Int. J. Nonlinear Anal. Appl. In Press, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2025.37562.5452

Generalized bivariate Fibonacci polynomials for a comprehensive subfamily of bi-univalent functions

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(Communicated by Ali Ebadian)

Abstract

Our present investigation is primarily motivated by the broad and impactful applications of special polynomials in geometric function theory. In particular, the generalized bivariate Fibonacci polynomials have recently attracted attention in the study of bi-univalent functions. In this article, we introduce and analyze a comprehensive subfamily of bi-univalent functions characterized by these generalized Fibonacci polynomials in two variables. This polynomial sequence is chosen due to its flexibility, as numerous other polynomial families can be derived through appropriate specialization of its parameters. By applying the subordination technique, we derive bounds for the initial coefficients of functions belonging to this subfamily and investigate the associated Fekete–Szegö problem. In addition to presenting several new findings, we also explore meaningful connections with previously established results in the theory of bi-univalent and subordinate functions, thereby extending and unifying existing literature in a novel direction.

Keywords: Holomorphic functions, generalized bivariate Fibonacci polynomials, bi-univalent functions, subordination
2020 MSC: Primary 30C45; Secondary 05A15, 11B39

1 Preliminaries

Geometric Function Theory (GFT) is a dynamic and productive branch of complex analysis that has increasingly captured the interest of researchers. A significant area of focus within GFT is the study of holomorphic univalent functions and their various subclasses. In recent years, particular attention has been devoted to the classees of bi-univalent functions. The investigation of such functions have attracted growing interest owing to their rich geometric structure and challenging coefficient problems. These bi-univalent classes open new avenues for exploring functional inequalities, coefficient estimates, and operator-based extensions. The present work contributes to this growing body of research by introducing and analyzing new subfamilies of bi-univalent functions governed by generalized bivariate Fibonacci polynomials (GBFP).

Let $\mathfrak{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. The family of holomorphic functions ϕ in $\mathfrak{U}$ of the form

$$\phi(\varsigma) = \varsigma + d_2\varsigma^2 + d_3\varsigma^3 + \cdots = \varsigma + \sum_{j=2}^{\infty} d_j\varsigma^j, \quad \varsigma \in \mathfrak{U}, \tag{1.1}$$

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Received: April 2025 *Accepted:* June 2025

is identified by $\mathcal{A}$. Let $\mathcal{S}$ be the subset of $\mathcal{A}$ defined by $\mathcal{S}$=$\{\phi \in \mathcal{A} : \phi$ is univalent in $\mathfrak{U}\}$. Biebereach conjectured in [5] that for every function $\phi \in \mathcal{S}$, $|d_j| \leq j, j \geq 2$. To settle the Biebereach conjecture, many new subclasses of $\mathcal{S}$ were defined, and several results were established. Branges finally resolved this hypothesis for each $j \geq 2$ in [10] after years of research into its proof. For each function $\phi \in \mathcal{S}$ [14], Fekete-Szegö Functional (FSF) $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ is another GFT problem. Researchers have published a large number of papers on the above problem for functions that are members of subclasses of $\mathcal{S}$. One of the most prominent subclasses of $\mathcal{S}$ is the bi-univalent function class σ. Levin presented the idea of σ in [22] and is defined by σ=$\{\phi \in \mathcal{A} : \phi$ and $\phi^{-1} = \psi$ are both univalent in $\mathfrak{U}\}$. The well-known Koebe theorem (see[13]) says that every function $\phi \in \mathcal{S}$ of the form (1.1) has an inverse given by

$$\phi^{-1}(w) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + \cdots = \psi(w) \tag{1.2}$$

such that $\varsigma = \psi(\phi(\varsigma)), \varsigma \in \mathfrak{U}$, and $w = \phi(\psi(w)), |w| < r_0(\phi)$, $1/4 \leq r_0(\phi), w \in \mathfrak{U}$. Since the functions $\frac{1}{2}\log\left(\frac{1+\varsigma}{1-\varsigma}\right)$, $\frac{\varsigma}{1-\varsigma}$, and $-\log(1-\varsigma)$ are members of the σ family, the class σ is not a null set. $\frac{2\varsigma-\varsigma^2}{2}$, $\frac{\varsigma}{1-\varsigma^2}$, and the Koebe function $\frac{\varsigma}{(1-\varsigma)^2}$, despite being in $\mathcal{S}$, are not elements of σ. For a brief analysis and to learn about some of the traits of the σ family, see [6, 7, 24, 36]. Research on the family of bi-univalent functions have recently gained momentum thanks to Srivastava and his co-authorss for an article[29]. Since this article revived the topic, numerous researchers have looked into a number of fascinating special σ families; see [8, 9, 11, 15, 16, 37] as well as the citation provided in these articles.

Many polynomials such as Bernoulli, Fibonacci, Gegenbauer, Horadam, Lucas-Balancing, Lucas-Lehmer, and their extensions are essential in a wide range of disciplines, including combinatorics, computer science, engineering, number theory, numerical analysis, and physics. Because of their extensive use in science and engineering, generalizations of these polynomials have been defined in the literature. Recently, a particular class of polynomials known as Generalized Bivariate Fibonacci Polynomials (GBFP) has drawn the attention of researchers (see[21]).

