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On the distribution modulo 1 of the sequence $w(n)\{en!\}$

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Abstract

In this paper, we study the distribution modulo 1 of sequences of the form $a_n = w(n)\{en!\}$. We show that if the weight function w is a nonlinear polynomial with a leading irrational coefficient, the a_n is uniformly distributed modulo 1. Also, we show that if $w(n) = \alpha n + \beta$ (linear polynomial), then $w(n)\{en!\} \to \alpha$ as $n \to \infty$, and if $w(n) = O(n^{1-\epsilon})$ (sublinear growth) for some $\epsilon > 0$, then $w(n)\{en!\} \to 0$ as $n \to \infty$.

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1 Introduction

Let $\lfloor t \rfloor$ be the largest integer not exceeding the real number t, and $\{t\}$ denotes its fractional part, i.e., $\{t\} = t - \lfloor t \rfloor$. A real sequence $(a_n)_{n\geqslant 1}$ is dense modulo 1 if the set $\{\{a_n\} : n \geqslant 1\}$ is sense in the interval $(0, 1)$. Also, $(a_n)_{n\geqslant 1}$ is uniformly distributed modulo 1 (see [14, Chapter 21] and [15]) if for all real numbers a, b with $0 \leqslant a < b \leqslant 1$ we have

$$\lim_{N\to\infty} \frac{1}{N} \#\Big\{ n \leqslant N : \{a_n\} \in [a, b] \Big\} = b - a.$$

Distribution modulo 1 and limit points of sequences of the form αa_n has been studied widely in literature (see [1, 2, 7, 8, 9, 10, 19, 22, 23, 24] and the references given there). In this paper we study distribution modulo 1 of sequences of the form $w(n)\{en!\}$, where the weight function w admits some certain conditions, precisely as follows.

Theorem 1.1. Let $w(x) = \alpha x^d + \cdots \in \mathbb{R}[x]$ be a polynomial with degree d and leading irrational coefficient α. Then:

(1) If $d \geqslant 2$, the sequence $w(n)\{en!\}$ is uniformly distributed modulo 1.

(2) If $w(n) = \alpha n + \beta$ (linear polynomial), then $w(n)\{en!\} \to \alpha$ as $n \to \infty$.

(3) If $w(n) = O(n^{1-\epsilon})$ for some $\epsilon > 0$, then $w(n)\{en!\} \to 0$ as $n \to \infty$.

In cases (2) and (3), the sequence is not dense modulo 1 and consequently not uniformly distributed modulo 1.

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Remark 1.2. If the weight function $w(n)$ satisfies the weaker condition $w(n) = O(n)$ as $n \to \infty$, then the assertion (2) may fails. For example, the sequence $w(n) = n \sin n$ satisfies $w(n) = O(n)$. From Proposition 1.4, in the following, we have

$$\{\mathrm{e}n!\} = \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Thus,

$$w(n)\{\mathrm{e}n!\} = (\sin n)\left(1 + O\Big(\frac{1}{n}\Big)\right).$$

The sequence $\sin n$ oscillates densely in $[-1, 1]$ due to the irrationality of π, and therefore does not converge to a limit. However, it is known [15, Ex. 2.7] that the sequence $\sin n$ is not uniformly distributed modulo 1.

Corollary 1.3. For $\alpha, \beta \in \mathbb{R}$, the sequences $(\alpha \log n)\{\mathrm{e}n!\}$ and $(\alpha \sin n + \beta \cos n)\{\mathrm{e}n!\}$ are not uniformly distributed modulo 1.

To determine distribution modulo 1 of a sequence we use Weyl criterion [27], asserting that the real sequence $(a_n)_{n\geqslant 1}$ is uniformly distributed modulo 1, if and only if for all integers $h \neq 0$,

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n\leqslant X} e(ha_n) = 0,$$

where $e(x) = \mathrm{e}^{2\pi i x}$. Originally, Weyl introduced and used the above criterion to show that if $P(x) = \alpha x^d + \cdots \in \mathbb{R}[x]$ is a polynomial with degree d and leading coefficient $\alpha \notin \mathbb{Q}$, then the sequence with general term $a_n = P(n)$ is uniformly distributed modulo 1 (see [14, Proposition 21.1] for a proof). Considering this result and Weyl criterion, Theorem 1.1 is a direct consequence of the asymptotic expansion, in sense of Poincaré (see [6, Section 1.5]), for the sequence with general term $\{\mathrm{e}n!\}$.

