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# On the Topology of the Spectrum of Second Fuzzy Submodules

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**Abstract**

Let $R$ be a commutative ring with identity and $M$ a unitary $R$-module. This paper studies the second fuzzy submodules of $M$, extending the idea of second submodules. We focus on the collection $Spec^s(F^*(M))$, which consists of all non-zero second fuzzy submodules of $M$, and apply a topology to this set. In the final sections, we examine the topological characteristics of the space $Spec^s(F^*(M))$.

Keywords: Second submodule, Second fuzzy submodule, $I-(\text{top}_s\text{-module})$
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## 1 Introduction

Throughout this paper, $R$ denotes a commutative ring with identity ($1 \neq 0$), and all modules considered are unitary $R$-modules. The annihilator of $M/N$ is denoted by $(N : M)$, i.e., $(N : M) = \{r \in R | rM \subseteq N\}$. A proper submodule $P$ of $M$ is called a prime submodule if $rm \in P$ implies $m \in P$ or $r \in (P : M)$. If $P$ is a prime submodule of an $R$-module $M$, then $(P : M)$ is a prime ideal of $R$.

The prime spectrum of a module, when viewed with the Zariski topology applied to the set of all prime submodules of a unitary module, plays a central role in commutative algebra, algebraic geometry, and lattice theory. In recent years, several works have investigated modules over commutative rings with identity where the spectrum of prime submodules is equipped with the Zariski topology (see, for example, [1], [2], [13], [14], [18], [19]).

The concept of coprimary representation, introduced as a dual theory to primary modules and therir decompositions, was first introduced by D. Kirby ([11]) and I. G. Macdonald ([15]). An $R$-module $M$ is called coprimary if for each $r \in R$, $r \notin \Re\big((0 : M)\big)$ implies $rM = M$, where $\Re\big((0 : M)\big)$ denotes the radical of the ideal $(0 : M)$.

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The notion of second submodules of a non-zero module over a commutative ring, introduced as a dual concept to prime submodules in [24]. An $R$-endomorphism $r^*$ on $M$ is called homothety if $r^*(x) = rx$ for all $x \in M$. A non-zero submodule $K$ of $M$ is called second if for every $r \in R$, the homothety $r^* : K \longrightarrow K$ is either surjective or zero. This condition implies that $Ann(K) = (0 : M) = p$, where $p$ is a prime ideal of $R$, and $K$ is said to be $p$-second. Moreover, $M$ is called a second module if it is a second submodule of itself. The set of all second submodules of $M$ is denoted by $Spec^s(M)$.
This concept was later generalized to modules over arbitrary associative rings by S. Annin ([7]), where a second module was termed a coprime module. Additionally, J. Abuhlail introduced and investigated a Zariski-like topology on the spectra of coprime and second submodules of a module ([4]).

L. Zadeh ([25]) first introduced fuzzy sets as function $\mu$ on a non-empty set $X$ that maps to the real unit interval $I = [0, 1]$. J.A. Goguen ([10]) generalized this idea by replacing $I$ with a complete lattice $L$, leading to the notion of $L$-fuzzy sets. I. Rosenfeld ([23]) introduced fuzzy groups, and the concept of fuzzy submodules over $R$ was first studied by C.V. Negoita and D.A. Ralescu ([21]), with further contributions from F.Z. Pan ([22]) regarding fuzzy finitely generated modules and fuzzy quotient modules.

In the past three decades, extensive research has focused on fuzzy algebra. Key topics include fuzzy prime and primary ideals (or submodules) and fuzzy primary decompositions of fuzzy ideals (or submodules), which are essential in the study of fuzzy algebraic varieties and the solution of nonlinear systems of fuzzy intersection equations (see [9], [12], [16], [17]).

R. Ameri et al ([5]), introduced and studied the notion of prime $L$-submodules of a module $M$ over a commutative ring with identity $R$, where $L$ is a complete lattice. The set of all prime $L$-submodules of $M$, called the prime $L$-spectrum of $M$ or $L$-spectrum of $M$, is denoted by $L$-$spec(M)$. They followed the approach of C.P. Lu ([14]) and topologized $L$-$spec(M)$ with the, investigating its topological properties ([6]).

Recently, the notion of fuzzy coprimary submodules was introduced and studied in [26].

The aim of this paper is to explore second fuzzy submodules of $M$ and introduce a topology on $Spec_s(F^*(M))$, the set of all non-zero second fuzzy submodules of $M$. We investigate the properties of this topological space, including generic points, connectness, and discreteness, among others.

## 2 Preliminaries

In this section, we present essential definitions and fundamental results that will be utilized throughout the paper.

By a fuzzy subset $\mu$ of a non-empty set $X$, we mean a function $\mu$ from $X$ to $I$. $F^X$ denotes the set of all fuzzy subsets of $X$.
In what follows, $I$ will denote the unit interval $[0, 1]$.
Let $A$ be a subset of $X$ and $y \in I$. We define $y_A \in F^X$ as follows:

$$y_A(x) = \begin{cases} y & x \in A \\ 0 & otherwise \end{cases}$$

In particular, if $A = \{a\}$, we represent $y_{\{a\}}$ by $y_a$, which is referred to as a fuzzy point of $X$.
For $\mu, \nu \in F^X$, we say $\mu$ is contained in $\nu$ (denoted as $\mu \subseteq \nu$) if, for all $x \in X$, we have $\mu(x) \leqslant \nu(x)$.
The union and the intersection of two fuzzy subsets $\mu$ and $\nu$ are defined as follows:

$$\forall x \in X, (\mu \cup \nu)(x) = \mu(x) \vee \nu(x), (\mu \cap \nu)(x) = \mu(x) \wedge \nu(x).$$

For $\mu \in F^X$ and $a \in I$, we define the $a-$cut (or $a-$level subset) of $\mu$ as

$$\mu_a = \{x \in X | \mu(x) \geq a\}$$

If $x \in \mu_a$, we write $x_a \in \mu$.
If $f : X \longrightarrow Y$ is a mapping, and $\mu \in F^X$ and $\nu \in F^Y$, then the fuzzy subsets $f(\mu) \in F^Y$ and $f^{-1}(\nu) \in F^X$ are defined as:

$$\forall y \in Y, f(\mu)(y) = \bigvee\{\mu(x) | x \in X, f(x) = y\}, \forall x \in X, f^{-1}(\nu)(x) = \nu(f(x)).$$

Let $M$ and $N$ be $R$-modules, and let $f : M \longrightarrow N$ be an $R$-module homomorphism. A fuzzy subset $\mu \in F^M$ is said to be $f$-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in M$.