It is known that the Fibonacci numbers have the recursive relation

$$\mathfrak{F}_j = \mathfrak{F}_{j-1} + \mathfrak{F}_{j-2},\ \mathfrak{F}_0 = 0, \mathfrak{F}_1 = 1, j \geq 2.$$

Fibonacci polynomials which generalize Fibonacci numbers is defined as (see [4])

$$\mathfrak{F}_j(\varkappa) = \varkappa\mathfrak{F}_{j-1}(\varkappa) + \mathfrak{F}_{j-2}(\varkappa), (\mathfrak{F}_0(\varkappa) = 0, \mathfrak{F}_1(\varkappa) = 1, j \geq 2).$$

A new generalization GBFP of $\mathfrak{F}_j(\varkappa)$ is introduced in [20] and is expressed as below:

$$\mathfrak{F}_j(\varkappa, \vartheta) = a(\varkappa, \vartheta)\mathfrak{F}_{j-1}(\varkappa, \vartheta) + b(\varkappa, \vartheta)\mathfrak{F}_{j-2}(\varkappa, \vartheta),\ \mathfrak{F}_0(\varkappa, \vartheta) = 0, \mathfrak{F}_1(\varkappa, \vartheta) = 1, \tag{1.3}$$

where $a(\varkappa, \vartheta)$, and $b(\varkappa, \vartheta)$ are real such that $a^2(\varkappa, \vartheta) + 4b(\varkappa, \vartheta) > 0$. The generating function (GF) of GBFP is (see [20])

$$\mathfrak{F}(\varkappa, \vartheta, z) = \sum_{j=0}^{\infty} \mathfrak{F}_j(\varkappa, \vartheta)z^j = \frac{z}{1 - a(\varkappa, \vartheta)z - b(\varkappa, \vartheta)z^2}. \tag{1.4}$$

The current emphasis is on functions associated with special polynomials or known number sequences that belong to a particular σ subfamily. Researchers have determined estimates of coefficients for members of σ subfamilies connected to known special polynomials or number sequences and FSF $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, (For more information, see [2, 3, 12, 17, 19, 25, 28, 30, 32, 34, 38]). Recently, GBFP has drawn the attention of researchers. In [1, 18, 26], Interesting results regarding the Fekete-Szego functional and coefficient estimates have been found for individuals belonging to particular subclasses of σ linked to GBFP.

For the sake of conciseness, we write $a(\varkappa, \vartheta) = a$ and $b(\varkappa, \vartheta)$=b from now on. Clearly

$$\mathfrak{F}_2(\varkappa, \vartheta) = a, \mathfrak{F}_3(\varkappa, \vartheta) = a^2 + b, \cdots \tag{1.5}$$

are evident from (1.3).

Building a more comprehensive and potent analytical framework for the study of bi-univalent functions, where GBFPs are adaptable, well-organized, and computationally tractable tools, is the main driving force. They give analysts the ability to create and study new families of functions that have deeper geometric meanings, stronger or more general coefficient bounds, and links to traditional mathematical structures. They are effective tools for creating and examining new subclasses of bi-univalent functions because of their parametric flexibility, recursive nature, and intrinsic generality. This allows for accurate coefficient estimates and unifies different previous results as special cases. Furthermore, GBFP are especially well-suited to recent advancements in geometric function theory since they support deeper geometric and combinatorial interpretations and make it easier to construct analytic operators.

Remark 1.1. It is possible to infer numerous polynomial sequences from GBFP by specializing a and b (see [20]). They are i). If $a = \varkappa$ and $b = \vartheta$, then we obtain bivariate Fibonacci polynmials, ii). If $a = 2\varkappa$ and $b = 1$, then we arrive at the Pell polynomials, iii). if $a = 1$ and $b = 2\varkappa$, then the Jacobsthal polynomials follow, iv). If $a = 3\varkappa$ and $b = -2$, the Fermat polynomials are then obtained, v). If $a = 2\varkappa$ and $b = -1$, then we have the kind two Chebyshev polynomials and vi). $a = 3\varkappa$ and $b = -2$, then we have the kind two Fermat polynomials.

For $\mathfrak{a}_1$, $\mathfrak{a}_2 \in \mathcal{A}$ holomorphic in $\mathfrak{U}$, $\mathfrak{a}_1$ is subordinate to $\mathfrak{a}_2$, if there is a Schwarz function $\varphi(\varsigma)$ that is holomorphic in $\mathfrak{U}$ with $\varphi(0) = 0$ and $|\varphi(\varsigma)| < 1$, such that $\mathfrak{a}_1(\varsigma) = \mathfrak{a}_2(\varphi(\varsigma))$, $\varsigma \in \mathfrak{U}$. This is represented by $\mathfrak{a}_1 \prec \mathfrak{a}_2$ or $\mathfrak{a}_1(\varsigma) \prec \mathfrak{a}_2(\varsigma)$. In case, if $\mathfrak{a}_2 \in \mathcal{S}$, then

$$\mathfrak{a}_1(\varsigma) \prec \mathfrak{a}_2(\varsigma) \;\Leftrightarrow\; \mathfrak{a}_1(0) = \mathfrak{a}_2(0),\; and\; \mathfrak{a}_1(\mathfrak{U}) \subset \mathfrak{a}_2(\mathfrak{U}).$$

Motivated by the earlier noted patterns in coefficient-related problems and the FSF [14] on specific subfamilies of σ, we present a comprehensive subfamily of σ that is connected to GBFP $\mathfrak{F}_j(\varkappa, \vartheta)$ as in (1.3), specifically $\mathfrak{T}^{\tau}_{\sigma}(\alpha, \gamma, \beta, \mathfrak{F})$.

