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Certain Subalgebras of Lipschitz Algebras of Infinitely Differentiable Functions and Their Maximal Ideal Spaces

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Dedicated to the Memory of Charalambos J. Papaioannou

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Abstract

We study an interesting class of Banach function algebras of infinitely differentiable functions on perfect, compact plane sets. These algebras were introduced by Honary and Mahyar in 1999, called *Lipschitz algebras of infinitely differentiable functions* and denoted by $Lip(X, M, \alpha)$, where X is a perfect, compact plane set, $M = \{M_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $M_0 = 1$ and $\frac{(m+n)!}{M_{m+n}} \leq (\frac{m!}{M_m})(\frac{n!}{M_n})$ for $m, n \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0, 1]$. Let $d = lim\ sup(\frac{n!}{M_n})^{\frac{1}{n}}$ and $X_d = \{z \in \mathbb{C} : dist(z, X) \leq d\}$. Let $Lip_{P,d}(X, M, \alpha)[Lip_{R,d}(X, M, \alpha)]$ be the subalgebra of all $f \in Lip(X, M, \alpha)$ that can be approximated by the restriction to X_d of polynomials [rational functions with poles off X_d]. We show that the maximal ideal space of $Lip_{P,d}(X, M, \alpha)$ is $\widehat{X_d}$, the polynomially convex hull of X_d, and the maximal ideal space of $Lip_{R,d}(X, M, \alpha)$ is X_d, for certain compact plane sets.. Using some formulae from combinatorial analysis, we find the maximal ideal space of certain subalgebras of Lipschitz algebras of infinitely differentiable functions.

Keywords: Infinitely differentiable functions, Function algebra, Lipschitz algebra, Maximal ideal space, Star-shaped set, Uniformly regular..
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1. Introduction and preliminaries

Let X be a compact Housdorff space. We denote by $C(X)$ the complex algebra of all continuous complex-valued functions on X. For $f \in C(X)$ and a closed subset E of X, we denote the uniform norm f on E by $||f||_E$; that is, $\|f\|_E = \sup\{|f(x)| : x \in X\}$. A *function algebra* on X is a subalgebra A of $C(X)$ that separates the points of X and contains the constant functions on X. If there is an algebra norm $||.||$ on A such that A is complete under the norm $||.||$ and $||1|| = 1$, then A is a *Banach function algebra* on X, and if the given norm is the uniform norm on X, then A is a *uniform algebra* on X.

Let A be a Banach function algebra on X. We denote by $\mathcal{M}(\mathcal{A})$ the maximal ideal space of A. We know that $\mathcal{M}(A)$ with the Gelfand topology is a compact Hausdorff space. For each $x \in X$, the map $e_x : A \to \mathbb{C}$, defined by $e_x(f) = f(x)$, is an element of $\mathcal{M}(A)$ and called the *evaluation character* on A at x. This fact implies that $\|f\|_X \leq \|\hat{f}\|_{\mathcal{M}(A)}$ for all $f \in A$, where $\hat{f}$ is the Gelfand transform of f. The map $J : X \to \mathcal{M}(A)$, defined by $J(x) = e_x$, is injective and continuous, and so X is homeomorphic to a compact subset of $\mathcal{M}(A)$. If the map J is surjective, then A is called a *natural Banach function algebra* on X and we write $\mathcal{M}(A) \approx X$. In this case, $\|f\|_X = \|\hat{f}\|_{\mathcal{M}(A)}$ for all $f \in A$.

It is known that if A is a Banach function algebra on X, then $\bar{A}$, the uniform closure A in $C(X)$, is a uniform algebra on X and $\mathcal{M}(\bar{A}) \subseteq \mathcal{M}(A)$. The following result is proved in [9] and we will use it in sequel.

Theorem 1.1. *Let X be a compact Hausdorff space and let A be a Banach function algebra on X. Then $\mathcal{M}(A) = \mathcal{M}(\bar{A})$ if and only if $||\hat{f}||_{\mathcal{M}(\mathcal{A})} = ||f||_X$, for all $f \in A$, where $\hat{f}$ is the Gelfand transform of f.*

For a compact plane set X, we denote the set of all complex-valued continuous functions on X that are analytic on $int(X)$ by $A(X)$, and the set of all complex-valued functions on X having an analytic extension to a neighborhood of X by $H_0(X)$. We denote the set of restriction to X of rational functions with poles off X by $R_0(X)$, and the restriction to X of polynomials by $P_0(X)$. The polynomial convex hull of X is denoted by $\widehat{X}$. By the coordinate functional on X we mean the function Z on X that maps any point to itself. We denote by $H(X)$, $R(X)$ and $P(X)$ the uniform closure of $H_0(X)$, $R_0(X)$ and $P_0(X)$, respectively. It is known that $R(X) = H(X)$, $R(X)$ is natural and $\mathcal{M}(P(X)) \approx \hat{X}$ (see[8]).

