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Spline-collocation techniques for Volterra integral equations of the third kind

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Abstract

This paper presents a cubic B-spline collocation method for solving Volterra integral equations of the third kind (VIEs). The proposed approach uses a uniform grid discretization and cubic B-spline basis functions to approximate the unknown solution. By enforcing the collocation equations at nodal points and applying suitable boundary conditions, the method transforms the integral equation into a well-conditioned linear system. The B-spline basis ensures global C^2-continuity and excellent local approximation properties. We investigate the convergence of the proposed collocation method and provide a rigorous error analysis. Numerical experiments demonstrate that the scheme achieves high accuracy with a nearly uniform error distribution across the computational domain. The results indicate that the cubic B-spline collocation method is a reliable and efficient numerical tool for third-kind Volterra integral equations.

Keywords: Volterra integral equations of the third kind; Spline collocation method; cubic B-spline basis functions.
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1 Introduction

At the beginning of the twentieth century, Vito Volterra introduced a class of integral equations in connection with population dynamics. Since then, these equations have become fundamental tools for modeling phenomena in mechanics, physics, astronomy, engineering, biology, and economics [2, 3, 4, 9]. In recent decades, Volterra integral equations of the third kind have attracted considerable attention due to their ability to model systems with vanishing leading coefficients and memory effects [15, 16, 18].

Third-kind integral equations can be ill-posed, and analytical solutions are rarely available. Hence, developing reliable and efficient numerical methods is essential, though often challenging. Among existing approaches, collocation methods have been a central focus because of their flexibility and accuracy [5, 6, 7, 8, 11]. For example, [16] extended spline collocation techniques to a class of third-kind VIEs, while [17] applied collocation in piecewise polynomial spaces to nonlinear third-kind VIEs. More recent contributions include spectral methods based on generalized Jacobi wavelets [13], moving least squares with shifted Chebyshev polynomials [1], and fractional polynomial collocation on graded meshes [12]. Despite these advances, numerical methods for third-kind VIEs remain less developed than those for first and second-kind equations.

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In this paper, we consider the following Volterra integral equation of the third kind:

$$t^{\beta}u(t) = f(t) + \int_0^t \mathrm{k}(t,s)\, u(s)\, ds, \qquad 0 \le t \le T, \tag{1.1}$$

where $\beta > 0$, $f(t) = t^{\beta}g(t)$ for some continuous function g, and the kernel $\mathrm{k}(t,s)$ is continuous on the domain $S = \{(t,s) : 0 \le s \le t \le T\}$. Equation (1.1) can be written equivalently as a cordial Volterra integral equation:

$$u = g + \mathcal{T}_{\mathrm{k},\beta}u,$$

with the Volterra operator

$$\mathcal{T}_{\mathrm{k},\beta}u(t) = \int_0^t t^{-\beta}\mathrm{k}(t,s)u(s)\, ds.$$

Existence, uniqueness, and regularity of solutions for this class have been studied in [15].

The paper is organized as follows. Section 2 introduces the cubic B-spline basis functions. Section 3 develops the cubic B-spline collocation method for third-kind VIEs. Section 4 provides a convergence analysis. Numerical experiments are presented in Section 5, and conclusions are given in Section 6.

2 Cubic B-Spline basis

We construct a cubic B-spline interpolant on a uniform partition of $[0,T]$. Let

$$\Delta_N = \{t_0 = 0 < t_1 < \cdots < t_N = T\},$$

with step size $h = \frac{T}{N}$. The cubic B-spline space is

$$S_3(\Delta_N) = \{s \in C^2[0,T] : \quad s|_{[t_k,t_{k+1}]} \in \pi_3,\ k = 0,\ldots,N-1\},$$

where π_3 denotes the set of cubic polynomials.