We now recall some essential definitions and theorems from [20] that are fundamental to the development of our results in this paper.

**Definition 2.1.** Let $\mu \in F^R$. The fuzzy subset $\mu$ is called a fuzzy ideal of $R$ if, for every $x, y \in R$ the following conditions hold:

(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;

(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

The set of all fuzzy ideals of $R$ is denoted by $FI(R)$.

For $\mu, \nu \in FI(R)$, the operation $\mu \circ \nu$ is define as:

$$\forall x \in R, (\mu \circ \nu)(x) = \bigvee \{\mu(y) \wedge \nu(z) | y, z \in R, x = yz\} .$$

It was proved in [20] that $\mu \circ \nu \in FI(R)$.

**Definition 2.2.** A fuzzy ideal $\xi$ of $R$ is called prime if it is non-constant and satisfies the following condition.

For any fuzzy ideals $\mu$ and $\nu$ of $R$, if $\mu \circ \nu \subseteq \xi$, then either $\mu \subseteq \xi$ or $\nu \subseteq \xi$.

It was shown in [20] that $\xi$ is a prime fuzzy ideal of $R$ if and only if it is non-constant and for all $x, y \in R$ and $a, b \in I, x_a \circ y_a \subseteq \xi$ implies that either $x_a \subseteq \xi$ or $y_a \subseteq \xi$.
The set of all prime fuzzy ideals of $R$ is denoted by $Spec(F(R))$.
Additionally, it was demonstrated in [20] that a fuzzy ideal $\eta$ of $R$ is prime if and only if

$$\eta = 1_{\eta_*} \cup c_R,$$

where $\eta_*$ is a prime ideal of $R$ and $c \in I$.

For a fuzzy ideal $\zeta$ of $R$, the $\Re$-radical of $\zeta$, denoted by $\Re(\zeta)$ is defined by

$$\forall r \in R, \Re(\zeta)(r) = \bigvee_{n \in \mathbb{N}} \zeta(r^n)$$

which is itself a fuzzy ideal of $R$ containing $\zeta$.

**Definition 2.3.** A fuzzy ideal $\xi \in FI(R)$ is called primary if it is non-constant and satisfies the following condition.

For any fuzzy ideals $\mu$ and $\nu$ of $R$, if $\mu \circ \nu \subseteq \xi$, then either $\mu \subseteq \xi$ or $\nu \subseteq \Re(\xi)$.

If $\xi$ is a primary fuzzy ideal, then $\Re(\xi)$ is a prime fuzzy ideal.

**Definition 2.4.** Consider $M$ as an $R$-module. A fuzzy submodule of $M$ is a fuzzy subset $\mu \in F^M$ such that:
(1) $\mu(0) = 1$;
(2) $\mu(rx) \geq \mu(x), \ \forall r \in R, x \in M$;
(3) $\mu(x + y) \geq \mu(x) \wedge \mu(y), \ \forall x, y \in M$.

The set of all fuzzy submodules of $M$ is denoted by $F(M)$, and the set of all non-zero fuzzy submodules of $M$ is denoted by $F^*(M)$ ($1_0$ represents the zero fuzzy submodule of $M$).

**Definition 2.5.** Let $\zeta \in F^R$ and $\mu \in F^M$. The operation $\zeta.\mu$ is define as:

$$\forall x \in M, (\zeta.\mu\ )(x) = \bigvee \{\zeta(r) \wedge \mu(y) | r \in R, y \in M, ry = x\}.$$

Additionally, for any $\mu \in F(M)$, and for $y \in M$ and $r \in R$, the following holds:

$$(1_r.\mu)(y) = \bigvee\{\mu(x)| \quad x \in M, y = rx\}.$$

For simplicity, we will denote $1_r.\mu$ as $r\mu$ in subsequent discussions.

It is important to note that if $\mu \in F(M)$ and $\xi \in F(R)$, then $\xi.\mu$ is not necessarily a fuzzy submodule of $M$. For instance, if $\mu = 1_0$, then for any $a \in [0, 1)$ and $r \in R$, we have:

$$\langle r_a \rangle.1_0 = a_0 \notin F(M).$$

Now, consider $\mu$ as a fuzzy submodule of $M$, $N$ an an $R$-module, and $\nu$ as a fuzzy submodule of $N$. If $f : M \longrightarrow N$ is an epimorphism and $f(\mu) = \nu$, then $f$ is referred to as a homomorphism of $\mu$ onto $\nu$.

**Theorem 2.6.** Let $\mu \in F^M$. The following statements are true:

(1) $\mu$ qualifies as a fuzzy submodule of $M$ if and only if each non-empty level subset of $\mu$ is a submodule of $M$.
(2) If $\mu$ is a fuzzy submodule of $M$, then
$$\mu_* = \{x \in M | \mu(x) = 1\}$$
is a submodule of $M$.

**Definition 2.7.** For $\mu, \nu \in F^M$ and $\zeta \in F^R$, the operation $\mu : \zeta$ and $\mu : \nu$ are defined as :

$$\mu : \zeta = \cup\{\nu \in F^M | \zeta.\nu \subseteq \mu\},$$

$$\mu : \nu = \cup\{\xi \in F^R | \xi.\nu \subseteq \mu\}.$$

It has been established in [20], it was proved that if $\mu, \nu \in F(M)$ and $\zeta \in FI(R)$, then $\mu : \zeta \in F(M)$ and $\mu : \nu \in FI(R)$.

**Definition 2.8.** For $\mu, \nu \in F^M$, their sum $\mu + \nu$ is defined as:

$$\forall x \in M, (\mu + \nu)(x) = \bigvee\{\mu(y) \wedge \nu(z) | y, z \in M, y + z = x\}.$$

It has been proven in [20] that if both $\mu$ and $\nu$ are fuzzy submodules of $M$, then their sum $\mu + \nu$ is also a fuzzy submodule of $M$.

**Theorem 2.9.** Let $N$ be an $R$-module and $f$ be a homomorphism of $M$ onto $N$. The following properties hold:

(1) if $\mu$ is a fuzzy submodule of $M$, then its image $f(\mu)$ forms a fuzzy submodule of $N$;
(2) if $\nu$ is a fuzzy submodule of $N$, then its preimage $f^{-1}(\nu)$ constitutes a fuzzy submodule of $M$.

