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Bilinear module actions over quasi-ideals

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Abstract

In this paper, we investigate certain properties and characterizations of bilinear module actions over quasi-ideals of a Banach algebra. Our results are applied to the quasi-ideals of the group algebra $M(G)$, where we demonstrate that $M(G)$ is isomorphic to a subspace of the space $\mathcal{B}_{mod}(L_1(G)^*)$ consisting of all bilinear and separately continuous right module actions over $L_1(G)^*$.

Keywords: Factorable Banach algebra, Module action, Strict topology, quasi-ideal
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1 Introduction

The notion of a *quasi-multiplier* generalizes that of a multiplier on a Banach algebra and was first introduced by Akemann and Pedersen [5] in the setting of C^*-algebras. McKennon [10] extended this concept to general complex Banach algebras possessing a bounded approximate identity (b.a.i.), by defining a quasi-multiplier as a bilinear map $m : A \times A \to A$ satisfying

$$m(ab, cd) = a \cdot m(b, c) \cdot d \qquad (a, b, c, d \in A).$$

Following McKennon's foundational work, the theory of quasi-multipliers has been further developed in various directions by numerous authors, including Kassem and Rowlands [7], Argün and Rowlands [6], Yilmaz and Rowlands [13], and others.

In our earlier works [1, 2, 3, 4], we investigated the extent to which the theory of quasi-multipliers can be extended beyond the class of Banach algebras. Specifically, we studied quasi-multipliers in the context of f-algebras, ℓ-algebras, the duals of Banach algebras, and more general topological algebraic frameworks, utilizing standard techniques to explore their behavior in these settings.

On a parallel line of research, Rieffel [11, 12] carried out an extensive study of the Banach module $\hom_A(A, X)$, which denotes the space of continuous homomorphisms from a Banach algebra A to a Banach left A-module X. More generally, if E and X are topological left A-modules and A is a topological algebra, the space $\hom_A(E, X)$ consists of all continuous A-module homomorphisms from E to X. If E is an A-bimodule, then the left A-module structure on $\hom_A(E, X)$ is defined via the action $(a * T)(x) = T(x \cdot a)$, for all $a \in A$, $T \in \hom_A(E, X)$, and $x \in E$. In particular, $\hom_A(A, A)$ is identified with the usual multiplier algebra $M(A)$.

These connections illustrate a fundamental interplay between module homomorphisms, multipliers, and quasi-multipliers. The structure of one often illuminates the properties of the others. Khan [8, 9] has contributed significantly

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to this line of inquiry by studying module homomorphisms and the general strict topology on topological left Banach modules.

In this work, we study bilinear module actions over the dual space A^* and extend the strict topology to the space $\mathcal{B}_{\mathrm{mod}}(A^*)$. We also investigate how these actions behave over quasi-ideals in Banach algebras.

2 Preliminaries

For any Banach space X, we denote its first dual space by X^*. For each $x \in X$ and $\zeta \in X^*$, the natural duality pairing between X and X^* is denoted by $\langle x, \zeta \rangle$ (or equivalently $\langle \zeta, x \rangle$).

We always identify X as naturally embedded into its second dual X^{**} via the canonical mapping π, defined by

$$\langle \pi(x), \zeta \rangle = \langle \zeta, x \rangle, \quad \text{for all } x \in X, \zeta \in X^*.$$

Let A be a Banach algebra. On the second dual space A^{**} of A, there exist two natural multiplications known as the first and second Arens products. In this paper, we consider A^{**} equipped with the first Arens product, whose definition we recall below.

For arbitrary elements $a \in A$, $\zeta \in A^*$, and $F, G \in A^{**}$, the module actions $\zeta \cdot a$ and $G \cdot \zeta$ are defined by

$$\langle \zeta \cdot a, b \rangle = \langle \zeta, ab \rangle, \quad \text{and} \quad \langle G \cdot \zeta, b \rangle = \langle G, \zeta \cdot b \rangle,$$

for all $b \in A$. The first Arens product of F and G is the element $F \circ G \in A^{**}$ defined by

$$\langle F \circ G, \zeta \rangle = \langle F, G \cdot \zeta \rangle,$$

for all $\zeta \in A^*$. The second Arens product, denoted by $\circ'$, is defined similarly. Equipped with these multiplications, A^{**} forms a Banach algebra containing A as a subalgebra. A Banach algebra A is called factorable if $AA = A$. We denote by A^*A the subspace of A^* given by

$$A^*A = \{\zeta \cdot a : \zeta \in A^*, a \in A\},$$

and similarly

$$AA^* = \{a \cdot \zeta : a \in A, \zeta \in A^*\}.$$

If $A^*A = A^*$, then A^* is said to factor on the left.