Let X be a perfect, compact plane set. We say that complex-valued function f on X is complex-differentiable at a point $a \in X$ if the limit

$$f'(a) = \lim_{\substack{z \to a \\ z \in X}} \frac{f(z) - f(a)}{z - a}$$

exists. We call $f'(a)$ the *complex-derivative* of f at a. We denoted the nth derivative of f at $a \in X$ by $f^{(n)}(a)$. We denote the set of n times continuously complex-differentiable functions on X by $D^n(X)$ and the set of infinitely complex-differentiable functions on X by $D^\infty(X)$.

Let $M = \{M_n\}_{n=0}^{\infty}$ be a sequence of positive numbers. We say that M is an *algebra sequence* if $M_0 = 1$ and

$$\frac{(m+n)!}{M_{m+n}} \leq \frac{m!}{M_m} \frac{n!}{M_n},$$

for all $m, n \in \mathbb{N} \cup \{0\}$. Let $M = \{M_n\}_{n=0}^{\infty}$ be an algebra sequence and let $d(M) = \limsup_{n\to\infty} (\frac{n!}{M_n})^{\frac{1}{n}}$. We say that $M = \{M_n\}_{n=0}^{\infty}$ is an *analytic algebra sequence* if $d(M) > 0$ and a *non-analytic algebra sequence* if $d(M) = 0$.

We need the following result about an algebra sequence $M = \{M_n\}_{n=0}^{\infty}$.

Lemma 1.2. *Let $M = \{M_n\}_{n=0}^{\infty}$ be an algebra sequence. Then*

(i) the sequence $\{(\frac{n!}{M_n})^{\frac{1}{n}}\}_{n=0}^{\infty}$ is convergent and

$$d(M) = \lim_{n\to\infty} (\frac{n!}{M_n})^{\frac{1}{n}} = inf\{(\frac{n!}{M_n})^{\frac{1}{n}} : n \in \mathbb{N} \cup \{0\}\},$$

(ii) the series $\sum_{n=0}^{\infty} \frac{n!}{M_n \rho^n}$ is convergent, if $\rho > d(M)$.

To see the proof of Lemma 1.2(i), we refer to [6, Proposition A.1.26].

For a perfect compact plane set X and an algebra sequence $M = \{M_n\}_{n=0}^{\infty}$, a *Dales-Davie algebra* associated with X and M is defined by

$$D(X, M) = \{f \in D^{\infty}(X) : \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_X}{M_n} < \infty\},$$

where the norm on $D(X, M)$ is given by

$$\|f\|_{D(X,M)} = \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_X}{M_n}.$$

Since $Z \in D(X, M)$ and M is an algebra sequence, $D(X, M)$ is a normed function algebra on X.

A compact subset X of the complex plane is *connected by rectifable arcs* if any two points of X can be joined by a rectifiable arc lying within X. For such a set, let $\delta(z, w)$ denote the *geodesic distance* between z and w; that is, the infimum of the lengths of the arcs joining z and w. Clearly δ defines a metric, the *geodesic metric*, on X.

Definition 1.3. Let X be a compact plane set which is connected by rectifiable arcs, and let $\delta(z, w)$ be the geodesic distance between z and w in X.

(i) X is *regular* if for every $z \in X$ there exists a constant C_z such that $\delta(z, w) \le C_z|z - w|$, for all $w \in X$.

(ii) X is *uniformly regular* if there exists a constant C such that $\delta(z, w) \le C|z-w|$, for all $z, w \in X$.

Dales and Davie in [6] proved that if X is finite of union of uniformly regular sets, then $D(X, M)$ is complete. In fact, if X is a finite union of regular sets, then $D^1(X)$ with the norm $||f||_{D^1(X)} = ||f||_X + ||f'||_X$ is complete and so $D(X, M)$ with the norm $|| \cdot ||_{D(X,M)}$ is complete. For some further results see [4].

We sometimes require the following condition on X which is called the $(*)$-condition.

$(*)$ There exists a constant C such that for every $z, w \in X$ and $f \in D^1(X)$,

$$|f(z) - f(w)| \leqslant C\,|z - w|\,(\|f\|_X + \|f'\|_X)\,.$$

For example, every uniformly regular set satisfies the $(*)$-condition (see [6]). The completeness of $D^1(X)$ is also concluded from the $(*)$-condition.

Let X be a compact plane set and let $\alpha \in (0,1]$. We denote by $Lip(X,\alpha)$ the complex algebra of complex-valued functions f on X for which $p_\alpha(f) = \sup\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} : z, w \in X, z \neq w\}$ is finite. For each $f \in Lip(X,\alpha)$, set $||f||_\alpha = ||f||_X + p_\alpha(f)$. Then $||.||_\alpha$ is an algebra norm on $Lip(X,\alpha)$ and $Lip(X,\alpha)$ with the norm $||.||_\alpha$ is a natural Banach function algebra on X. These algebras are called *Lipschitz algebras* of order α and were first studied by Sherbert in [13, 14].