This paper employs the GF $\mathfrak{F}(\varkappa, \vartheta, z)$ as in (1.4), the function $\phi^{-1}(w) = \psi(w)$ as in (1.2), and $\alpha \in \mathbb{C}\backslash\{0\}$, unless otherwise noted.

Definition 1.2. Let $\tau \geq 1$, $0 \leq \beta \leq 1$, and $0 \leq \gamma \leq 1$. If $\phi \in \sigma$ satisfies

$$1 + \frac{1}{\alpha}\left(\gamma\left(\frac{[(\varsigma\phi'(\varsigma))']^{\tau}}{1-\beta+\beta\phi'(\varsigma)}\right) + (1-\gamma)\left(\frac{\varsigma(\phi'(\varsigma))^{\tau}}{(1-\beta)\varsigma+\beta\phi(\varsigma)}\right) - 1\right) \prec F(\varsigma), \tag{1.6}$$

and

$$1 + \frac{1}{\alpha}\left(\gamma\left(\frac{[(w\psi'(w))']^{\tau}}{1-\beta+\beta\psi'(w)}\right) + (1-\gamma)\left(\frac{w(\psi'(w))^{\tau}}{(1-\beta)w+\beta\psi(w)}\right) - 1\right) \prec F(w), \tag{1.7}$$

then we say that $\phi \in \mathfrak{T}^{\tau}_{\sigma}(\alpha, \gamma, \beta, \mathfrak{F})$, where

$$F(\varsigma) = \frac{\mathfrak{F}(\varkappa, \vartheta, \varsigma)}{\varsigma} = \frac{1}{1 - a\varsigma - b\varsigma^2}, a \neq 0, a^2 + 4b > 0, \tag{1.8}$$

and $\varsigma, w \in \mathfrak{U}$.

Remark 1.3. When $\beta = 1$, the class $\mathfrak{T}^{\tau}_{\sigma}(\alpha, \gamma, 1, \mathfrak{F})$ was considered in [33].

For particular choices of τ, γ, and β, the class $\mathfrak{T}^{\tau}_{\sigma}(\alpha, \gamma, \beta, F)$ produces the following subfamilies of σ:

Example 1.4. If $\tau = 1$ in the family $\mathfrak{T}^{\tau}_{\sigma}(\alpha, \gamma, \beta, \mathfrak{F})$, then we get a subfamily $\mathfrak{J}_{\sigma}(\alpha, \gamma, \beta, \mathfrak{F}) \equiv \mathfrak{T}^{1}_{\sigma}(\alpha, \gamma, \beta, \mathfrak{F})$ of elements $\phi \in \sigma$ satisfying

$$1 + \frac{1}{\alpha}\left(\gamma\left(\frac{(\varsigma\phi'(\varsigma))'}{1-\beta+\beta\phi'(\varsigma)}\right) + (1-\gamma)\left(\frac{\varsigma\phi'(\varsigma)}{(1-\beta)\varsigma+\beta\phi(\varsigma)}\right) - 1\right) \prec F(\varsigma),$$

and

$$1 + \frac{1}{\alpha}\left(\gamma\left(\frac{(w\psi'(w))'}{1-\beta+\beta\psi'(w)}\right) + (1-\gamma)\left(\frac{w\psi'(w)}{(1-\beta)w+\beta\psi(w)}\right) - 1\right) \prec F(w),$$

where $F(\varsigma)$ is as given in (1.8), $0 \leq \gamma \leq 1$, $0 \leq \beta \leq 1$, and $\varsigma, w \in \mathfrak{U}$.

Example 1.5. Allowing $\gamma = 0$ in the family $\mathfrak{T}^{\tau}_{\sigma}(\alpha, \gamma, \beta, \mathfrak{F})$, we get a subfamily $\mathfrak{P}^{\tau}_{\sigma}(\alpha, \beta, \mathfrak{F}) \equiv \mathfrak{T}^{\tau}_{\sigma}(\alpha, 0, \beta, \mathfrak{F})$ of elements $\phi \in \sigma$ fulfilling

$$1 + \frac{1}{\alpha}\left(\left(\frac{\varsigma(\phi'(\varsigma))^{\tau}}{(1-\beta)\varsigma+\beta\phi(\varsigma)}\right) - 1\right) \prec F(\varsigma),$$

and

$$1 + \frac{1}{\alpha}\left(\left(\frac{w(\psi'(w))^{\tau}}{(1-\beta)w+\beta\psi(w)}\right) - 1\right) \prec F(w),$$

where $F(\varsigma)$ is as stated in (1.8), $\tau \geq 1$, $0 \leq \beta \leq 1$, and $\varsigma, w \in \mathfrak{U}$.

Remark 1.6. When $\beta = 1$, the class $\mathfrak{P}^{\tau}_{\sigma}(\alpha, 1, \mathfrak{F})$ was considered in [33].