3 Module Actions over the Dual Space

Definition 3.1. A bilinear mapping $m : A^* \times A \to A^*$ is called a *right bilinear module action* over A^* if for all $\zeta \in A^*$ and $a, b \in A$, the following conditions hold:

$$m(a \cdot \zeta, b) = a \cdot m(\zeta, b) \quad \text{and} \quad m(\zeta, ba) = m(\zeta, b) \cdot a.$$

Similarly, a bilinear mapping $m' : A \times A^* \to A^*$ is called a *left bilinear module action* over A^* if for all $\zeta \in A^*$ and $a, b \in A$,

$$m'(ab, \zeta) = a \cdot m'(b, \zeta) \quad \text{and} \quad m'(b, \zeta \cdot a) = m'(b, \zeta) \cdot a.$$

Let $\mathcal{B}_{mod}(A^*)$ denote the collection of all bilinear, separately continuous right bilinear module actions over A^*. It is straightforward to verify that $\mathcal{B}_{mod}(A^*)$ forms a vector space. Moreover, equipped with the norm

$$\|m\| = \sup\{\|m(\zeta, a)\| : \zeta \in A^*,\ a \in A,\ \|\zeta\| \leq 1,\ \|a\| \leq 1\},$$

this space becomes a Banach space.

Now, let A be an arbitrary Banach algebra. A map $T : A^* \to A^*$ is called a *right module action* over A^* if

$$T(a \cdot \zeta) = a \cdot T(\zeta)$$

for all $\zeta \in A^*$ and $a \in A$. We denote by $\mathcal{M}_{mod}(A^*)$ the space of all bounded linear right module actions over A^*.

Definition 3.2. A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a Banach algebra A is called an *ultra*-approximate identity* if for every $m \in \mathcal{B}_{mod}(A^*)$ and $\zeta \in A^*$, the net $\{m(\zeta, e_\lambda) : \lambda \in I\}$ is Cauchy in A^*.

Theorem 3.3. Let A be factorable and possess an ultra*-approximate identity $\{e_\lambda\}$. Then the map

$$\rho : \mathcal{M}_{mod}(A^*) \to \mathcal{B}_{mod}(A^*), \quad \rho_T(\zeta, a) = (T\zeta) \cdot a,$$

for all $T \in \mathcal{M}_{mod}(A^*), \zeta \in A^*, a \in A$, is bijective and satisfies $\|\rho\| \leq 1$. Moreover, if $\{e_\lambda\}$ is of norm one, then ρ is an isometry.

Proof . Let $T \in \mathcal{M}_{mod}(A^*)$ be arbitrary. It is clear that ρ_T defines a bilinear map from $A^* \times A$ into A^*, bounded by $\|T\|$. For $\zeta \in A^*$ and $a, b \in A$, we have

$$\rho_T(a \cdot \zeta, b) = T(a \cdot \zeta) \cdot b = (a \cdot T\zeta) \cdot b = a \cdot (T\zeta \cdot b) = a \cdot \rho_T(\zeta, b),$$

and

$$\rho_T(\zeta, ba) = (T\zeta) \cdot (ba) = (T\zeta \cdot b) \cdot a = \rho_T(\zeta, b) \cdot a.$$

Thus, $\rho_T \in \mathcal{B}_{mod}(A^*)$. The linearity of ρ follows directly from the definition, and clearly $\|\rho_T\| \leq \|T\|$, hence $\|\rho\| \leq 1$. If $\rho_T = 0$, then for all $\zeta \in A^*$ and $a \in A$, we have

$$(T\zeta) \cdot a = 0,$$

which implies

$$\langle T\zeta \cdot a, b\rangle = \langle T\zeta, ab\rangle = 0 \quad \text{for all } b \in A.$$