In the following result and through the paper, by $f = \widetilde{O}(g)$ we mean $|f| \leqslant g$. Also, we let B_k be the k-th Bell number, counting the number of partitions of a set of size k, and $S(k, j)$ be the Stirling numbers of the second kind, counting the number of partitions of a set of size k into j disjoint non-empty subsets.

Proposition 1.4. Given any positive integer r, there exist computable integer constants $c_1, \ldots, c_r$ such that for any integer $n \geqslant 1$ we have

$$\{\mathrm{e}n!\} = \sum_{k=1}^{r} \frac{c_k}{n^k} + \widetilde{O}\left(\frac{\mathrm{e}^2 B_{r+1}}{n^{r+1}}\right). \tag{1.1}$$

More precisely,

$$c_k = (-1)^k \sum_{j=0}^{k} (-1)^j\, S(k, j) \qquad (k \geqslant 1). \tag{1.2}$$

Accordingly, some initial values of c_k is $1, 0, -1, 1, 2, -9, 9, 50, -267, 413, 2180, -17731, 50533$.

Remark 1.5. The constants c_k in (1.1) match the k-th term of the sequence A014182 on OEIS [21], and satisfies

$$\frac{(-1)^k}{\mathrm{e}}\, c_k = \sum_{j=0}^{\infty} (-1)^j\, \frac{j^k}{j!} \qquad (k \geqslant 1), \tag{1.3}$$

which is an alternating form of Dobinski's formula [26, p. 178] concerning Bell numbers, asserting that

$$\mathrm{e}\, B_k = \sum_{j=0}^{\infty} \frac{j^k}{j!}. \tag{1.4}$$

The series representation (1.2) is a special value of generating function for the finite sequence $(S(k, j))_{0\leqslant j\leqslant k}$, which is known as the Touchard polynomials $T_k(x)$. These polynomials studied by Jacques Touchard [25] and defined by

$$T_k(x) = \sum_{j=0}^{k} S(k, j) x^j.$$

We refer the interested reader to [3, 4, 5, 16, 17, 18, 28] for the above definition and some properties, including the identity $B_k = T_k(1)$. More precisely, it is known (see the second page of [18]) that if X is a random variable with a Poisson distribution with expected value x, then its k-th moment is $E(X^k) = T_k(x)$. This gives the following analogue of Dobinski's formula

$$\mathrm{e}^x\, T_k(x) = \sum_{j=0}^{\infty} x^j \frac{j^k}{j!}. \tag{1.5}$$

Note that for $x = -1$ the relation (1.5) coincides with (1.3). Hence, $c_k = (-1)^k T_k(-1)$, and this gives (1.2), as more as, guarantees that the constants c_k all are integers.

Remark 1.6. The sequence $\{en!\}$ coincides with sequence

$$\delta_n = \Big(\mathrm{e} - \sum_{j=0}^{n} \frac{1}{j!}\Big) n!,$$

which has been studied in [13]. Although, Proposition 1.4 is Theorem 1.1 of [13], because of our new view in this paper and also to keep completeness of the proof of our results in uniform distribution study of $w(n)\{en!\}$, we reproved it by following a more simplified argument. Finally, we mention that the sequence with general term $\lfloor en! \rfloor$ actually counts the number of all distinct paths between a specific pair of vertices in a complete graph on $n+2$ vertices (see [11, 12] and the references given there).