Example 1.7. Allowing $\gamma = 1$ in the family $\mathfrak{T}_\sigma^\tau(\alpha,\gamma,\beta,\mathfrak{F})$, we get a subfamily $\mathfrak{Q}_\sigma^\tau(\alpha,\beta,\mathfrak{F}) \equiv \mathfrak{T}_\sigma^\tau(\alpha,1,\beta,\mathfrak{F})$ of elements $\phi \in \sigma$ satisfying

$$1 + \frac{1}{\alpha}\left(\left(\frac{[(\varsigma\phi'(\varsigma))']^\tau}{1-\beta+\beta\phi'(\varsigma)}\right) - 1\right) \prec F(\varsigma),$$

and

$$1 + \frac{1}{\alpha}\left(\left(\frac{[(w\psi'(w))']^\tau}{1-\beta+\beta\psi'(w)}\right) - 1\right) \prec F(w),$$

where $F(\varsigma)$ is as stated in (1.8), $0 \le \beta \le 1$, $\tau \ge 1$, and $\varsigma, w \in \mathfrak{U}$.

Remark 1.8. When $\beta = 1$, the class $\mathfrak{Q}_\sigma^\tau(\alpha,1,\mathfrak{F})$ was considered in [33].

Letting $\beta = 0$ in the class $\mathfrak{T}_\sigma^\tau(\alpha,\gamma,\beta,\mathfrak{F})$, we get the new class $\mathfrak{Z}_\sigma^\tau(\alpha,\gamma,\mathfrak{F}) \equiv \mathfrak{T}_\sigma^\tau(\alpha,\gamma,0,\mathfrak{F})$ as defined below:.

Example 1.9. If a function $\phi \in \sigma$ satisfies

$$1 + \frac{1}{\alpha}(\gamma[(\varsigma\phi'(\varsigma))']^\tau + (1-\gamma)(\phi'(\varsigma))^\tau - 1) \prec F(\varsigma),$$

and

$$1 + \frac{1}{\alpha}(\gamma[(w\psi'(w))']^\tau + (1-\gamma)(\psi'(w))^\tau - 1) \prec F(w),$$

then we say that $\phi \in \mathfrak{Z}_\sigma^\tau(\alpha,\gamma,\mathfrak{F})$, where $F(\varsigma)$ is as mentioned in (1.8), $\tau \ge 1$, $0 \le \gamma \le 1$, $\varsigma \in \mathfrak{U}$, and $w \in \mathfrak{U}$.

Remark 1.10. When $\gamma = 0$, the class $\mathfrak{Z}_\sigma^\tau(\alpha,0,\mathfrak{F})$ was considered in [33] and when $\gamma = 1$ in $\mathfrak{Z}_\sigma^\tau(\alpha,\gamma,\mathfrak{F})$, we have the following:

Example 1.11. Let $\tau \ge 1$. If a function $\phi \in \sigma$ satisfies

$$\frac{1}{\alpha}([(\varsigma\phi'(\varsigma))']^\tau - 1) + 1 \prec F(\varsigma),$$

and

$$\frac{1}{\alpha}([(w\psi'(w))']^\tau - 1) + 1 \prec F(w),$$

then we say that $\phi \in \mathfrak{E}_\sigma^\tau(\alpha,F) \equiv \mathfrak{Z}_\sigma^\tau(\alpha,1,\mathfrak{F})$, where $F(\varsigma)$ is as mentioned in (1.8), $\varsigma \in \mathfrak{U}$, and $w \in \mathfrak{U}$.

The following is the structure of the paper's content. For functions in the class $\mathfrak{T}_\sigma^\tau(\alpha,\gamma,\beta,\mathfrak{F})$, the estimates for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, are found in Section 2. In Section 3, we highlight pertinent associations between some of the particular cases and the key conclusions. We also go over some observations regarding our findings. In Section 4, we conclude the study with some observations.

2 Main Results

For any function $\phi \in \mathfrak{T}_\sigma^\tau(\alpha,\gamma,\beta,\mathfrak{F})$, we find the estimates associated with the coefficients.

Theorem 2.1. Let $\tau \ge 1$, $\xi \in \mathbb{R}$, $0 \le \gamma \le 1$, $0 \le \beta \le 1$, and $F(\varsigma)$ is as in (1.8). If $\phi \in \sigma$ is a member of $\mathfrak{T}_\sigma^\tau(\alpha,\gamma,\beta,\mathfrak{F})$, then

$$|d_2| \le |\alpha a|\sqrt{\frac{|a|}{|((1+2\gamma)(3\tau-\beta) + A(1+3\gamma))\alpha a^2 - (1+\gamma)^2(2\tau-\beta)^2(a^2+b)|}}, \tag{2.1}$$

$$|d_3| \le \frac{|\alpha a|}{(1+2\gamma)(3\tau-\beta)} + \frac{|\alpha a|^2}{(1+\gamma)^2(2\tau-\beta)^2}, \tag{2.2}$$

and

$$|d_3 - \xi d_2^2| \le \begin{cases} \frac{|\alpha a|}{(1+2\gamma)(3\tau-\beta)} & ; |1-\xi| \le \daleth \\ \frac{|\alpha|^2|a|^3\,|1-\xi|}{|((1+2\gamma)(3\tau-\beta)+A(1+3\gamma))\alpha a^2-(1+\gamma)^2(2\tau-\beta)^2(a^2+b)|} & ; |1-\xi| \ge \daleth, \end{cases} \tag{2.3}$$

where

$$\daleth = \left| \frac{((1+2\gamma)(3\tau-\beta)+A(1+3\gamma))\alpha a^2 - (1+\gamma)^2(2\tau-\beta)^2(a^2+b)}{(3\tau-\beta)(1+2\gamma)\alpha a^2} \right|, \tag{2.4}$$