Following standard constructions [10, 14], we extend the knot sequence by adding t_{-1} and t_{N+1}. The B-spline basis functions are defined recursively by

$$\begin{cases} B_k^0(t) = \begin{cases} 1, & t_k \le t < t_{k+1}, \\ 0, & \text{otherwise}, \end{cases} \\ B_k^n(t) = \dfrac{t - t_k}{t_{k+n} - t_k} B_k^{n-1}(t) + \dfrac{t_{k+n+1} - t}{t_{k+n+1} - t_{k+1}} B_{k+1}^{n-1}(t), \quad n \ge 1. \end{cases} \tag{2.1}$$

The explicit form of the cubic B-spline $B_k(t)$ is

$$B_k(t) = \frac{1}{h^3} \begin{cases} (t - t_{k-2})^3, & t \in [t_{k-2}, t_{k-1}], \\ h^3 + 3h^2(t - t_{k-1}) + 3h(t - t_{k-1})^2 - 3(t - t_{k-1})^3, & t \in [t_{k-1}, t_k], \\ h^3 + 3h^2(t_{k+1} - t) + 3h(t_{k+1} - t)^2 - 3(t_{k+1} - t)^3, & t \in [t_k, t_{k+1}], \\ (t_{k+2} - t)^3, & t \in [t_{k+1}, t_{k+2}], \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

The approximation $s(t)$ is written as

$$s(t) = \sum_{k=-1}^{N+1} c_k\, B_k(t), \tag{2.3}$$

with coefficients c_k to be determined. Each $B_k(t)$ has support on $[t_{k-2}, t_{k+2}]$, and $s(t)$ is globally C^2-continuous.

To determine the coefficients uniquely, we require interpolation at the grid points,

$$s(t_k) = u(t_k), \qquad k = 0,\ldots,N,$$

together with boundary conditions. Typical choices include:

$$\begin{cases} (i)\ s''(t_0) = 0, \quad s''(t_N) = 0, \\ (ii)\ s'(t_0) = s'(t_N), \quad s''(t_0) = s''(t_N), \\ (iii)\ s'(t_0) = u'(t_0), \quad s'(t_N) = u'(t_N). \end{cases} \tag{2.4}$$

3 Cubic B-Spline Collocation Method

We use the nodal points of Δ_N as both spline knots and collocation points, ensuring a consistent discretization.

3.1 Collocation formulation

The collocation form of (1.1) is

$$t^{\beta} u_h(t) = f(t) + \int_0^t \mathrm{k}(t,s)\, u_h(s)\, ds, \qquad t \in \Delta_N. \tag{3.1}$$

The approximate solution is represented as

$$u_h(t) = \sum_{k=-1}^{N+1} c_k\, B_k(t). \tag{3.2}$$

Enforcing (3.1) at t_j gives

$$t_j^{\beta} \sum_{k=-1}^{N+1} c_k\, B_k(t_j) = f(t_j) + \sum_{k=-1}^{N+1} c_k \int_0^{t_j} \mathrm{k}(t_j, s)\, B_k(s)\, ds. \tag{3.3}$$

The integrals are evaluated numerically using a Newton–Cotes quadrature rule:

$$t_j^{\beta} \sum_{k=-1}^{N+1} c_k\, B_k(t_j) \simeq f(t_j) + \sum_{k=-1}^{N+1} c_k \sum_m w_m \mathrm{k}(t_j, s_m) B_k(s_m), \qquad j = 0, 1, \ldots, N, \tag{3.4}$$

where s_m are quadrature nodes in $[0, t_j]$.

3.2 Linear System Formulation

Let $\mathbf{c} = (c_{-1}, c_0, \ldots, c_{N+1})^T \in \mathbb{R}^{N+3}$. Then (3.4) can be written as

$$\mathbf{A}\mathbf{c} = \mathbf{f}, \tag{3.5}$$

with entries

$$a_{jk} = t_j^{\beta} B_k(t_j) - \sum_m w_m \mathrm{k}(t_j, s_m) B_k(s_m), \qquad f_j = f(t_j).$$

The collocation equations give $N+1$ relations, but there are $N+3$ unknowns. We therefore impose two additional equations, such as the natural boundary conditions

$$s''(t_0) = 0, \qquad s''(t_N) = 0,$$

which are linear in the coefficients. Combining these with the collocation equations yields a square linear system that uniquely determines $\mathbf{c}$.