We denote the complex algebra of complex-valued functions f on a perfect compact plane set X whose derivatives of all orders exist and $f^{(n)} \in Lip(X,\alpha)$ for all $n \in \mathbb{N} \cup \{0\}$, by $Lip^\infty(X,\alpha)$.

For a perfect compact plane set X, an algebra sequence $M = \{M_n\}_{n=0}^\infty$ and $\alpha \in (0,1]$, a *Lipschitz algebra of infinitely differentiable functions* associated with X , M and α is defined by

$$Lip(X,M,\alpha) = \{f \in Lip^\infty(X,\alpha) : \sum_{n=0}^\infty \frac{\|f^{(n)}\|_X + p_\alpha(f^{(n)})}{M_n} < \infty\},$$

where the norm on $Lip(X,M,\alpha)$ is given by

$$\|f\|_{Lip(X,M,\alpha)} = \sum_{n=0}^\infty \frac{\|f^{(n)}\|_X + p_\alpha(f^{(n)})}{M_n}.$$

Since $Z \in Lip(X,M,\alpha)$ and M is an algebra sequence, $Lip(X,M,\alpha)$ is a normed function algebra on X. If $D^1(X)$ under the norm $||f||_{D^1(X)} = ||f||_X + ||f'||_X$ is complete, then $Lip(X,M,\alpha)$ with the norm $||\cdot||_{Lip(X,M,\alpha)}$ is complete and so a Banach function algebra on X [10]. Some properties of these algebras have studied in [3, 10, 11, 12].

NOTATION. Let X be a compact plane set. For a point $\zeta \in \mathbb{C}$, the distance between ζ and X is defined by $dist(\zeta,X) = inf\{|\zeta - z| : z \in X\}$. For a non-negative real number d, we set $X_d = \{\zeta \in \mathbb{C} : dist(\zeta,X) \leq d\}$. For $z \in \mathbb{C}$ and $r > 0$, $\mathbb{D}(z,r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$ and $\Delta(z,r) = \{\zeta \in \mathbb{C} : |\zeta - z| \leq r\}$ are the open and closed disc with center at z and radius r.

Abtahi and Honary studied some properties of certain subalgebras of Dales-Davie algebras in [2]. In this paper we introduce certain subalgebras of Lipschitz algebras of infinitely differentiable functions and determine their maximal ideal spaces.

2. Certain subalgebras

Throughout this section, we assume that X is a perfect compact plane set, $M = \{M_n\}_{n=0}^\infty$ is an algebra sequence and $\alpha \in (0,1]$.

Clearly, $Lip(X,M,\alpha)$ contains the polynomials on X; that is,

$$P_0(X) \subseteq Lip(X,M,\alpha).$$

Proposition 2.1. *Suppose that $d = d(M)$ and X satisfies the $(*)$-condition. Then $H_0(X_d)$ is contained in $Lip(X,M,\alpha)$ and, moreover, the embedding of $H_0(X_d)$ in $Lip(X,M,\alpha)$ is continuous in the sense that if $f \in H_0(X_d)$ and $\{f_n\}_{n=1}^\infty$ is a sequence in $H_0(X)$ and $f_n \to f$ uniformly on a neighborhood U of X_d, then $f_n \to f$ in $Lip(X,M,\alpha)$.*

Proof . Let $f \in H_0(X_d)$. Then, there exists a neighborhood U of X_d such that f and f' are analytic on U. Choose $\rho > d$ so that $X_\rho \subseteq U$. Suppose that $z \in X$. Then $C(z,\rho) \subseteq U$, where $C(z,\rho)$ is the circle with center at z and radius ρ. Let $n \in \mathbb{N} \cup \{0\}$. By the Cauchy integral formula, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int\limits_{C(z,p)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta,$$

$$f^{(n+1)}(z) = \frac{n!}{2\pi i} \int\limits_{C(z,p)} \frac{f'(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Hence,

$$\left|f^{(n)}(z)\right| \leq \frac{n!\,\|f\|_{X_\rho}}{\rho^n}, \quad \left|f^{(n+1)}(z)\right| \leq \frac{n!\,\|f'\|_{X_\rho}}{\rho^n}.$$

Therefore,

$$\left\|f^{(n)}\right\|_X \leq \frac{n!\,\|f\|_{X_\rho}}{\rho^n}, \quad \left\|f^{(n+1)}\right\|_X \leq \frac{n!\,\|f'\|_{X_\rho}}{\rho^n}. \tag{2.1}$$

