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A PERIOD 5 DIFFERENCE EQUATION

WITOLD A. J. KOSMALA[1*]

Abstract. The main goal of this note is to introduce another second-order difference equation where every nontrivial solution is of minimal period 5, namely the difference equation:

$$x_{n+1} = \frac{1 + x_{n-1}}{x_n x_{n-1} - 1}, \quad n = 1, 2, 3, \dots$$

with initial conditions x_0 and x_1 any real numbers such that $x_0 x_1 \neq 1$.

1. Introduction

Rational difference equations of second order have been studied for over 10 years, and solutions have very unique and beautiful behaviors. In the references we give a very limited list of papers that study behavior of such equations. See [1]–[16]. It is very intriguing to find an equation, which possesses solutions that are always periodic. For example, the equation:

$$x_{n+1} = \frac{1}{x_n x_{n-1}}$$

has only period 3 solutions; the equation:

$$x_{n+1} = \frac{1}{x_{n-1}}$$

has only period 4 solutions; and so on. It is believed that up to now Lyness's equation of the form

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 1, 2, 3, \dots,$$

with nonzero initial conditions, was the only equation for which nontrivial solutions were periodic with the minimal period of 5. In this paper we want to change this fact by introducing another second–order equation with this property, namely:

$$x_{n+1} = \frac{1 + x_{n-1}}{x_n x_{n-1} - 1}, \quad n = 1, 2, 3, \dots \tag{1.1}$$

with real initial conditions x_0 and x_1 such that $x_0 x_1 \neq 1$.

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*: Corresponding author.

This equation (1.1) is rooted in the difference equation:

$$x_{n+1} = \frac{1 + x_{n-2}}{x_n}, \quad n = 1, 2, 3, \ldots \tag{1.2}$$

with three initial conditions x_{-1}, x_0, and x_1. The difference equation (1.2) has been extensively studied by many mathematicians around the world. In particular, Camouzis and Ladas in [4] introduced an identity which was used to prove that in the equation (1.2) all solutions must converge to a period 5 solution. Trichotomy properties of the equation (1.2) as well as its generalities are also discussed in [3].

To see where the difference equation (1.1) comes from, we need to look at the period 5 solution of the equation (1.2) which all solutions converge to. A simple calculation shows that a solution of the equations (1.2) is of period 5 if and only if the initial conditions are $x_{-1} = \alpha$, $x_0 = \beta$ with α and β real such that $\alpha\beta \neq 1$, and $x_1 = \dfrac{1+\alpha}{\alpha\beta - 1}$. In addition, the other two terms of such a solution are $x_2 = \alpha\beta - 1$ and $x_3 = \dfrac{1+\beta}{\alpha\beta - 1}$. The expressions for x_1, x_2, and x_3 above can be written as difference equations in their own right. That is, x_1 can be written as our equation (1.1); x_2 can be written as:

$$x_{n+1} = x_{n-2}x_{n-1} - 1, \quad n = 1, 2, 3, \ldots \tag{1.3}$$

x_{-1}, x_0, and x_1 real numbers; and x_3 can be written as

$$x_{n+1} = \frac{1 + x_{n-2}}{x_{n-3}x_{n-2} - 1}, \quad n = 1, 2, 3, \ldots, \tag{1.4}$$

with appropriate initial conditions. The difference equation (1.3) has been studied in [8]. In addition, the equation (1.3) with reduced delay appears in [2]. Many wonderful properties of (1.3) have been presented in [9]. The equation (1.4) is yet to be investigated.

2. Main results

Theorem 2.1. *All the difference equations (1.1) – (1.4) have two equilibrium points: the golden number* $\dfrac{1+\sqrt{5}}{2}$ *and its conjugate* $\dfrac{1-\sqrt{5}}{2}$.

Theorem 2.2. *Every non-equilibrium solution of the difference equation (1.1) is periodic with the minimal period 5.*

Proof. Let α and β be real numbers such $\alpha\beta \neq 1$. Define $x_0 = \alpha$ and $x_1 = \beta$. Then, by simple calculations we obtain $x_2 = \dfrac{1+\alpha}{\alpha\beta - 1}$, $x_3 = \alpha\beta - 1$, $x_4 = \dfrac{1+\beta}{\alpha\beta - 1}$, and the sequence repeats. $\square$

3. Future Work

Continuation of the study of the equation (1.1) for generalizations

$$x_{n+1} = \frac{p + x_{n-1}}{x_n x_{n-1} - q}$$

with varies values of p and q is encouraged. An increased delay was already proposed as the equation (1.4). Investigation of (1.4) with values of p and q replacing the values of 1 is also of interest. Moreover, the difference equations that belong to the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-l} - 1, \quad n \in N$$

and particular choice of $k, l \in N$, other than those presented in [8] and [9], are also of great interest.

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[1] Department of Math. Sci., Appalachian State University, Boone, NC 28608, USA
E-mail address: wak@math.appstate.edu