and

$$A = 2\tau(\tau-1) - 2\tau\beta + \beta^2. \tag{2.5}$$

Proof . Let $\phi \in \mathfrak{T}_\sigma^\tau(\alpha,\gamma,\beta,\mathfrak{F})$. Then, from subordinations (1.6) and (1.7), we can write

$$1 + \frac{1}{\alpha}\left(\gamma\left(\frac{[(\varsigma\phi'(\varsigma))']^\tau}{1-\beta+\beta\phi'(\varsigma)}\right) + (1-\gamma)\left(\frac{\varsigma(\phi'(\varsigma))^\tau}{(1-\beta)\varsigma+\beta\phi(\varsigma)}\right) - 1\right) = F(\mathfrak{u}(\varsigma)), \tag{2.6}$$

and

$$1 + \frac{1}{\alpha}\left(\gamma\left(\frac{[(w\psi'(w))']^\tau}{1-\beta+\beta\psi'(w)}\right) + (1-\gamma)\left(\frac{w(\psi'(w))^\tau}{(1-\beta)w+\beta\psi(w)}\right) - 1\right) = F(\mathfrak{v}(w)), \tag{2.7}$$

where $\mathfrak{u}(\varsigma) = \mathfrak{u}_1\varsigma + \mathfrak{u}_2\varsigma^2 + \cdots$, and $\mathfrak{v}(w) = \mathfrak{v}_1 w + \mathfrak{v}_2 w^2 + \cdots$, $(\varsigma \in \mathfrak{U},\ w \in \mathfrak{U})$, are Schwarz functions satisfying (see[13])

$$|\mathfrak{u}_j| \le 1,\ \textit{and}\ |\mathfrak{v}_j| \le 1\ (j \in \mathbb{N}). \tag{2.8}$$

Using standard mathematical manipulations, equations (2.6) and (2.7) can be rewritten as follows:

$$\begin{aligned} &1 + \frac{1}{\alpha}\left(\gamma\left(\frac{[(\varsigma\phi'(\varsigma))']^\tau}{1-\beta+\beta\phi'(\varsigma)}\right) + (1-\gamma)\left(\frac{\varsigma(\phi'(\varsigma))^\tau}{(1-\beta)\varsigma+\beta\phi(\varsigma)}\right) - 1\right) \\ &= 1 + \frac{1}{\alpha}(1+\gamma)(2\tau-\beta)d_2\varsigma + \frac{1}{\alpha}\left((1+2\gamma)(3\tau-\beta)d_3 + A(1+3\gamma)d_2^2\right)\varsigma^2 + \cdots, \end{aligned} \tag{2.9}$$

where A is as stated in (2.5),

$$F(\mathfrak{u}(\varsigma)) = 1 + \mathfrak{F}_2(\varkappa,\vartheta)\mathfrak{u}_1\varsigma + \left[\mathfrak{F}_2(\varkappa,\vartheta)\mathfrak{u}_2 + \mathfrak{F}_3(\varkappa,\vartheta)\mathfrak{u}_1^2\right]\varsigma^2 + \cdots, \tag{2.10}$$

and

$$\begin{aligned} &1 + \frac{1}{\alpha}\left(\gamma\left(\frac{[(w\psi'(w))']^\tau}{1-\beta+\beta\psi'(w)}\right) + (1-\gamma)\left(\frac{w(\psi'(w))^\tau}{(1-\beta)w+\beta\psi(w)}\right) - 1\right) \\ &= 1 - \frac{1}{\alpha}(1+\gamma)(2\tau-\beta)d_2 w + \frac{1}{\alpha}\left([(1+2\gamma)(3\tau-\beta)(2d_2^2-d_3) + A(1+3\gamma)d_2^2]w^2 + \cdots\right), \end{aligned} \tag{2.11}$$

where A is as stated in (2.5),

$$F(\mathfrak{v}(w)) = 1 + \mathfrak{F}_2(\varkappa,\vartheta)\mathfrak{v}_1 w + \left[\mathfrak{F}_2(\varkappa,\vartheta)\mathfrak{v}_2 + \mathfrak{F}_3(\varkappa,\vartheta)\mathfrak{v}_1^2\right]w^2 + \cdots. \tag{2.12}$$

Comparing the coefficients of like powers in (2.9) and (2.11), and invoking the equality (2.6), we arrive at the following conclusion:

$$(1+\gamma)(2\tau-\beta)d_2 = \alpha\mathfrak{F}_2(\varkappa,\vartheta)\mathfrak{u}_1, \tag{2.13}$$

$$(1+2\gamma)(3\tau-\beta)d_3 + A(1+3\gamma)d_2^2 = \alpha[\mathfrak{F}_2(\varkappa,\vartheta)\mathfrak{u}_2 + \mathfrak{F}_3(\varkappa,\vartheta)\mathfrak{u}_1^2], \tag{2.14}$$

Comparing the coefficients of like powers in (2.10) and (2.12), and invoking the equality (2.7), we arrive at the following conclusion:

$$-(1+\gamma)(2\tau-\beta)\,d_2 = \alpha\mathfrak{F}_2(\varkappa,\vartheta)\mathfrak{v}_1, \tag{2.15}$$

and

$$(1+2\gamma)(3\tau-\beta)(2d_2^2-d_3) + A(1+3\gamma)d_2^2 = \alpha[\mathfrak{F}_2(\varkappa,\vartheta)\mathfrak{v}_2 + \mathfrak{F}_3(\varkappa,\vartheta)\mathfrak{v}_1^2]. \tag{2.16}$$