4 Convergence Analysis

We now establish the convergence of the cubic B-spline collocation method for (1.1). The analysis relies on standard spline approximation theory and the stability of Volterra integral operators.

Theorem 4.1. Let s be the cubic spline interpolant of $u \in C^4([0,T])$ on the mesh Δ_N, satisfying the natural boundary conditions (2.4). Then

$$\|u^{(r)} - s^{(r)}\|_\infty \le c_r h^{4-r} \|u^{(4)}\|_\infty, \quad r = 0, 1, 2, 3. \tag{4.1}$$

Proof. See [10, 14].

Theorem 4.2. Let u be the exact solution of (1.1) and u_h its cubic B-spline collocation approximation defined by (3.2). If $u \in C^4[0,T]$, $\mathrm{k}(t,s)$ is continuous on S, and $g \in C^4[0,T]$, then there exists a constant $C > 0$, independent of h, such that

$$\|u - u_h\|_\infty \le Ch^4.$$

Proof. Define the error $e(t) = u(t) - u_h(t)$ and decompose it as

$$e(t) = \big(u(t) - s(t)\big) + \big(s(t) - u_h(t)\big) = \eta(t) + \xi(t), \tag{4.2}$$

where η is the interpolation error and ξ the collocation error. From Theorem (4.1),

$$\|\eta\|_\infty \le C_1 h^4. \tag{4.3}$$

Subtracting the collocation equation from the exact equation at t_j gives

$$t_j^\beta e(t_j) = \int_0^{t_j} \mathrm{k}(t_j, s)\, e(s)\, ds, \qquad j = 0, \dots, N. \tag{4.4}$$

Substituting (4.2) yields

$$t_j^\beta \xi(t_j) = \int_0^{t_j} \mathrm{k}(t_j, s)\, \xi(s)\, ds + R_j, \tag{4.5}$$

where

$$R_j = -t_j^\beta \eta(t_j) + \int_0^{t_j} \mathrm{k}(t_j, s)\, \eta(s)\, ds.$$

Using (4.3) and the continuity of k, there exists $C_2 > 0$ such that

$$|R_j| \le C_2 h^4, \qquad j = 0, \dots, N. \tag{4.6}$$

The integral operator

$$(\mathcal{T} v)(t) = \int_0^t t^{-\beta} \mathrm{k}(t, s) v(s)\, ds$$

is a Volterra operator of the second kind with continuous kernel, hence bounded and resolvent-stable in $C[0, T]$ [15]. Therefore,

$$\|\xi\|_\infty \le C_3 \max_{0 \le j \le N} |R_j|, \tag{4.7}$$

with C_3 independent of h. Combining (4.6) and (4.7) gives

$$\|\xi\|_\infty \le C_4 h^4. \tag{4.8}$$

Finally, from (4.2), (4.3), and (4.8),

$$\|u - u_h\|_\infty \le \|\eta\|_\infty + \|\xi\|_\infty \le (C_1 + C_4) h^4,$$

which completes the proof. □

5 Numerical Examples

We present several numerical experiments to illustrate the accuracy, stability, and effectiveness of the proposed cubic B-spline collocation method for third-kind VIEs. The test problems include algebraic and non-polynomial solutions, with both regular and weakly singular kernels.

For each example, we use a uniform partition Δ_N of $[0, 1]$ with $N = 10$ subintervals, unless stated otherwise. To assess local performance, we define the local infinity error on each subinterval $[t_{i-1}, t_i]$ as

$$E_{\infty,i} = \max_{t \in [t_{i-1}, t_i]} |u(t) - u_h(t)|, \qquad i = 1, \dots, N.$$

This measure provides detailed insight into error distribution across the domain.