Since A is factorable, it follows that $T = 0$. Hence, ρ is injective. Now let $m \in \mathcal{B}_{mod}(A^*)$ be arbitrary. Define

$$T\zeta := \lim_\lambda m(\zeta, e_\lambda),$$

which belongs to $\mathcal{M}_{mod}(A^*)$ by the ultra*-approximate identity property. Then for all $\zeta \in A^*$ and $a \in A$,

$$\rho_T(\zeta, a) = (T\zeta) \cdot a = \lim_\lambda m(\zeta, e_\lambda) \cdot a = m(\zeta, a),$$

showing ρ is surjective. Finally, suppose $\{e_\lambda\}$ is an approximate identity of norm one. For any $T \in \mathcal{M}_{mod}(A^*)$ and $\varepsilon > 0$, there exists $\zeta \in A^*$ with $\|\zeta\| \leq 1$ such that

$$\|T\| - \varepsilon < \|T\zeta\|.$$

Since for every $a \in A$,

$$\lim_\lambda \langle T\zeta \cdot e_\lambda, a\rangle = \lim_\lambda \langle T\zeta, e_\lambda \cdot a\rangle = \langle T\zeta, a\rangle,$$

we get

$$\|\rho_T\| \geq \lim_\lambda \|\rho_T(\zeta, e_\lambda)\| = \lim_\lambda \|T\zeta \cdot e_\lambda\| > \|T\| - \varepsilon,$$

and since ε is arbitrary, ρ is an isometry. □

Let A be factorable with an ultra*-approximate identity. By virtue of Theorem 3.3, we may define a multiplication on $\mathcal{B}_{mod}(A^*)$ that endows it with the structure of a Banach algebra. We outline the construction as follows.

For $m_1, m_2 \in \mathcal{B}_{mod}(A^*)$, there exist $T_1, T_2 \in \mathcal{M}_{mod}(A^*)$ such that $m_1 = \rho_{T_1}$ and $m_2 = \rho_{T_2}$ by the above theorem. Then the product

$$m_1 \circ_\rho m_2 := \rho_{T_2 T_1}$$

defines a well-defined multiplication on $\mathcal{B}_{mod}(A^*)$.

4 Strict Topology and Modules

The space $\mathcal{B}_{mod}(A^*)$ can be equipped with the structure of an A-bimodule as follows. For each $m \in \mathcal{B}_{mod}(A^*)$ and $a \in A$, define the products $a * m$ and $m * a$ by

$$(a * m)(\zeta, b) = m(\zeta \cdot a, b), \quad \text{and} \quad (m * a)(\zeta, b) = m(\zeta, ab),$$

for all $\zeta \in A^*$ and $b \in A$.

It is straightforward to verify that $a * m$ and $m * a$ belong to $\mathcal{B}_{mod}(A^*)$, thereby making $\mathcal{B}_{mod}(A^*)$ an A-bimodule.

Definition 4.1. The *strict topology* β on $\mathcal{B}_{mod}(A^*)$ is defined by the family of seminorms

$$m \mapsto \|m * a\| \qquad (a \in A,\ m \in \mathcal{B}_{mod}(A^*)).$$

Let A and B be two factorable Banach algebras each possessing an ultra*-approximate identity, and let $\varphi : B \to A$ be a homomorphism such that the adjoint map $\varphi^* : A^* \to B^*$ is surjective. We define

$$\widetilde{\varphi} : \mathcal{B}_{mod}(A^*) \to \mathcal{B}_{mod}(B^*)$$

by the rule

$$[\widetilde{\varphi}(m)](\varphi^*(\zeta), b) := \varphi^*(m(\zeta, \varphi(b))) \quad (\zeta \in A^*,\ b \in B).$$

Theorem 4.2. Let A, B, φ, φ^*, and $\widetilde{\varphi}$ be as above, and let β_A and β_B denote the strict topology on $\mathcal{B}_{mod}(A^*)$ and $\mathcal{B}_{mod}(B^*)$, respectively. If $\varphi : B \to A$ is continuous, then the map

$$\widetilde{\varphi} : (\mathcal{B}_{mod}(A^*), \beta_A) \to (\mathcal{B}_{mod}(B^*), \beta_B)$$

is a continuous algebra homomorphism.

Proof . Let $m, m' \in \mathcal{B}_{mod}(A^*)$. By Theorem 3.3, there exist $T, T' \in \mathcal{M}_{mod}(A^*)$ such that $m = \rho_T$ and $m' = \rho_{T'}$. Then

$$m \circ_\rho m' = \rho_{T'T},$$

and we compute:

$$\begin{aligned}
[\widetilde{\varphi}(m \circ_\rho m')](\varphi^*(\zeta), b) &= \varphi^*((m \circ_\rho m')(\zeta, \varphi(b))) \\
&= \varphi^*(\rho_{T'T}(\zeta, \varphi(b))) \\
&= \varphi^*(T'T(\zeta) \cdot \varphi(b)).
\end{aligned}$$