2 Proofs

Proof of Proposition 1.4

First we show that for any integer $n \geqslant 0$,

$$\int_1^{\infty} t^n \mathrm{e}^{-t}\, dt = \frac{1}{\mathrm{e}} \sum_{j=0}^{n} \frac{n!}{j!}. \tag{2.1}$$

For, we let $\mathcal{I}_n$ be the improper integral in the left hand side of (2.1). Integrating by parts gives

$$\int t^r \mathrm{e}^{-t}\, dt = -t^r \mathrm{e}^{-t} + r \int t^{r-1} \mathrm{e}^{-t}\, dt.$$

Thus, $\mathcal{I}_j = \mathrm{e}^{-1} + j\mathcal{I}_{j-1}$, and dividing the sides by $j!$ we get $\mathcal{I}_j/j! = \mathrm{e}^{-1}/j! + \mathcal{I}_{j-1}/(j-1)!$ or $\mathcal{I}_j/j! - \mathcal{I}_{j-1}/(j-1)! = \mathrm{e}^{-1}/j!$. Summing the sides of the last identity over $1 \leqslant j \leqslant n$, and considering $\mathcal{I}_0 = \mathrm{e}^{-1}$, we get (2.1).

Now we obtain an integral representation for $\{en!\}$. Following ideas of the proof of irrationality of the number e (see [20, p. 65]), for any integer $n \geqslant 1$ we have

$$0 < \mathrm{e}n! - \sum_{j=0}^{n} \frac{n!}{j!} = \sum_{j=n+1}^{\infty} \frac{n!}{j!} = \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{1}{n+i} < \sum_{j=1}^{\infty} \frac{1}{(n+1)^j} = \frac{1}{n}. \tag{2.2}$$

Therefore, $\lfloor en! \rfloor = \sum_{j=0}^{n} n!/j!$, and consequently,

$$\{\mathrm{e}n!\} = \mathrm{e}n! - \sum_{j=0}^{n} \frac{n!}{j!}.$$

By using $n! = \int_0^{\infty} t^n \mathrm{e}^{-t}\, dt$ and (2.1) we deduce that

$$\{\mathrm{e}n!\} = \mathrm{e} \int_0^1 t^n \mathrm{e}^{-t}\, dt \qquad (n \geqslant 0). \tag{2.3}$$

The rest of proof is to approximate the integral in (2.3). Uniform convergence of Taylor expansion of the exponential function e^{-t} on the interval $[0,1]$ allows us to write

$$\int_0^1 t^n \mathrm{e}^{-t}\,dt = \int_0^1 t^n \sum_{j=0}^{\infty} \frac{(-t)^j}{j!}\,dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_0^1 t^{n+j}\,dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j+1)}.$$

The following identity holds for any real c and any positive integer r, provided $n+c \neq 0$,

$$\frac{1}{n+c} = \sum_{k=1}^{r} (-1)^{k-1} \frac{c^{k-1}}{n^k} + \frac{(-1)^r}{n+c} \left(\frac{c}{n}\right)^r.$$

Since $n > 0$, taking $c = j+1$ with $j \geqslant 0$ fulfills the above condition. Thus, considering (2.3) we deduce that

$$\frac{\{\mathrm{e}n!\}}{\mathrm{e}} = \sum_{j=0}^{\infty} \sum_{k=1}^{r} \frac{(-1)^{k-1}}{n^k} \frac{(-1)^j (j+1)^{k-1}}{j!} + (-1)^r \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j+1)} \left(\frac{j+1}{n}\right)^r.$$

Rearranging the sums implies

$$\frac{\{\mathrm{e}n!\}}{\mathrm{e}} = \sum_{k=1}^{r} \frac{(-1)^{k-1}}{n^k} \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)^{k-1}}{j!} + \mathcal{E}_r(n),$$

where

$$\mathcal{E}_r(n) = \frac{(-1)^r}{n^r} \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)^r}{j!(n+j+1)}.$$

By using alternating Dobinski's formula (1.3) we have

$$\sum_{j=0}^{\infty} \frac{(-1)^j (j+1)^{k-1}}{j!} = \sum_{j=0}^{\infty} (-1)^j \frac{(j+1)^k}{(j+1)!} = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{j^k}{j!} = \frac{(-1)^{k-1}}{\mathrm{e}} c_k,$$

where the constants c_k are as in (1.2). Hence,

$$\frac{\{\mathrm{e}n!\}}{\mathrm{e}} = \frac{1}{\mathrm{e}} \sum_{k=1}^{r} \frac{c_k}{n^k} + \mathcal{E}_r(n).$$

