ACIs for β and γ are computed. Let q^* be an estimator of q. Then, the empirical bias or the estimated bias (bias for short) and the estimated root mean squared error (ERMSE) for q^* are given by

$$\text{Bias}(q^*) = \frac{1}{N}\sum_{i=1}^{N}(q_i^* - q) \quad \text{and} \qquad \text{ERMSE}(q^*) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(q_i^* - q)^2},$$

respectively, where q_i^* is the estimate of q in the i-th replication.

Similarly, we can calculate the biases and ERMSEs for the point estimators of β and γ. The interval estimators are evaluated based on average width (AW) and estimated coverage probability (CP for short). Let $CI_q = \big(L_q, U_q\big)$ be an interval estimator for q, and $CI_q(i) = \big(L_q(i), U_q(i)\big)$ be the corresponding interval estimate for q obtained in the i-th iteration. Let further $I_A(q)$ be an indicator function of q for a set A in the sense that $I_A(q) = 1$ if $q \in A$, and otherwise $I_A(q) = 0$. Then, the AW and CP for CI_q are given by

$$\text{AW}(CI_q) = \frac{1}{N}\sum_{i=1}^{N}(U_q(i) - L_q(i)) \quad \text{and} \qquad \text{CP}(CI_q) = \frac{1}{N}\sum_{i=1}^{N} I_{CI_q(i)}(q),$$

respectively.

Analogously, we can obtain the AWs and CPs for the interval estimators of β and γ. Some badly generated replications were removed from the simulation. These replications mostly included cases where the variance of at least one of the ML estimators was obtained to be negative. Moreover, when $q = 0.75$ and $n = 100$, we encountered several cases where all the generated numbers equaled zero, and we removed these cases. Note that if all the observations become zero, then the ML estimate of q becomes 1, which does not belong to the parameter space as $0 < q < 1$ (see [42]), and the ML estimates of γ and β are not unique and can take any positive values. In such cases, we recommend increasing the sample size until at least one positive observation occurs. We noticed that when the sample size increased to $n = 200$, we did not encounter such cases. Finally, when $q = 0.75$, $\beta = \gamma = 2$, and $n = 200$, we observed that the upper bounds of 95% ACIs for q became "NA" (namely not available or missing value) for a few replications (exactly 10 replications here), and we removed those replications as well. As we sought the reason, we found in such cases, the value of U in (4.1) resulted in a large number, causing the value of e^U in (4.2) to be evaluated as infinity. The numerical results of the simulation for ML estimators are given in Table 3 and the results for the 95% interval estimators are given in Table 4.

From the numerical results of Tables 3 and 4, we can observe several cases where the ML estimators perform well, but the 95% ACIs for q and 95% modified ACIs for β and γ have large AWs. However, as the sample size increases, the AWs decrease (except for a few cases).

6 A real data example

In this section, we analyze a real data set in order to show the applicability and flexibility of the new distribution. Table 5 includes the number of outbreaks of strikes in the UK coal mining industry over four consecutive week periods during 1948–1959. The data were reported originally by [26] and many authors have analyzed this data set using different discrete distributions, see for example [6, 21]. In this data set, the last group with frequecny 1, is related to 4 outbreaks or more. However, since we need a number to calculate the ML estimates of the parameters and the goodness-of-fit criteria (as "4 or more" cannot be considered as a number), we suppose that the frequency of the last group, which equals 1, is related to 4 outbreaks for convenience. In other words, we suppose that the last group represents "4 outbreaks" rather than "4 or more outbreaks" for computational purposes.

We compare the fit of the EDIW model with those of the following distributions:

- The exponentiated discrete inverse Rayleigh (EDIR) distribution [22].
- The discrete inverse Weibull (DIW) distribution [3].
- The discrete inverse Rayleigh (DIR) distribution, see [23].

Table 3: The ERMSEs and biases in parentheses of the ML estimators of the parameters based on the simulated samples.