From (2.13) and (2.15), we get

$$\mathfrak{u}_1 = -\mathfrak{v}_1, \tag{2.17}$$

and

$$2(1+\gamma)^2(2\tau-\beta)^2 d_2^2 = \alpha^2(\mathfrak{u}_1^2 + \mathfrak{v}_1^2)\mathfrak{F}_2^2(\varkappa,\vartheta). \tag{2.18}$$

Addition of (2.14) and (2.16) yield

$$2((1+2\gamma)(3\tau-\beta) + A(1+3\gamma))d_2^2 = \alpha\mathfrak{F}_2(\varkappa,\vartheta)(\mathfrak{u}_2+\mathfrak{v}_2) + \alpha\mathfrak{F}_3(\varkappa,\vartheta)(\mathfrak{u}_1^2+\mathfrak{v}_1^2). \tag{2.19}$$

Substituting $\mathfrak{u}_1^2+\mathfrak{v}_1^2$ from (2.18) into (2.19), we get

$$d_2^2=\frac{\alpha^2\mathfrak{F}_2^3(\varkappa,\vartheta)(\mathfrak{u}_2+\mathfrak{v}_2)}{2\left[((1+2\gamma)(3\tau-\beta)+A(1+3\gamma))\alpha\mathfrak{F}_2^2(\varkappa,\vartheta)-(1+\gamma)^2(2\tau-\beta)^2\mathfrak{F}_3(\varkappa,\vartheta)\right]}. \tag{2.20}$$

By applying (2.8) to $\mathfrak{u}_2$ and $\mathfrak{v}_2$, and employing (1.5) for $\mathfrak{F}_2(\varkappa,\vartheta)$ and $\mathfrak{F}_3(\varkappa,\vartheta)$, we obtain (2.1). The bound on $|d_3|$ is obtained by subtracting equation (2.16) from equation (2.14).

$$d_3=d_2^2+\frac{\alpha\mathfrak{F}_2(\varkappa,\vartheta)(\mathfrak{u}_2-\mathfrak{v}_2)}{2(1+2\gamma)(3\tau-\beta)}. \tag{2.21}$$

If we replace d_2^2 from (2.18) into (2.21), we get

$$|d_3|=\frac{\alpha^2\mathfrak{F}_2^2(\varkappa,\vartheta)(\mathfrak{u}_1^2+\mathfrak{v}_2^2)}{2(1+\gamma)^2(2\tau-\beta)^2}+\frac{\alpha\mathfrak{F}_2(\varkappa,\vartheta)(\mathfrak{u}_2-\mathfrak{v}_2)}{2(1+2\gamma)(3\tau-\beta)}. \tag{2.22}$$

We deduce equation (2.2) from (2.22) by applying (1.5) and (2.8). Finally, we compute the bound on $|d_3-\xi d_2^2|$ using the values of d_2^2 and d_3 from (2.20) and (2.21), respectively. Consequently, we have

$$|d_3-\xi d_2^2|=\frac{|\alpha||\mathfrak{F}_2(\varkappa,\vartheta)|}{2}\left|\left(\mathfrak{B}+\frac{1}{(3\tau-\beta)(1+2\gamma)}\right)\mathfrak{u}_2+\left(\mathfrak{B}-\frac{1}{(3\tau-\beta)(1+2\gamma)}\right)\mathfrak{v}_2\right|$$

where

$$\mathfrak{B}=\frac{\alpha(1-\xi)\mathfrak{F}_2^2(\varkappa,\vartheta)}{\left[((3\tau-\beta)(1+2\gamma)+A(1+3\gamma))\alpha\mathfrak{F}_2^2(\varkappa,\vartheta)-(1+\gamma)^2(2\tau-\beta)^2\mathfrak{F}_3(\varkappa,\vartheta)\right]}.$$

Clearly

$$|d_3-\xi d_2^2|\leq\begin{cases}\frac{|\alpha||\mathfrak{F}_2(\varkappa,\vartheta)|}{(3\tau-\beta)(1+2\gamma)} & ;\ |\mathfrak{B}|\leq\frac{1}{(3\tau-\beta)(1+2\gamma)}\\ |\alpha||\mathfrak{F}_2(\varkappa,\vartheta)||\mathfrak{B}| & ;\ |\mathfrak{B}|\geq\frac{1}{(3\tau-\beta)(1+2\gamma)}.\end{cases} \tag{2.23}$$

We derive (2.3) from (2.23), with $\daleth$ is the same as in (2.4). □

The following would result if $\xi=1$, according to Theorem 2.1:

Corollary 2.2. If a function $\phi\in\sigma$ is a member of $\mathfrak{T}_\sigma^\tau(\alpha,\gamma,\beta,\mathfrak{F})$, then $|d_3-d_2^2|\leq\frac{|\alpha a|}{(1+2\gamma)(3\tau-\beta)}$.

Remark 2.3. By allowing $\beta=1$, we can infer Theorem 2.1 in [33] from Theorem 2.1.

Remark 2.4. With reference to the above described definition, we can obtain multiple subfamilies of bi-univalent functions governed by GBFP for specific parameters, such as τ, γ, and β. We address some of these in the section that follows.

3 Special cases

The outcome of Theorem 2.1 when $\tau=1$ is as follows.