For the remainder term $\mathcal{E}_r(n)$, we use Dobinski's formula (1.4) to approximate it as follows

$$\begin{aligned}\mathcal{E}_r(n) = \widetilde{O}\Big(\frac{1}{n^r} \sum_{j=0}^{\infty} \frac{(j+1)^r}{j!(n+j+1)}\Big) &= \widetilde{O}\Big(\frac{1}{n^{r+1}} \sum_{j=0}^{\infty} \frac{(j+1)^r}{j!}\Big) \\ &= \widetilde{O}\Big(\frac{1}{n^{r+1}} \sum_{j=0}^{\infty} \frac{(j+1)^{r+1}}{(j+1)!}\Big) = \widetilde{O}\Big(\frac{\mathrm{e}\, B_{r+1}}{n^{r+1}}\Big).\end{aligned}$$

This completes the proof of (1.1).

Proof of Theorem 1.1

For every real sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ we have

$$\begin{aligned}\Big|\sum_{n\leqslant X} e(a_n+b_n) - \sum_{n\leqslant X} e(a_n)\Big| &= \Big|\sum_{n\leqslant X} e(a_n)(e(b_n)-1)\Big| \\ &\leqslant \sum_{n\leqslant X} \big|e(a_n)(e(b_n)-1)\big| = \sum_{n\leqslant X} \big|e(b_n)-1\big|.\end{aligned}$$

Note that $\big|e(b_n)-1\big| = \big|\cos(2\pi b_n) - 1 + i\sin(2\pi b_n)\big| = 2\big|\sin(\pi b_n)\big|$. Also, the inequality $|\sin(x)| \leqslant |x|$ holds for every $x \in \mathbb{R}$. Hence,

$$\Big|\sum_{n\leqslant X} e(a_n+b_n) - \sum_{n\leqslant X} e(a_n)\Big| \leqslant \sum_{n\leqslant X} 2\big|\sin(\pi b_n)\big| \leqslant 2\pi \sum_{n\leqslant X} \big|b_n\big|.$$

Therefore, as X tends to infinity,

$$\sum_{n\leqslant X} e(a_n+b_n) = \sum_{n\leqslant X} e(a_n) + O\Big(\sum_{n\leqslant X} |b_n|\Big). \tag{2.4}$$

(1) Since $d \geqslant 2$, by using the expansion (1.1) we deduce that

$$w(n)\{\mathrm{e}n!\} = \alpha n^{d-1} + \cdots + O\Big(\frac{1}{n}\Big).$$

This expansion, splitter identity (2.4) and the result of Weyl concerning uniform distribution of sequences at polynomial arguments yield the result for the case $d \geqslant 2$.

(2) (Linear polynomial) If $w(n) = \alpha n + \beta$, then

$$w(n)\{\mathrm{e}n!\} = (\alpha n + \beta)\left(\frac{1}{n} + O\Big(\frac{1}{n^2}\Big)\right) = \alpha + O\left(\frac{1}{n}\right) \to \alpha.$$

(3) (Sublinear growth) If $w(n) = O(n^{1-\epsilon})$ for some $\epsilon > 0$, then

$$|w(n)\{\mathrm{e}n!\}| = O(n^{1-\epsilon}\cdot n^{-1}) = O(n^{-\epsilon}) \to 0.$$

In both cases (2) and (3), convergence to a limit implies the sequence cannot be dense modulo 1. This completes the proof.

Proof of Corollary 1.3

Both weight functions satisfy $w(n) = O(n^{1-\epsilon})$ for any $0 < \epsilon < 1$. More precisely, $\alpha \log n = O(n^{1-\epsilon})$ for any $\epsilon > 0$, and $\alpha \sin n + \beta \cos n = O(1) = O(n^{1-\epsilon})$. Thus, by Theorem 1.1 (3) both sequences converge to 0.

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