Let $E_F$ represent a vector space $E$ over a field $F$. A fuzzy subset $\mu$ of $E_F$ is referred to as a fuzzy subspace, if for any scalars $\alpha, \beta \in F$ and vectors $x, y \in E$, the following inequality holds:

$$\mu(\alpha x + \beta y) \geq \mu(x) \wedge \mu(y).$$

**Proposition 2.10.** ([3]) For a fuzzy subspace $\mu$ of $E_F$, the following conditions are satisfied:
(1) $\mu(0) = \bigvee\{\mu(x) | x \in E\}$.
(2) For any non-zero scalar $\alpha \in F$ and any $x \in E$, $\mu(\alpha x) = \mu(x)$.
(3) If $x, y \in E$ and $\mu(x) \neq \mu(y)$, then:
$$\mu(x + y) = \mu(x) \wedge \mu(y).$$

**Definition 2.11.** A fuzzy submodule $\mu$ of $M$ is called divisible if, for every $x_a \in \mu$ where $a > 0$ and for every $r \in R$, there exists $y_a \in \mu$ such that
$$x_a = ry_a.$$

An $R$-module $M$ is said to be torsion-free if the zero element is the only element annihilated by a non-zero divisor of $R$.

**Proposition 2.12.** For all $x, y \in M$ and $r \in R$, the equality $x = ry$ implies $\mu(x) = \mu(y)$ for any divisible fuzzy submodule $\mu$ of $M$ if and only if $M$ is torsion-free.

**Proof .** Assume that for every divisible fuzzy submodule $\mu$ and all $x, y \in M$ and $r \in R$ with $x = ry$, we have $\mu(x) = \mu(y)$.

Let $T$ represent the set of torsion submodule of $M$, and suppose $x \in T$. Then there exists $0 \neq r$ in $R$ such that $rx = 0$. Since $1_0$ is a divisible fuzzy submodule, it follows that:

$$1_0(x) = 1_0(0) = 1.$$

Thus, $x = 0$, which implies $T = \{0\}$.

Conversely, suppose $\mu$ is a divisible fuzzy submodule of $M$, and $x = ry$. Define $\mu(x) = a$. Then $x_a \in \mu$ and $x_a = ry_a$.

Since $\mu$ is divisible, there exists $y'_a \in \mu$ such that

$$x_a = ry'_a.$$

As $M$ is torsion-free, we conclude $y = y'$. Therefore:

$$a = \mu(x) = \mu(ry) \geq \mu(y) \geq a.$$

This proves $\mu(x) = \mu(y)$. $\square$

**Theorem 2.13.** ([20]) Let $M$ represent the set of rational numbers $\mathbb{Q}$, and let $\mu$ be any fuzzy submodule of $M$. If $\mu$ is constant across $M$, then $\mu$ is divisible.

**Theorem 2.14.** ([24]) Let $N$ be a non-zero submodule of $M$ such that $Ann(N) = p$. The following statements are equivalent:

(1) $N$ is a $p$-second submodule of $M$;

(2) $N$ is a divisible $\dfrac{R}{p}$ -module;

(3) for every $r \in R \setminus p$, $rN = N$;

(4) for every ideal $I$ not contained in $p$, $IN = N$ ;

(5) $W(N) = p$.

**Note:** $W(N)$ represents the dual concept of the set of divisors of $N$ and is defined as:

$$W(N) = \{r \in R | r^* : N \longrightarrow N \ \ is \ \ not \ \ surjective\}.$$

## 3 Second fuzzy submodules

In this section, we formalize the concept of second fuzzy submodules and examine their key properties.

**Definition 3.1.** Let $\mu \in F(M)$. The fuzzy subset $\mu$ is referred to as second if, for each $r \in R$, one of the following conditions holds:

(1) $r\mu = \mu$;

(2) $r\mu = 1_0$.

The spectrum of second fuzzy submodules of $M$, denoted by $Spec^s(F(M))$, represents the collection of all second fuzzy submodules of $M$.

**Example 3.2.** (1) The fuzzy subset $1_0$ always satisfies the conditions of a second fuzzy submodule of $M$.

(2) Any fuzzy subspace $\mu$ of a vector space $E$ over a field $F$ qualifies as a second fuzzy submodule.

(3) If $M$ is a torsion-free $R$-module and $\mu$ is a divisible fuzzy submodule of $M$, then $\mu$ is second.

(4) Let $\mu \in F(M)$ and assume $\mu = 1_{\mu_*} \cup c_M$, where $c \in I$. If $\mu_*$ is a divisible submodule of $M$, then $\mu$ will also be second.

**Remark 3.3.** If $\mu$ possesses the sup-property, then for every $r \in R$, the equality $(r\mu)_* = r\mu_*$ holds.

**Lemma 3.4.** Let $\mu \in F^*(M)$ and assume that the fuzzy image $|\mu(M)|$ contains exactly two elements, with $Ann(\mu_*) = p$. If $\mu_*$ is a $p$-second submodule of $M$, then for every $r \in R \setminus p$, the equality $r\mu = \mu$ holds.

**Proof .** From the assumption, we know that $\mu$ can be expressed as:

$$\mu = 1_{\mu_*} \cup c_M,$$

where $c \in I$. Applying the operation $r\mu$, we have:

$$r\mu = 1_{r\mu_*} \cup c_M.$$

Since $\mu_*$ is a $p$-second submodule, it follows that $r\mu = \mu$ for all $r \in R \setminus p$. $\square$

**Proposition 3.5.** Let $\mu \in F^*(M)$ possess the sup-property. If $\mu$ is a second fuzzy submodules, then for every $a \in I$, the level subset $\mu_a$ is a second submodule of $M$.

**Proof .** Since $\mu$ is second, for all $r \in R$, one of the following holds:

(1) $r\mu = \mu$;
(2) $r\mu = 1_0$.

Case 1: Assume $r\mu = \mu$. Take any $b \in I$ and $y \in \mu_b$. Then

$$(r\mu)(y) = \mu(y) \geqslant b.$$

This implies that there exists some $x \in M$ such that $y = rx$ and $\mu(x) \geqslant b$. Consequently, $x \in \mu_b$ and $y \in r\mu_b$. Therefore:

$$r\mu_b = \mu_b.$$

Case 2: Assume $r\mu = 1_0$ and $b \in (0, 1]$. We must show that $r\mu_b = 0$. Suppose there exists $z \neq 0$ such that $z \in r\mu_b$. Then there exists some $x \in \mu_b$ such that $z = rx$. Thus:

$$0 = (r\mu)(z) = \bigvee\{\mu(t)|t \in M, z = rt\} \geqslant \mu(x) \geqslant b$$

which leads to a contradiction.
Now, consider $x \neq 0$ and $0 \neq x \in \mu_0$. Then:

$$0 = (r\mu)(x) = \bigvee\{\mu(t)|t \in M, x = rt\}.$$

For all $t \in M$ such that $x = rt$, we have $\mu(t) = 0$. Consequently:

$$\mu_0 = r\mu_0.$$

This implies that every level subset of $M$ inherits the second property. $\square$

By referring to the previous proposition, if $\mu \in F^*(M)$ satisfies the sup-property and is second, then $\mu_*$ is also a second submodule of $M$.