Corollary 3.1. Let $\xi\in\mathbb{R}$, $0\leq\gamma\leq1$, $0\leq\beta\leq1$, and $F(\varsigma)$ be as in (1.8). If a function $\phi\in\mathfrak{J}_\sigma(\alpha,\gamma,\beta,\mathfrak{F})$, then

$$|d_2|\leq|\alpha a|\sqrt{\frac{|a|}{|((3-\beta)(1+2\gamma)-\beta(2-\beta)(1+3\gamma))\alpha a^2-(2-\beta)^2(1+\gamma)^2(a^2+b)|}},$$

$$|d_3|\leq\frac{|\alpha a|}{(3-\beta)(1+2\gamma)}+\frac{|\alpha a|^2}{(2-\beta)^2(1+\gamma)^2},$$

and

$$|d_3-\xi d_2^2|\leq\begin{cases}\frac{|\alpha a|}{(3-\beta)(1+2\gamma)} & ;\ |1-\xi|\leq\daleth_1\\ \frac{|\alpha|^2|a|^3\,|1-\xi|}{|((3-\beta)(1+2\gamma)-\beta(2-\beta)(1+3\gamma))\alpha a^2-(2-\beta)^2(1+\gamma)^2(a^2+b)|} & ;\ |1-\xi|\geq\daleth_1,\end{cases}$$

where

$$\daleth_1=\left|\frac{((3-\beta)(1+2\gamma)-\beta(2-\beta)(1+3\gamma))\alpha a^2-(2-\beta)^2(1+\gamma)^2(a^2+b)}{(3-\beta)(1+2\gamma)\alpha a^2}\right|.$$

Remark 3.2. By allowing $\alpha = 1$ and $\beta = 1$, we can infer Theorem 1 and Theorem 3 in [39] from Corollary 3.1.

The outcome of Theorem 2.1 would be as follows, if $\gamma = 0$.

Corollary 3.3. Let $\tau \geq 1$, $\xi \in \mathbb{R}$, $0 \leq \beta \leq 1$, and $F(\varsigma)$ be as in (1.8). If a function $\phi \in \mathfrak{P}_\sigma^\tau(\alpha, \beta, \mathfrak{F})$, then

$$|d_2| \leq |\alpha a| \sqrt{\frac{|a|}{|((\tau - \beta)(1 + 2\tau) + \beta^2)\alpha a^2 - (2\tau - \beta)^2(a^2 + b)|}},$$

$$|d_3| \leq \frac{|\alpha a|^2}{(2\tau - \beta)^2} + \frac{|\alpha a|}{3\tau - \beta},$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\alpha a|}{3\tau - \beta} & ; |1 - \xi| \leq \daleth_2 \\ \frac{|\varrho|^2 |a|^3 \, |1-\xi|}{|((\tau-\beta)(1+2\tau)+\beta^2)\alpha a^2-(2\tau-\beta)^2(a^2+b)|} & ; |1 - \xi| \geq \daleth_2, \end{cases}$$

where

$$\daleth_2 = \left| \frac{((\tau - \beta)(1 + 2\tau) + \beta^2)\alpha a^2 - (2\tau - \beta)^2(a^2 + b)}{(3\tau - \beta)\alpha a^2} \right|.$$

Remark 3.4. Corollary 3.1 in [33] can be obtained by applying $\beta = 1$ in Corollary 3.3. Furtermore, Corollaries 2 and 6 in [39] can be obtained by taking $\alpha = 1$ and $\tau = 1$.

According to Theorem 2.1, the following result holds when $\gamma = 1$.

Corollary 3.5. Let $\tau \geq 1$, $\xi \in \mathbb{R}$, $0 \leq \beta \leq 1$, and $F(\varsigma)$ be as in (1.8). If a function $\phi \in \mathfrak{Q}_\sigma^\tau(\alpha, \beta, \mathfrak{F})$, then

$$|d_2| \leq |\alpha a| \sqrt{\frac{|a|}{|((\tau - \beta)(1 + 8\tau) + 2\beta(2\beta - 1))\alpha a^2 - 4(2\tau - \beta)^2(a^2 + b)|}},$$

$$|d_3| \leq \frac{|\alpha a|^2}{4(2\tau - \beta)^2} + \frac{|\alpha a|}{3(3\tau - \beta)},$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\alpha a|}{3(3\tau - \beta)} & ; |1 - \xi| \leq \daleth_3 \\ \frac{|\alpha|^2 |a|^3 \, |1-\xi|}{|((\tau-\beta)(1+8\tau)+2\beta(2\beta-1))\alpha a^2-4(2\tau-\beta)^2(a^2+b)|} & ; |1 - \xi| \geq \daleth_3, \end{cases}$$

where

$$\daleth_3 = \left| \frac{((\tau - \beta)(1 + 8\tau) + 2\beta(2\beta - 1))\alpha a^2 - 4(2\tau - \beta)^2(a^2 + b)}{3(3\tau - \beta)\alpha a^2} \right|.$$

Remark 3.6. Corollary 3.2 in [33] is recovered by setting $\beta = 1$ in Corollary 3.5. Furthermore, Corollaries 3 and 7 of [39] emerge as special cases when $\alpha = 1$, and $\tau = 1$.