On the other hand, $\widetilde{\varphi}(m) = \rho_S$ and $\widetilde{\varphi}(m') = \rho_{S'}$ for some $S, S' \in \mathcal{M}_{mod}(B^*)$, since $\widetilde{\varphi}(m), \widetilde{\varphi}(m') \in \mathcal{B}_{mod}(B^*)$. Then,

$$\varphi^*(T(\zeta) \cdot \varphi(b)) = S(\varphi^*(\zeta)) \cdot b$$

and similarly,

$$\varphi^*(T'T(\zeta) \cdot \varphi(b)) = S'S(\varphi^*(\zeta)) \cdot b = [\rho_{S'S}](\varphi^*(\zeta), b).$$

Therefore,

$$[\widetilde{\varphi}(m) \circ_\rho \widetilde{\varphi}(m')](\varphi^*(\zeta), b) = [\widetilde{\varphi}(m \circ_\rho m')](\varphi^*(\zeta), b),$$

showing that $\widetilde{\varphi}$ is an algebra homomorphism. To prove continuity, let $\{m_\alpha\} \subseteq \mathcal{B}_{mod}(A^*)$ be a net such that $m_\alpha \xrightarrow{\beta_A} m \in \mathcal{B}_{mod}(A^*)$. Let $\zeta \in B^*$ be arbitrary. Since φ^* is surjective, there exists $\eta \in A^*$ such that $\varphi^*(\eta) = \zeta$. For any $b, c \in B$, we have

$$\begin{aligned}
\|(\widetilde{\varphi}(m_\alpha - m) * b)(\zeta, c)\| &= \|\widetilde{\varphi}(m_\alpha - m)(\zeta, bc)\| \\
&= \|\widetilde{\varphi}(m_\alpha - m)(\varphi^*(\eta), bc)\| \\
&= \|\varphi^*((m_\alpha - m)(\eta, \varphi(bc)))\| \to 0,
\end{aligned}$$

by continuity of φ^* and the assumption $m_\alpha \xrightarrow{\beta_A} m$. Thus, $\widetilde{\varphi}(m_\alpha) \to \widetilde{\varphi}(m)$ in β_B, completing the proof. $\square$

Definition 4.3. A subalgebra A of an algebra B is called a *quasi-ideal* of B if

$$ABA \subseteq A.$$

Definition 4.4. Let A be a quasi-ideal of the algebra B. Then there is a linear map

$$\Psi_B : B \to \mathcal{B}_{mod}(A^*)$$

defined for each $b \in B$ by

$$\Psi_B(b)(\zeta, a) := \zeta \cdot (ba), \qquad \text{for all } \zeta \in A^*,\ a \in A.$$

Note that for arbitrary $a' \in A$, we can define the dual action by

$$\langle \zeta \cdot (ba), a' \rangle = \langle \zeta, a'ba \rangle.$$

Definition 4.5. The topology on B generated by the inverse images $\{\Psi_B^{-1}(\vartheta) : \vartheta \in \beta_A\}$ is denoted by $u_{(B,A)}$. A net $\{b_\alpha\} \subset B$ is said to converge to $b \in B$ in the $u_{(B,A)}$-topology, written as

$$b_\alpha \xrightarrow{u_{(B,A)}} b,$$

if and only if

$$\lim_\alpha \|\zeta \cdot (b_\alpha a) - \zeta \cdot (ba)\| = 0 \quad \text{for all } a \in A,\ \zeta \in A^*.$$

Theorem 4.6. Let A be factorable and possess an ultra*-approximate identity $\{e_\lambda\}$. If A is a quasi-ideal of an algebra B, then the mapping

$$\Psi_B : B \to \mathcal{B}_{mod}(A^*)$$

is a $(u_{(B,A)}, \beta_A)$-continuous algebra homomorphism.