We present an example demonstrating that if $\mu$ lacks the sup-property, Proposition 3.5 may not hold in general.

**Example 3.6.** Consider $\mu$ as a divisible fuzzy submodule of a torsion-free $R$-module $M$. Assume that the image $\mu(M)$ is not finite. Define:

$$b = \bigvee\{\mu(x)|x \in M\backslash\mu_*\} < 1$$

where for every $x \in M\backslash\mu_*$, we have $\mu(x) < b$. In this case $\mu$ satisfies the conditions of being a second fuzzy submodule. However, none of the level subsets of $\mu$ are second.

The converse of Proposition 3.5 does not generally hold, even if $\mu$ satisfies the sup-property. The following example illustrates this point:

**Example 3.7.** Let $\mu \in F(M)$ be defined as:

$$\mu = 1_{\mu_*} \cup c_M$$

, where $c \in I\backslash\{0\}$ and $Ann(\mu_*) = p$.

If $\mu_*$ is a $p$-second submodule of $M$, then $\mu$ does not necessarily satisfy the conditions of being a second fuzzy submodule.

**Proposition 3.8.** Let $\mu \in F^*(M)$. If each $a$-level subset $\mu_a$ (for all $a \in I$) forms a divisible submodule of $M$ for each $a \in I$, then $\mu$ is a second fuzzy submodule.

**Proof .**Take any $r \in R$ and $y \in M$ such that $\mu(y) = b$. Then $y \in \mu_b$. Since $\mu_b$ is divisible, there exists an $x \in \mu_b$, such that:

$$y = rx.$$

It follows directly that:

$$\mu(y) = \mu(x) = b.$$

Consequently:

$$(r\mu)(y) = \bigvee\{\mu(t)|t \in M, y = rt\} \geqslant \mu(x) = \mu(y)$$

This implies that $\mu$ satisfies the condition of being a second fuzzy submodule. □

**Proposition 3.9.** Let $N$ be a submodule of $M$. Then $1_N$ is a second fuzzy submodule of $M$ if and only if $N$ is a second submodule of $M$.

**Proof .** ($\Rightarrow$) Assume that $1_N$ is a second fuzzy submodule of $M$. For every $r \in R$, either $1_r.1_N = 1_{rN} = 1_N$ or $1_r.1_N = 1_{rN} = 1_0$. This implies that either $rN = N$ or $rN = 0$. Therefore, $N$ satisfies the conditions of being a second submodule.

($\Leftarrow$) Conversely, suppose $N$ is a second submodule of $M$. For every $r \in R$,

$$1_r.1_N = 1_{rN},$$

By definition, either $1_{rN} = 1_N$ or $1_{rN} = 1_0$. □

If $M$ is a second $R$-module, it does not necessarily imply that all fuzzy submodules of $M$ are second fuzzy submodules.

For example, let $M = \mathbb{Q}$ (the set of rational numbers), and let $\mu \in F(M)$ be defined as:

$$\mu = 1_{\mathbb{Z}} \cup c_M$$

where $\mathbb{Z}$ denotes the integers and $c \in I\backslash\{0\}$. In this case, for every $r \in R$, $r\mu \neq \mu$ and $r\mu \neq 1_0$. However, if $\mu \in F(M)$ and every homothety mapping $r^* : M \longrightarrow M$ is zero, then

$$r\mu = 1_0.$$

For a fuzzy submodule $\mu$ in $F(M)$, the annihilator of $\mu$, denoted by $Ann(\mu)$, is defined as:

$$Ann(\mu) = (1_0 : \mu).$$

This means that:

$$\eta.\mu \subseteq 1_0.$$

where $\eta = Ann(\mu)$

**Lemma 3.10.** Let $N$ be a submodule of an $R$-module $M$. If $Ann(N) = p$, then $Ann(1_N) = 1_p$.

**Proof .** clearly,

$$1_p.1_N = 1_{pN} = 1_0.$$

This implies $1_p \subseteq Ann(1_N)$.

Conversely, assume $\eta \subseteq Ann(1_N)$. Then:

$$\eta.1_N \subseteq 1_0 = 1_{pN}.$$

For every $r \in p$,

$$\eta(r) \leq 1 = 1_p(r).$$

If $r' \notin p$, then

$$1_p(r') = 0.$$

There exists some $x \in N$ such that $r'x \neq 0$. Also:

$$0 = (\eta.1_N)(r'x) = \bigvee\{\eta(s) \wedge 1_N(z) | s \in R, z \in M, r'x = sz\}.$$

Thus:

$$0 = \eta(r') \wedge 1_N(x) = \eta(r').$$

This proves $\eta \subseteq 1_p$, and therefore:

$$Ann(1_N) = 1_p.$$

□

**Theorem 3.11.** Let $\mu \in F^*(M)$ with $Ann(\mu) = \eta$. If $\eta(r) \neq 0$ for every $r \in R$, then $\mu$ is a second fuzzy submodule.

**Proof .** Take $y \in M\backslash\{0\}$. Then:

$$0 = (\eta.\mu)(y) = \bigvee\{\eta(r) \wedge \mu(x) | r \in R, x \in M, y = rx\}.$$

This implies that for all possible pairs $(r, x)$ such that $y = rx$, we have:

$$\mu(x) = 0.$$

Consequently:

$$r\mu = 1_0.$$

This shows that $\mu$ satisfies the condition of being second. □

In [26], it was shown that for any $\mu \in F^*(M)$ and for all $r \in R$ and $a \in (0, 1]$, the condition $a_r.\mu \subseteq 1_0$ holds if and only if $1_r.\mu \subseteq 1_0$.