By setting $\beta = 0$, Theorem 2.1 yields the following outcome:

Corollary 3.7. Let $F(\varsigma)$ be as in (1.8), $\tau \geq 1$, $\xi \in \mathbb{R}$, and $0 \leq \gamma \leq 1$. If a function $\phi \in \mathfrak{Z}_\sigma^\tau(\alpha, \gamma, \mathfrak{F})$, then

$$|d_2| \leq |\alpha a| \sqrt{\frac{|a|}{|\tau(1 + 2\tau + 6\gamma\tau)\alpha a^2 - 4\tau^2(1 + \gamma)^2(a^2 + b)|}},$$

$$|d_3| \leq \frac{|\alpha a|^2}{4\tau^2(1 + \gamma)^2} + \frac{|\alpha a|}{3\tau(1 + 2\gamma)},$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\alpha a|}{3\tau(1+2\gamma)} & ; |1 - \xi| \leq \daleth_4 \\ \frac{|\alpha|^2 |a|^3 \, |1-\xi|}{|\tau(1+2\tau+6\gamma\tau)\alpha a^2-4\tau^2(1+\gamma)^2(a^2+b)|} & ; |1 - \xi| \geq \daleth_4, \end{cases}$$

where

$$\daleth_4 = \left| \frac{(1 + 2\tau + 6\gamma\tau)\alpha a^2 - 4\tau(1 + \gamma)^2(a^2 + b)}{3\alpha a^2(1 + 2\gamma)} \right|.$$

If $\xi = 1$ is taken in Corollary 3.7, the following outcome arises:

Corollary 3.8. If a function $\phi \in \sigma$ is a member of $\mathfrak{E}_\sigma^\tau(\alpha, \mathfrak{F})$, then $|d_3 - d_2^2| \leq \frac{|\alpha a|}{3\tau(1+2\gamma)}$.

Remark 3.9. If $\gamma = 0$ is taken in Corollary 3.7, the result corresponds to [33, Corollary 3.3].

Applying $\gamma = 1$ in Corollary 3.7 yields the following result:

Corollary 3.10. Let $\xi \in \mathbb{R}$, $\tau \geq 1$, and $F(\varsigma)$ be as in (1.8). If a function $\phi \in \mathfrak{E}_\sigma^\tau(\alpha, \mathfrak{F})$, then

$$|d_2| \leq |\alpha a|\sqrt{\frac{|a|}{|\tau(1+8\tau)\alpha a^2 - 16\tau^2(a^2+b)|}}, \quad |d_3| \leq \frac{|\alpha a|^2}{16\tau^2} + \frac{|\alpha a|}{9\tau},$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\alpha a|}{9\tau} & ;\ |1-\xi| \leq \left|\dfrac{(1+8\tau)\alpha a^2 - 16\tau(a^2+b)}{9\alpha a^2}\right| \\ \frac{|\alpha|^2|a|^3\,|1-\xi|}{|\tau(1+8\tau)\alpha a^2 - 16\tau^2(a^2+b)|} & ;\ |1-\xi| \geq \left|\dfrac{(1+8\tau)\alpha a^2 - 16\tau(a^2+b)}{9\alpha a^2}\right| \end{cases}.$$

Setting $\xi = 1$ in Corollary 3.10 yields the following outcome.

Corollary 3.11. If a function $\phi \in \sigma$ is a member of $\mathfrak{E}_\sigma^\tau(\alpha, \mathfrak{F})$, then $|d_3 - d_2^2| \leq \frac{|\alpha a|}{9\tau}$.

Remark 3.12. With $\tau = 1$ in the classes $\mathfrak{P}_\sigma^\tau(\alpha, \beta, \mathfrak{F}), \mathfrak{Q}_\sigma^\tau(\alpha, \beta, \mathfrak{F}), \mathfrak{Z}_\sigma^\tau(\alpha, \gamma, \mathfrak{F})$, and $\mathfrak{E}_\sigma^\tau(\alpha, \mathfrak{F})$, we get bounds for $|d_2|, |d_3|,$ and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for function belonging to $\mathfrak{P}_\sigma^1(\alpha, \beta, \mathfrak{F})$, $\mathfrak{Q}_\sigma^1(\alpha, \beta, \mathfrak{F})$, $\mathfrak{Z}_\sigma^1(\alpha, \gamma, \mathfrak{F})$, and $\mathfrak{E}_\sigma^1(\alpha, \mathfrak{F})$ from Corolary 3.2 to Corolary 3.5, respectively.

4 Conclusions

An extensive bi-univalent functions subfamily related to GBFP is introduced in this presentation and is represented by the notation $\mathfrak{T}_\sigma^\tau(\alpha, \gamma, \beta, \mathfrak{F})$. In Section 2, we have estimated Maclaurin coefficients $|d_2|$ *and* $|d_3|$ for functions that belong to the family $\mathfrak{T}_\sigma^\tau(\alpha, \gamma, \beta, \mathfrak{F})$. We have also found the FSF for members in this subfamily. As discussed in Section 3, specialized parameters applied to our findings result in intested outcomes. Finally, we note for readers who are interested that the defined subfamily can be studied for Hankel determinant problems of higher order. The estimates of the coefficients $|d_2|$ *and* $|d_3|$ and the FSF $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for functions that are members of the defined subfamily of bi-univalent functions linked to any of the special polynomials mentioned in the introduction can be found.

Acknowledgments: The authors express their gratitude to the reviewer of this article for their insightful feedback, which helped them refine and enhance the paper's presentation. The authors made necessary changes in response to comments of the reviewer.

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