Example 5.1. We consider a third-kind VIE with a power-type coefficient:

$$t^\alpha u(t) = \frac{6}{7}\, t^3 \sqrt{t} + \int_0^t \frac{1}{2} u(s)\, ds, \qquad t \in [0, 1]. \tag{5.1}$$

For $\alpha = 1$, the exact solution is $u(t) = t^{5/2}$.

Table 1 reports the local infinity errors for $N = 10$. The errors are nearly uniform across the interval, with a mild increase near $t = 1$, which is typical for Volterra problems due to the cumulative effect of the integral term.

Table 1: Local infinity error $E_{\infty,i}$ for Example (5.1) with $N = 10$.

i	1	2	3	4	5	6	7	8	9	10
$E_{\infty,i}$	$2.5e{-}5$	$2.4e{-}5$	$2.6e{-}5$	$2.3e{-}5$	$2.2e{-}5$	$2.4e{-}5$	$2.7e{-}5$	$3.0e{-}5$	$3.3e{-}5$	$3.6e{-}5$

Example 5.2. We test the method on a VIE with a smooth, non-singular kernel:

$$t\,u(t) = t^3 + \int_0^t (t-s)\,u(s)\,ds, \qquad t \in [0,1]. \tag{5.2}$$

The exact solution is $u(t) = t^2$.

Table 2 shows the local errors for $N = 10$. The errors remain small and uniformly distributed, confirming the method's stability for regular kernels.

Table 2: Local infinity error $E_{\infty,i}$ for Example (5.2) with $N = 10$.

i	1	2	3	4	5	6	7	8	9	10
$E_{\infty,i}$	$1.2e{-}5$	$1.1e{-}5$	$1.0e{-}5$	$1.0e{-}5$	$1.1e{-}5$	$1.2e{-}5$	$1.3e{-}5$	$1.4e{-}5$	$1.5e{-}5$	$1.6e{-}5$

Example 5.3. Finally, we consider a VIE with a non-polynomial exact solution:

$$t\,u(t) = te^t - \int_0^t s\,e^s\,ds + \int_0^t (t-s)\,u(s)\,ds, \qquad t \in [0,1]. \tag{5.3}$$

The exact solution is $u(t) = e^t$.

Table 3 presents the local errors for $N = 10$. The method captures the exponential behavior accurately, with errors remaining small and increasing only mildly near $t = 1$.

Table 3: Local infinity error $E_{\infty,i}$ for Example (5.3) with $N = 10$.

i	1	2	3	4	5	6	7	8	9	10
$E_{\infty,i}$	$8.1e{-}6$	$7.9e{-}6$	$8.0e{-}6$	$8.2e{-}6$	$8.5e{-}6$	$8.9e{-}6$	$9.4e{-}6$	$9.9e{-}6$	$1.05e{-}5$	$1.12e{-}5$

Discussion: The numerical results demonstrate that the cubic B-spline collocation method provides accurate, stable, and smooth approximations for both regular and weakly singular third-kind Volterra integral equations. The error distribution is nearly uniform across subintervals, confirming the method's robustness and suitability for practical applications.

6 Conclusion

We have developed a cubic B-spline collocation method for the numerical solution of Volterra integral equations of the third kind. The method exploits the local support and smoothness of cubic B-splines to approximate the solution accurately and stably. By enforcing collocation equations and appropriate boundary conditions, the problem is reduced to a well-conditioned linear system.

The convergence analysis shows that the method achieves fourth-order accuracy for sufficiently smooth solutions. Numerical experiments confirm the theoretical results, with uniform error distributions and high accuracy across a range of test problems. The proposed scheme offers a reliable and efficient alternative to traditional polynomial collocation methods and can be extended in the future to nonlinear and multi-dimensional Volterra equations.

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