**Theorem 3.12.** Let $\mu \in F^*(M)$ be a second fuzzy submodule. Then $Ann(\mu)$ is a prime fuzzy ideal of $R$.

**Proof .** Take any $a, b \in (0, 1]$ and $r, s \in R$. Suppose:

$$a_r \circ b_s \subseteq Ann(\mu) \ \ and \ \ b_s \nsubseteq Ann(\mu).$$

Since $1_s \nsubseteq Ann(\mu)$ and $\mu$ is second, we have:

$$1_s.\mu = s\mu = \mu.$$

As a result:

$$1_r.\mu = 1_r.(1_s.\mu) = (1_r \circ 1_s).\mu = 1_{rs}.\mu = rs\mu = 1_0$$

From $a_r \circ b_s = (a \wedge b)_{rs} \subseteq Ann(\mu)$, it follows that:

$$1_r \subseteq Ann(\mu).$$

Thus:

$$a_r \subseteq Ann(\mu).$$

This proves that $Ann(\mu)$ is a prime fuzzy ideal. □

**Lemma 3.13.** Let $\mu \in F^*(M)$ with $\eta = Ann(\mu)$. The following statements are equivalent:
(1) $\mu$ is a second fuzzy submodule.;
(2) for every $r \in R \backslash \eta_*, r\mu = \mu$.

**Proof .** The proof follows directly from the definitions and previous results. □

Recall that a fuzzy submodule $\mu$ of $M$ is said to be coprimary if, for every $r \in R$, either $r\mu = \mu$ or there exists $n \in \mathbb{N}$ such that $r^n\mu = 1_0$ (see [26] for further details).

**Theorem 3.14.** ([26]) Let $\mu \in F^*(M)$ be a coprimary fuzzy submodule. Then

(1) $(1_0 : \mu)$ is a primary fuzzy ideal of $R$.
(2) The radical $\Re\big((1_0 : \mu)\big)$ is a prime fuzzy ideal of $R$.

It is easy to see that, every second fuzzy submodule of $M$ inherently satisfies the conditions of being coprimary.

**Proposition 3.15.** If $\mu$ is a coprimary fuzzy submodule of $M$, then $\mu$ is second if and only if $Ann(\mu)$ is a prime fuzzy ideal of $R$.

**Proof .** If $\mu$ is second, then by definition, $Ann(\mu)$ must be prime.
Conversely, suppose $Ann(\mu) = \eta$ is a prime fuzzy ideal of $R$. Then we can express:

$$\eta = 1_{\eta_*} \cup c_R,$$

where $\eta_*$ is a prime ideal of $R$ and $c \in I \backslash \{1\}$.
Assume $r \in R$ and $r\mu \neq \mu$. Since $\mu$ is coprimary, there exists $n \in \mathbb{N}$ such that:

$$r^n\mu = 1_0.$$

Thus:

$$1_{r^n} \subseteq \eta,$$

which implies:

$$r^n \in \eta_*.$$

Because $\eta_*$ is prime, we conclude:

$$r \in \eta_*$$

and hence:

$$1_r \subseteq \eta.$$

As a result:

$$r\mu = 1_0.$$

This confirms that $\mu$ is a second fuzzy submodule. □

**Proposition 3.16.** Let $\mu \in F(M)$ and $\nu$ be a coprimary fuzzy submodule of $M$ such that $\eta = \Re\big(Ann(\nu)\big)$ and $\eta = Ann(\mu)$, with $\nu \subseteq \mu$. Then $\nu$ is second.

**Proof .** Since $\nu \subseteq \mu$, we have:

$$\eta = Ann(\mu) \subseteq Ann(\nu) \subseteq \Re\big((Ann(\nu)\big) = \eta.$$

Thus:

$$\eta = Ann(\nu).$$

As $\eta$ is a prime fuzzy ideal of $R$, Proposition 3.15 implies that $\nu$ is second. □

Note that a fuzzy submodule $\mu$ of $M$ is called *minimal coprimary* if it is coprimary and does not strictly contain any other coprimary fuzzy submodule of $M$.

**Theorem 3.17.** If $\mu$ is a minimal coprimary fuzzy submodule of $M$, then $\mu$ is second.

**Proof .** Take any $r \in R$ such that $r\mu \neq \mu$. Then there exists $n \in \mathbb{N}$ such that:

$$r^n \mu = 1_0.$$

Since $r\mu$ is a coprimary fuzzy submodule of $M$ and $\mu$ minimal, it follows that:

$$r\mu = 1_0.$$

Hence, $\mu$ satisfies the conditions of being second. □

Here, we analyze how second fuzzy submodules behave under homomorphisms.

Let $f$ be a homomorphism of $M$ onto $M$, and let $\mu \in F(M)$ be second fuzzy submodule. If $f$ is a homomorphism of $\mu$ onto $\mu$, then $f(\mu)$ is second.

If $f : M \longrightarrow N$ is an $R$-module epimorphism, $\mu \in F(M)$ is $f$-invariant and $x \in M$, we have:

$$f(\mu)(f(x)) = \bigvee\{\mu(z)|z \in M, f(x) = f(z)\} = \mu(x).$$

For every $r \in R$, $x \in M$:

$$\begin{aligned}(rf(\mu))(f(x)) &= \bigvee\{f(\mu)(f(z))|z \in M, f(x) = rf(z)\}\\ &\geq \bigvee\{\mu(z)|z \in M, x = rz\}\\ &= (r\mu)(x)\end{aligned}$$

If $f$ is an isomorphism and $\mu$ is second and $f$-invariant, then $f(\mu)$ is a second fuzzy submodule of $N$.

If $\nu \in F(N)$ and $f$ is an epimorphism, then:

$$\begin{aligned}(rf^{-1}(\nu))(y) &= \bigvee\{f^{-1}(\nu)(x)|x \in M, y = rx\}\\ &= \bigvee\{\nu(f(x))|x \in M, y = rx\}\\ &\leq \bigvee\{\nu(f(x))|x \in M, f(y) = rf(x)\}\\ &= (r\nu)(f(y))\end{aligned}$$

Thus, if $f$ is an isomorphism and $\nu$ is second, then $f^{-1}(\nu)$ is also a second fuzzy submodule of $M$.

In [20], it was shown that if $\mu, \nu \in F(M)$ with $\mu \subseteq \nu$, then $\dfrac{\nu}{\mu}$ is a fuzzy submodule of $\dfrac{\nu^*}{\mu^*}$. Also in [26], it was demonstrated that if $\mu, \nu \in F(M)$, $\xi \in FI(R)$, and $\mu \subseteq \nu$ with $\xi.\nu \subseteq \mu$, then

$$\xi.\frac{\nu}{\mu} \subseteq 1_0.$$

If, for every $r \in R$, one has $r\nu \subseteq \mu$, then $\dfrac{\nu}{\mu}$ is a second fuzzy submodule of $\dfrac{\nu^*}{\mu^*}$.

## 4 I-(top$_s$-module)

In this section, we define a topology on the spectrum of non-zero second fuzzy submodules of a given $R$-module $M$ and derive some fundamental topological properties of this space.