	$n = 100$		
(q, β, γ)	q	β	γ
$(0.25, 3, 3)$	$0.12827\,(-0.08586)$	$2.59183\,(1.34011)$	$6.44661\,(3.79426)$
$(0.25, 3, 5)$	$0.04899\,(-0.00054)$	$1.34058\,(1.28697)$	$0.93991\,(0.17786)$
$(0.25, 5, 3)$	$0.05591\,(-0.03848)$	$0.94805\,(0.84276)$	$0.96813\,(0.72096)$
$(0.25, 5, 5)$	$0.05637\,(0.02682)$	$0.39641\,(-0.31804)$	$0.62169\,(-0.46521)$
$(0.5, 1, 3)$	$0.40440\,(-0.19255)$	$2.04302\,(0.55391)$	$24.36236\,(17.63035)$
$(0.5, 2, 3)$	$0.24757\,(-0.17288)$	$2.34819\,(1.41432)$	$11.45044\,(6.47221)$
$(0.5, 3, 2)$	$0.31246\,(-0.23481)$	$2.77229\,(0.84003)$	$11.22822\,(7.25742)$
$(0.5, 3, 3)$	$0.13793\,(-0.11362)$	$1.62206\,(1.35066)$	$4.30035\,(1.97193)$
$(0.5, 5, 3)$	$0.07803\,(-0.06354)$	$0.28436\,(0.18101)$	$1.01840\,(0.66415)$
$(0.75, 1, 2)$	$0.53071\,(-0.40297)$	$1.83218\,(0.40016)$	$23.85277\,(17.63779)$
$(0.75, 2, 2)$	$0.43595\,(-0.34849)$	$2.32771\,(0.94724)$	$16.42191\,(10.70589)$
$(0.75, 3, 2)$	$0.30207\,(-0.22906)$	$1.85199\,(0.96704)$	$9.48603\,(4.64662)$
	$n = 200$		
(q, β, γ)	q	β	γ
$(0.25, 3, 3)$	$0.13820\,(-0.07917)$	$2.52661\,(0.80005)$	$5.23602\,(3.22514)$
$(0.25, 3, 5)$	$0.02468\,(-0.01561)$	$1.50948\,(1.43465)$	$0.80066\,(0.48797)$
$(0.25, 5, 3)$	$0.05905\,(-0.05655)$	$0.71393\,(0.53369)$	$1.10777\,(1.03427)$
$(0.25, 5, 5)$	$0.02341\,(0.01351)$	$0.38812\,(-0.36961)$	$0.30430\,(-0.29148)$
$(0.5, 1, 3)$	$0.40165\,(-0.11913)$	$1.81642\,(0.50136)$	$22.83387\,(14.86689)$
$(0.5, 2, 3)$	$0.29014\,(-0.16875)$	$2.56145\,(0.99562)$	$11.38707\,(7.16958)$
$(0.5, 3, 2)$	$0.36320\,(-0.22754)$	$3.11450\,(-0.01613)$	$12.24758\,(8.68164)$
$(0.5, 3, 3)$	$0.14296\,(-0.10921)$	$1.98179\,(1.53127)$	$3.89402\,(1.95687)$
$(0.5, 5, 3)$	$0.05949\,(-0.04462)$	$0.43865\,(0.39210)$	$0.60219\,(0.48600)$
$(0.75, 1, 2)$	$0.52658\,(-0.37030)$	$1.17139\,(-0.02594)$	$23.49291\,(16.78337)$
$(0.75, 2, 2)$	$0.44955\,(-0.35980)$	$2.08948\,(0.48715)$	$14.89097\,(9.96325)$
$(0.75, 3, 2)$	$0.31593\,(-0.27016)$	$1.93956\,(1.06188)$	$7.72847\,(4.25849)$

Table 4: The AWs and CPs in parentheses of the 95% ACIs for q and 95% modified ACIs for β and γ based on the simulated samples.