Proof . Let $\{b_\alpha\}$ be a net in B such that $b_\alpha \xrightarrow{u_{(B,A)}} b \in B$. By the definition of the $u_{(B,A)}$-topology, for all $a, c \in A$ and $\zeta \in A^*$, we have

$$\|(\Psi_B(b_\alpha) * a)(\zeta, c) - (\Psi_B(b) * a)(\zeta, c)\| = \|\zeta \cdot (b_\alpha ac) - \zeta \cdot (bac)\| \to 0.$$

Hence, $\Psi_B(b_\alpha) \xrightarrow{\beta_A} \Psi_B(b)$, showing that Ψ_B is $(u_{(B,A)}, \beta_A)$-continuous. To show that Ψ_B is a homomorphism, let $b_1, b_2 \in B$. By Theorem 3.3, there exist $T, S \in \mathcal{M}_{mod}(A^*)$ such that $\Psi_B(b_1) = \rho_T$ and $\Psi_B(b_2) = \rho_S$. This means that for all $a \in A$, $\zeta \in A^*$,

$$\zeta \cdot (b_1 a) = T(\zeta) \cdot a, \quad \zeta \cdot (b_2 a) = S(\zeta) \cdot a.$$

Since A is a quasi-ideal of B, and $\{e_\alpha\}$ is an approximate identity, we have (see [10], p. 111):

$$\lim_\alpha e_\alpha ba = ba, \qquad \text{for all } b \in B,\ a \in A.$$

Then,

$$\begin{aligned}
[\Psi_B(b_1) \circ_\rho \Psi_B(b_2)](\zeta, a) &= (\rho_T \circ_\rho \rho_S)(\zeta, a) = \rho_{ST}(\zeta, a) \\
&= S(T(\zeta)) \cdot a = T(\zeta) \cdot (b_2 a) \\
&= \lim_\alpha T(\zeta) \cdot (e_\alpha b_2 a) = \lim_\alpha \zeta \cdot (b_1 e_\alpha b_2 a).
\end{aligned}$$

On the other hand,

$$\Psi_B(b_1 b_2)(\zeta, a) = \zeta \cdot (b_1 b_2 a).$$

Thus,

$$\begin{aligned}
\|[\Psi_B(b_1) \circ_\rho \Psi_B(b_2)](\zeta, a) - \Psi_B(b_1 b_2)(\zeta, a)\| &= \lim_\alpha \|\zeta \cdot (b_1 e_\alpha b_2 a) - \zeta \cdot (b_1 b_2 a)\| \\
&\leq \lim_\alpha \|\zeta \cdot b_1\| \cdot \|e_\alpha b_2 a - b_2 a\| = 0.
\end{aligned}$$

Therefore, $\Psi_B(b_1 b_2) = \Psi_B(b_1) \circ_\rho \Psi_B(b_2)$, and so Ψ_B is multiplicative. $\square$

Corollary 4.7. If the mapping Ψ_B is injective, then the algebra B may be identified with a subalgebra of $\mathcal{B}_{mod}(A^*)$.

Example 4.8. Let G be a compact group and let $A = L_1(G)$. Then the mapping

$$\Psi_\mu(\zeta, f) := (\zeta * \mu) * f \qquad (\mu \in M(G),\ \zeta \in L_\infty(G),\ f \in L_1(G))$$

defines a linear isomorphism from $M(G)$ into a subspace of $\mathcal{B}_{mod}(A^*)$.

Proof . Recall that $M(G)$ denotes the convolution algebra of all bounded regular Borel measures on G, and for $f \in L_1(G)$ and $\mu \in M(G)$, the convolution is defined by

$$f * \mu(x) = \int_G f(xy^{-1})\, d\mu(y).$$

Given $\mu \in M(G)$, $\zeta \in L_\infty(G) \cong L_1(G)^*$, and $f \in L_1(G)$, the expression $\Psi_\mu(\zeta, f) = (\zeta * \mu) * f$ defines a bilinear mapping. Since convolution is associative, we have:

$$(\zeta * \mu) * f = \zeta * (\mu * f),$$

which shows that $\Psi_\mu \in \mathcal{B}_{mod}(L_1(G)^*)$. Thus, $\Psi : M(G) \to \mathcal{B}_{mod}(L_1(G)^*)$ is a well-defined linear map.

We claim that Ψ is injective. Suppose $\Psi_\mu = 0$. Then for all $\zeta \in L_\infty(G)$, $f \in L_1(G)$, we have:

$$(\zeta * \mu) * f = 0.$$

Since $L_1(G)$ has a bounded approximate identity $\{e_\alpha\}$, we obtain:

$$\zeta * \mu = \lim_\alpha (\zeta * \mu) * e_\alpha = 0.$$

Therefore, $\zeta * \mu = 0$ for all $\zeta \in L_\infty(G)$, which implies that $\mu = 0$ as a measure, because $L_\infty(G)$ separates points in $M(G)$, and $M(G)$ is the dual space of $C_0(G)$, which also has a bounded approximate identity. Hence, Ψ is injective.
□

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