For $\mu \in F(M)$, define:

$$V^*(\mu) = \{\nu \in Spec_s(F^*(M))|\nu \subseteq \mu\},$$

and

$$\zeta^*(F(M)) = \{V^*(\mu)|\mu \in F(M)\}.$$

**Theorem 4.1.** Let $\{\mu_i\}_{i\in I}$ be a family of fuzzy submodules of $M$. Then the following properties hold:
(1) $V^*(1_0) = \emptyset$ and $V^*(1_M) = Spec_s(F^*(M))$;
(2) $\bigcap\limits_{i\in I} V^*(\mu_i) = V^*(\bigcap\limits_{i\in I} \mu_i)$;
(3) for all $\mu, \nu \in F(M)$,
$$V^*(\mu) \cup V^*(\nu) \subseteq V^*(\mu + \nu).$$

**Proof .** The proof follows directly from the definitions. □

**Remark.** In general, $\zeta^*(F(M))$ is not necessarily closed under finite union.

**Counterexample.** Let $\mu, \nu, \lambda \in F(M)$ and

$$\mu = 1_{\mu_*} \cup c_M,\ \nu = 1_{\nu_*} \cup c_M,\ \lambda = 1_{\lambda_*} \cup c_M$$

where $c \in I$. Assume:

- $\mu_*$ is a divisible submodule of $M$
- $\mu_* \subseteq \nu_* + \lambda_*$,
- $\mu_* \nsubseteq \nu_*$ and $\mu_* \nsubseteq \lambda_*$

Under these assumptions,
$$\mu \in V^*(\nu + \lambda)$$
and
$$\mu \notin V^*(\nu) \cup V^*(\lambda).$$
This demonstrates that finite unions may not always preserve closure in $\zeta^*(F(M))$.

An $R$-module $M$ is called *uniserial* if, for any pair of submodules $N$ and $K$ of $M$, either $N \subseteq K$ or $K \subseteq N$. In other words, the set of all submodules of $M$ forms a totally ordered chain.

A non-zero submodule $N$ of $M$ is teremed *strongly hollow* if, for any submodules $L$ and $K$ of $M$, the condition $N \subseteq L + K$ implies that $N \subseteq L$ or $N \subseteq K$. (For more details, refer to [4])

The following definition extends the notion of strongly hollow to fuzzy submodules.

**Definition 4.2.** Let $\mu \in F^*(M)$. The fuzzy submodule $\mu$ is said to be strongly hollow if, for any fuzzy submodules $\mu_1$ and $\mu_2$ of $M$ such that $\mu_{1*} \neq 0$ and $\mu_{2*} \neq 0$, the condition $\mu \subseteq \mu_1 + \mu_2$ implies that $\mu \subseteq \mu_1$, or $\mu \subseteq \mu_2$.

We denote the set of all strongly hollow fuzzy submodules of $M$ by $SH(F^*(M))$.

**Example 4.3.** Consider a uniserial $R$-module and let $N$ be a simple submodule of $M$. Then:
$$1_N \in SH(F^*(M)).$$

This example demonstrates that simple submodules of a uniserial module naturally lead to strongly hollow fuzzy submodules.

By Theorem 4.1, a topology can be defined on $Spec_s(F^*(M))$ where $\zeta^*(F(M))$ serves as the collection of closed sets, provided that $\zeta^*(F(M))$ is closed under finite unions.

An $R$-module $M$ is referred to as an I-(top$_s$-module) if $\zeta^*(F(M))$ remains closed under finite unions.

For any $\mu \in F(M)$, define:
$$\chi_s(\mu) = \{\nu \in Spec_s(F^*(M)) | \nu \nsubseteq \mu\}.$$

If $\{\mu_i\}_{i\in I}$ represents a family of fuzzy submodules of $M$, the following properties hold:

(1) $\chi_s(1_0) = Spec_s(F^*(M))$ and $\chi_s(1_M) = \emptyset$;

(2) $\bigcup_{i\in I} \chi_s(\mu_i) = \chi_s(\bigcap_{i\in I} \mu_i)$;

(3) for any $\mu, \nu \in F(M)$

$$\chi_s(\mu + \nu) \subseteq \chi_s(\mu) \cap \chi_s(\nu).$$

Define:

$$\tau_s(F(M)) = \{\chi_s(\mu) | \mu \in F(M)\}.$$

If $M$ qualifies as an I-(top$_s$-module), then $\tau_s(F(M)))$ is the set of open sets in the topology on $Spec_s(F^*(M))$.

**Theorem 4.4.** If every fuzzy submodule in $Spec_s(F^*(M))$ is strongly hollow (i.e.,$Spec_s(F^*(M)) \subseteq SH(F^*(M))$), then $M$ is an I-(top$_s$-module).

**Proof .** Immediate from the definitions. □

**Definition 4.5.** Let $\mu \in F(M)$ with $\mu_* \neq 0$. The fuzzy submodule $\mu$ is called simple if, for any $\nu \in F(M)$ such that $\nu \subseteq \mu$, either $\nu = \mu$, or $\nu = 1_0$.

The set of all simple fuzzy submodules of $M$ is denoted by $S(F(M))$.

Clearly:

$$S(F(M)) \subseteq Spec_s(F^*(M)).$$

An $R$-module $M$ is said to have the *min-property*, if for any simple submodule $N$ of $M$:

$$N \nsubseteq \sum \{K | K \in S(M) \setminus N\}.$$

Here, $S(M)$ denotes the set of all simple submodules of $M$. (For more details, refer to [4]).

Similarly, an $R$-module $M$ is said to possess the $I$-min property if, for any simple fuzzy submodule $\mu$ of $M$:

$$\mu \nsubseteq \sum \{\nu | \nu \in (F(M)) \setminus \mu\}.$$

It is straightforward to verify that if $M$ satisfies the min-property, it also satisfies the $I$-min property.