$n = 100$			
(q, β, γ)	q	β	γ
(0.25, 3, 3)	0.9981 (0.9958)	110.7347 (0.9434)	136.5763 (0.9924)
(0.25, 3, 5)	1.0000 (1.0000)	105.6776 (1.0000)	158.5149 (1.0000)
(0.25, 5, 3)	1.0000 (1.0000)	152.5317 (0.9988)	124.2070 (1.0000)
(0.25, 5, 5)	1.0000 (1.0000)	134.7217 (1.0000)	170.6075 (1.0000)
(0.5, 1, 3)	0.9633 (0.9952)	18.1568 (0.7604)	164.6267 (0.8902)
(0.5, 2, 3)	0.9965 (0.9996)	96.8216 (0.9194)	168.8607 (0.9906)
(0.5, 3, 2)	0.9881 (0.9974)	71.1973 (0.8684)	111.5572 (0.9890)
(0.5, 3, 3)	0.9998 (1.0000)	130.7502 (0.9624)	153.7191 (0.9990)
(0.5, 5, 3)	1.0000 (1.0000)	200.7873 (0.9984)	169.9534 (0.9998)
(0.75, 1, 2)	0.9902 (0.9966)	53.7110 (0.6826)	218.2456 (0.9920)
(0.75, 2, 2)	0.9980 (0.9998)	129.1692 (0.7188)	228.5331 (0.9994)
(0.75, 3, 2)	0.9997 (1.0000)	139.4383 (0.8550)	156.5544 (1.0000)
$n = 200$			
(q, β, γ)	q	β	γ
(0.25, 3, 3)	0.9776 (0.9896)	54.3068 (0.9560)	78.1065 (0.9832)
(0.25, 3, 5)	0.9999 (1.0000)	126.1706 (1.0000)	182.1682 (1.0000)
(0.25, 5, 3)	0.9977 (0.9998)	115.5713 (0.9996)	107.7730 (0.9998)
(0.25, 5, 5)	1.0000 (1.0000)	91.6425 (1.0000)	115.2683 (1.0000)
(0.5, 1, 3)	0.9219 (0.9740)	20.1565 (0.7080)	109.1625 (0.8336)
(0.5, 2, 3)	0.9913 (0.9998)	46.6989 (0.8768)	106.1893 (0.9666)
(0.5, 3, 2)	0.9612 (0.9862)	48.0733 (0.6942)	109.1397 (0.9734)
(0.5, 3, 3)	0.9995 (1.0000)	130.4841 (0.9834)	136.3901 (0.9972)
(0.5, 5, 3)	1.0000 (1.0000)	152.4484 (0.9986)	114.3566 (1.0000)
(0.75, 1, 2)	0.9734 (0.9850)	7.2291 (0.6384)	143.0424 (0.9826)
(0.75, 2, 2)	0.9952 (1.0000)	51.0465 (0.7936)	135.8319 (0.9984)
(0.75, 3, 2)	0.9993 (1.0000)	97.2467 (0.8896)	112.6414 (1.0000)

Table 5: The real data of the example.

Number of outbreaks	0	1	2	3	4 or more
Frequency	46	76	24	9	1

The comparison of the fits is based on the Akaike information criterion (AIC) and Kolmogorov-Smirnov (K-S) test statistic. The AIC measure was first proposed by [4, 5]. Let $\widehat{\ell}$ be the maximized value of the log-likelihood function of the parameters and k denote the number of parameters of the distribution. Then, the AIC measure, as a goodness-of-fit measure, is defined as $-2\widehat{\ell}+2k$. It is clear that the AIC is an increasing function of k. A lower value of the AIC measure indicates a superior fit. In other words, among the various candidate distributions for a data set, the distribution that is considered best is the one that has the lowest AIC value, see [14] for more details on the AIC measure. The K-S statistic measures the difference between the empirical cumulative distribution of a data set and the CDF of the distribution of interest. Thus, it is clear that the smaller the value of the K-S statistic, the better the fit, see [16] for more details on the K-S measure.

The ML estimates of the parameters, AIC measures, and the K-S test statistics are computed and presented in Table 6. From Table 6, we observe that though the EDIW distribution involves the most number of parameters, it possesses the smallest AIC, which establishes its best fit compared to its sub-models. Moreover, the new model outperforms its sub-models w.r.t. the K-S measure.

Table 6: The ML estimates, as well as the goodness of fit measures for the real data example

Distribution	ML estimates	AIC	K-S statistic
EDIW	$(\widehat{q},\widehat{\beta},\widehat{\gamma})=(0.04166, 0.88566, 7.99444)$	382.8124	0.011791
EDIR	$(\widehat{q},\widehat{\gamma})=(0.19250, 1.55295)$	385.6556	0.032737
DIW	$(\widehat{q},\widehat{\beta})=(0.28614, 2.59662)$	387.6523	0.031074
DIR	$\widehat{q}=0.31232$	393.9354	0.063741

7 Concluding remarks

In this paper, we introduced a novel extension of the discrete inverse Weibull distribution, called the exponentiated discrete inverse Weibull distribution, and discussed some of its distributional properties. We emphasize that the new distribution can be suitable for modeling platykurtic data sets. We also studied the problem of parameter estimation for the new model. Then, we evaluated the point and interval estimators with the help of a simulation study. Besides, we compared the fit of the new model with those of several newly introduced distributions using a real data set. Our new distribution showed the best fit among the considered models w.r.t. the AIC measure even though it possesses the most number of parameters (note that the AIC is increasing w.r.t. the number of parameters), so we can conclude that the new model can be a good candidate for modeling real data due to its high flexibility. All the computations of this paper were done using Maple 16 and the statistical software R [39] and the package `dgof` [8, 9] therein.

Acknowledgements

We would like to sincerely thank the anonymous referee for his/her valuable comments.

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