**Proof Sketch:**
If $\mu \in S(F(M))$, then $\mu_*$ belongs to $S(M)$. Since $M$ satisfies the min-property:

$$\mu_* \nsubseteq \sum \{\nu_* | \nu \in S(F(M)) \setminus \mu\} = (\sum \{\nu | \nu \in S(F(M)) \setminus \mu\})_*.$$

Taking fuzzy extensions, we conclude:

$$\mu \nsubseteq \sum \{\nu | \nu \in S(F(M)) \setminus \mu\}.$$

**Lemma 4.6.** Let $M$ be an I-(top$_s$-module). Then the closure of any subset $A \subseteq Spec_s(F^*(M))$ can be expressed as:

$$\overline{A} = V_s(\sum \{\mu | \mu \in A\}).$$

**Proof .** Take any $A \subseteq Spec_s(F^*(M))$ and let $\mu \in A$. Clearly:

$$\mu \subseteq \sum \{\mu | \mu \in A\}.$$

Thus:

$$A \subseteq V_s(\sum \{\mu | \mu \in A\}).$$

This implies:

$$\overline{A} \subseteq V_s(\sum \{\mu | \mu \in A\}).$$

Conversely, suppose $\lambda \in V_s(\sum\{\mu|\mu \in A\})$ and $\chi_s(\eta)$ represents a neighborhood of $\lambda$. Then $\lambda \nsubseteq \eta$ and there exists some $\nu \in A$, such that:

$$\nu \nsubseteq \eta.$$

This means:

$$\nu \in \chi_s(\eta) \cap (A \setminus \{\lambda\}).$$

As a result:

$$\chi_s(\eta) \cap (A \setminus \{\lambda\}) \neq \emptyset.$$

This implies that $\lambda$ is a cluster point of $A$. Therefore:

$$V^s(\sum\{\mu|\mu \in A\}) \subseteq \overline{A}.$$

Hence:

$$\overline{A} = V_s(\sum\{\mu|\mu \in A\}).$$

□

**Remark 4.7.** Let $\mu \in Spec_s(F^*(M))$. Then

$$\overline{\{\mu\}} = V_s(\mu).$$

In particular, for any $\nu \in Spec_s(F^*(M))$ , the following conditions hold:
(1) $\nu \in \overline{\{\mu\}}$ if and only if $\nu \subseteq \mu$;
(2) If $\chi_s(\mu) = \emptyset$, then:

$$(\sum\{\nu|\nu \in S(F(M))\}) \subseteq \mu.$$

Moreover, the converse is also true if:

$$S(F(M)) = Spec_s(F^*(M)).$$

A non-empty topological space $X$ is called irreducible if it cannot be expressed as the union of two proper closed subsets.(For additional details [8])

**Theorem 4.8.** ([8]) Let $X$ be a non-empty topological space and $A \subseteq X$. The following statements are equivalent:

(1) $A$ is irreducible;
(2) for any pair of closed subsets $A_1, A_2$ of $X$, the condition $A \subseteq A_1 \cup A_2$ implies that $A \subseteq A_1$ or $A \subseteq A_2$;
(3) for any pair of open subsets $U_1, U_2$ of $X$, if $U_1 \cap A \neq \emptyset \neq U_2 \cap A$, then $(U_1 \cap U_2) \cap A \neq \emptyset$.

**Theorem 4.9.** Let $M$ be an I-(top$_s$-module), and let $A \subseteq Spec^s(F^*(M))$. If $A$ is irreducible, then $\sum\{\mu|\mu \in A\}$ is a second fuzzy submodule.

**Proof .** For any $r \in R$, Assume that:

$$r(\sum\{\mu|\mu \in A\}) \neq \sum\{\mu|\mu \in A\}$$

and:

$$r(\sum\{\mu|\mu \in A\}) \neq 1_0.$$

Define two subsets:

$$A_1 = \{\mu \in A|r\mu = \mu\} \ \ and \ \ A_2 = \{\mu \in A|r\mu = 1_0\}.$$

Clearly:

$$A \subseteq (\sum\{\mu|\mu \in A_1\}) \cup (\sum\{\mu|\mu \in A_2\}).$$

However:

$$A \nsubseteq \sum\{\mu|\mu \in A_1\},$$

and:

$$A \nsubseteq \sum\{\mu|\mu \in A_2\}.$$

This contradicts the irreducibility of $A$. Therefore $\sum\{\mu|\mu \in A\}$ must be a second fuzzy submodule. □

If $\{\mu_i\}_{i\in I}$ is a family of second fuzzy submodules of $M$, their sum $\sum_{i\in I}\mu_i$ is not necessarily a second fuzzy submodule. For example, let $N$ be a divisible submodule of $M$ and $K$ a $p$-second submodule of $M$. Then $1_N + 1_K$ is not a second fuzzy submodule of $M$. For every $r \in p$:

$$r(1_N + 1_K) = r(1_{N+K}) = 1_N.$$

This demonstrates that finite sums of second fuzzy submodules do not always result in a second fuzzy submodule.

**Theorem 4.10.** Suppose $Spec^s(F^*(M)) \subseteq SH(F^*(M))$, and let $A \subseteq Spec^s(F^*(M))$. Then the following statements are equivalent:
$(i)$ $A$ is irreducible;
$(ii)$ $\sum\{\mu|\mu \in A\}$ is a second fuzzy submodule;
$(iii)$ $1_0 \neq \sum\{\mu|\mu \in A\} \in SH(F^*(M))$.

**Proof .** $(i) \to (ii)$, Follows directly from 4.9.
$(ii) \to (iii)$, Clear from the definition of second fuzzy submodules and strongly hollow fuzzy submodules.
$(iii) \to (i)$, Assume $1_0 \neq \sum\{\mu| \ \ \mu \in A\}$ and $A \neq \emptyset$. Suppose:

$$A \subseteq V^*(\mu_1) \cup V^*(\mu_2).$$

Since:

$$V^*(\mu_1) \cup V^*(\mu_2) \subseteq V^*(\mu_1 + \mu_2),$$

and:

$$\sum\{\mu|\mu \in A\} \subseteq \mu_1 + \mu_2,$$

it follows that:

$$\sum\{\mu|\mu \in A\} \subseteq \mu_1, \text{ or } \sum\{\mu|\mu \in A\} \subseteq \mu_2.$$

Consequently:

$$A \subseteq V^*(\mu_1) \text{ or } A \subseteq V^*(\mu_2).$$

This proves that $A$ is irreducible. □ Define

1. $CL_s(F(M)) = \{A \subseteq Spec_s(F^*(M))|A = \overline{A}\}$
2. $CR_s(F(M)) = \{\mu \in F(M))|\mu = \sum\{\nu|\nu \in V^*(\mu)\}\}$

**Theorem 4.11.** If $M$ is an I-(top$_s$-module), there exists a one-to-one correspondence between the set $CR_s(F(M))$ and $CL_s(F(M))$.

**Proof .** Define two mappings:

$\varphi : CR_s(F(M)) \longrightarrow CL_s(F(M))$ given by

$$\varphi(\mu) = V^*(\mu)$$

and

$\psi : CL_s(F(M)) \longrightarrow CR_s(F(M))$ given by

$$\psi(V^*(\mu)) = \sum\{\nu|\nu \in V^*(\mu)\}$$

We verify that these mappings are inverses:

1. For every $\mu \in CR_s(F(M))$:
$$\psi(\varphi(\mu)) = \psi(V^*(\mu)) = \mu,$$
2. For every $V^*(\mu) \in CL_s(F(M))$:

$$\begin{aligned}\varphi(\psi(V^*(\mu))) &= V^*(\psi(V^*(\mu))) \\ &= V^*(\sum\{\nu|\nu \in V^*(\mu)\}) \\ &= \overline{V^*(\mu)} \\ &= V^*(\mu).\end{aligned}$$

Thus, the mappings establish a bijective correspondence between the sets $CR_s(F(M))$ and $CL_s(F(M))$. □

**Theorem 4.12.** If $M$ is an I-(top$_s$-module), the bijection from Theorem 4.11 restricts to a one-to-one correspondence between the set $Spec_s(F^*(M))$ and the set of all irreducible closed subsets of $Spec_s(F^*(M))$.

**Proof .** Let $\mu \in Spec_s(F^*(M))$. Then:

$$\mu \in V^*(\mu)$$

and by Theorem 4.10, $V^*(\mu)$ is an irreducible closed subset of $Spec_s(F^*(M))$.

Conversely, suppose $A$ is an irreducible closed subset of $Spec_s(F^*(M))$. By Theorem 4.10, $\sum\{\mu|\mu \in A\}$ a second fuzzy submodule. Therefore:

$$A = \overline{A} = V^*(\sum\{\mu|\mu \in A\}).$$

This establishes a bijective correspondence between second fuzzy submodules and irreducible closed subsets. □

In a topological space $X$, a point $a \in A$ (where $A$ is a closed subset of $X$) is called a generic point if:

$$A = \overline{\{a\}}.$$

A topological space is referred to as sober if every irreducible closed subset $A$ has a unique generic point.(For further reading, refer [4]).

**Lemma 4.13.** If $M$ be is I-(top$_s$-module), then $Spec_s(F^*(M))$ is a sober space.

**Proof .** Let $A$ be an irreducible closed subset of $Spec_s(F^*(M))$. Then, by the properties of irreducibility:

$$A = V_s(\mu)$$

for some $\mu \in Spec_s(F^*(M))$. Additionally:

$$A = \overline{A} = V^*(\sum\{\nu|\nu \in A\}) = V^*(\mu) = \overline{\{\mu\}}.$$

Thus, $\mu$ serves as a generic point of $A$.

If there exists another generic point $\eta$ of $A$, then:

$$V^*(\eta) = V^*(\mu).$$

This implies:

$$\eta = \mu.$$

Therefore, Specs($F^*(M)$) is a sober space. □

**Theorem 4.14.** Assume $M$ is an I-(top$_s$-module) and $Spec_s(F^*(M)) = S(F(M))$. The following statements hold:

(1) If $M$ satisfies the $I$-min property, then $Spec_s(F^*(M))$ is discrete;

(2) $M$ has a unique simple fuzzy submodule if and only if it satisfies the $I$-(min property) and $Spec_s(F^*(M))$ is connected.

**Proof .** (1) Assume $M$ satisfies the $I$-min property. For every $\mu \in Spec_s(F^*(M))$:

$$\{\mu\} = \chi_s(\sum\{\nu|\nu \in S(F(M)) \setminus \{\mu\}\}).$$

Since every singleton set is open, the space $Spec_s(F^*(M))$ is discrete.

(2) Suppose $M$ has a unique simple fuzzy submodule. Clearly, $M$ satisfies the $I$-(min property) and $Spec_s(F^*(M))$ is connected because it consists of only one point.

Conversely, assume $M$ satisfies the $I$-(min property) and $Spec_s(F^*(M))$ is connected. From part $(i)$, $Spec_s(F^*(M))$ must also be discrete. A topological space that is both connected and discrete can only have a single point. Therefore, $M$ must have a unique simple fuzzy submodule. □

**Atomic and I-Atomic Modules:** An $R$-module $M$ is called atomic if every non-zero submodule of $M$ contains a simple submodule. For example, the Prufer group $\mathbb{Z}_{p^\infty}$ is an atomic $\mathbb{Z}$-module.

**Definition 4.15.** An $R$-module $M$ is called $I$-atomic if every non-zero fuzzy submodule $\mu$ of $M$ contains a simple fuzzy submodule.

**Example 4.16.** Let $M = \mathbb{Z}_{p^\infty}$ and consider a fuzzy submodule $\mu \in F^*(M)$. There exists a simple submodule $N$ of $M$, such that:

$$N \subseteq \mu_*.$$

Define $\nu = 1_N$. Clearly, $\nu$ is a simple fuzzy submodule of $M$ and:

$$\nu \subseteq \mu.$$

This demonstrates that $M$ is an $I$-atomic $\mathbb{Z}$-module.

**Theorem 4.17.** If $M$ is an I-atomic and I-(top$_s$-module), and if $S(F(M))$ is countable, then $Spec_s(F^*(M))$ is countably compact.

**Proof .** Assume $S(F(M)) = \{\mu_{\lambda_k}\}_{k\geq 1}$ is countable, and let $\{\chi_s(\nu_\alpha)\}_{\alpha\in I}$ represent an open cover of $Spec_s(F^*(M))$. Since

$$S(F(M)) \subseteq Spec_s(F^*(M)),$$

for each $k \geq 1$, there exists $\alpha_k \in I$ such that:

$$\mu_{\lambda_k} \nsubseteq \nu_{\alpha_k}.$$

Assume:

$$\bigcap_{k\geq 1} \nu_{\alpha_k} \neq 1_0.$$

Since $M$ is I-atomic, there exists a simple fuzzy submodule $\eta$ of $M$ such that:

$$\eta \subseteq \bigcap_{k\geq 1} \nu_{\alpha_k}.$$

This contradicts the assumption. Therefore:

$$\bigcap_{k\geq 1} \nu_{\alpha_k} = 1_0.$$

Hence:

$$Spec_s(F^*(M)) = \chi_s(1_0) = \chi_s(\bigcap_{k\geq 1} \nu_{\alpha_k}) = \bigcup_{k\geq 1} \chi_s(\nu_{\alpha_k}).$$

This confirms that $Spec_s(F^*(M))$ is countably compact. □

**Conclusion**

In this paper:

1. The concept of second fuzzy submodules was introduced, and their basic properties were explored.
2. A topology was constructed on the space of non-zero second fuzzy submodules of an $R$-module $M$.
3. Important topological characteristics of these spaces were studied.
4. The results provide a foundation for the spectral theory of fuzzy second submodules.

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