Since X satisfies the $(*)$-condition, there exists a positive constant C such that for each $g \in D^1(X)$ and for all $z, w \in X$,

$$|g(z) - g(w)| \leq C\,|z - w(|\,\|g\|_X + \|g'\|_X). \tag{2.2}$$

Applying (2.2) for $f^{(n)}$ and then (2.1), we obtain

$$\begin{aligned} |f^{(n)}(z) - f^{(n)}w| &\leq C|z-w|(\|f^{(n)}\|_X + \|(f^{(n)})'\|_X) \\ &= C|z-w|^\alpha|z-w|^{1-\alpha}(\|f^{(n)}\|_X + \|(f^{(n+1)})\|_X) \\ &\leq C|z-w|^\alpha(diamX)^{1-\alpha}(\frac{n!\,\|f\|_{X_\rho}}{\rho^n} + \frac{n!\,\|f'\|_{X_\rho}}{\rho^n}) \\ &= |z-w|^\alpha(\frac{n!C(diamX)^{1-\alpha}}{\rho^n})(\|f\|_{X_\rho} + \|f'\|_{X_\rho}), \end{aligned}$$

for all $z, w \in X$ with $z \neq w$. This implies that

$$p_\alpha(f^n) \leq (\frac{n!C(diamX)^{1-\alpha}}{\rho^n})(\|f\|_{X_\rho} + \|f'\|_{X_\rho}). \tag{2.3}$$

From (2.1) and (2.3), we give

$$\frac{\left\|f^{(n)}\right\|_X + p_\alpha\left(f^{(n)}\right)}{M_n} \leq \frac{n!}{\rho^n M_n}[\|f\|_{X_\rho} + C(diamX)^{1-\alpha}(\|f\|_{X_\rho} + \|f'\|_{X_\rho})]. \tag{2.4}$$

Since the series $\sum\limits_{n=0}^{\infty} \frac{n!}{\rho^n M_n}$ is convergent by Lemma 1.2(ii) and (2.4) holds for all $n \in \mathbb{N} \cup \{0\}$, we conclude that the series $\sum\limits_{n=0}^{\infty} \frac{\left\|f^{(n)}\right\|_X + p_\alpha\left(f^{(n)}\right)}{M_n}$ is convergent and

$$\sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_X + p_\alpha\left(f^{(n)}\right)}{M_n} \leq \lambda \sum_{n=0}^{\infty} \frac{n!}{\rho^n M_n},$$

where

$$\lambda = \|f\|_{X_\rho} + C(diamX)^{1-\alpha}(\|f\|_{X_\rho} + \|f'\|_{X_\rho}).$$

This implies that $f \in Lip(X, M, \alpha)$ and

$$\|f\|_{Lip(X,M,\alpha)} \le \lambda \sum_{n=0}^{\infty} \frac{n!}{\rho^n M_n}. \tag{2.5}$$

Therefore, $H_0(X_d)$ is contained in $Lip(X, M, \alpha)$.

Now, suppose that $f \in H_0(X_d)$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence on $H_0(X_d)$ such that $f_n \to f$ uniformly in a neighborhood U of X_d. By [5, Theorem VII.2.1], ${f_n}' \to f'$ uniformly on every compact subset of U. We can choose $\rho > d$ such that $X_d \subseteq X_\rho \subseteq U$, so that $\lim_{n\to\infty} \|f_n - f\|_{X_\rho} = 0$ and $\lim_{n\to\infty} \|{f_n}' - f'\|_{X_\rho} = 0$. Let $n \in \mathbb{N}$. Then $f_n - f \in Lip(X, M, \alpha)$ and, by above argument, we have

$$0 \le \|f_n - f\|_{Lip(X,M,\alpha)} \le \gamma_n \sum_{m=0}^{\infty} \frac{m!}{\rho^m M_m},$$

where

$$\gamma_n = \|f_n - f\|_{X_\rho} + C(diamX)^{1-\alpha}(||f_n - f||_{X_\rho} + \|{f_n}' - f'\|_{X_\rho}).$$

Clearly, $\lim_{n\to\infty} \gamma_n = 0$. Therefore,

$$\lim_{n\to\infty} \|f_n - f\|_{Lip(X,M,\alpha)} = 0,$$

and so the proof is complete. □

Proposition 2.2. *Suppose that X satisfies the $(*)$-condition. If d is a real number with $d(M) \le d$, then $R_0(X_d)$ is contained in $Lip(X, M, \alpha)$.*

Proof . Since $M = \{M_n\}_{n=0}^{\infty}$ is an algebra sequence, X satisfies the $(*)$-condition and $\alpha \in (0, 1]$, we conclude that $H_0(X_{d(M)}) \subseteq Lip(X, M, \alpha)$ by Theorem 2.1. Now, let d be a real number with $d(M) \le d$. Then $R_0(X_d) \subseteq R_0(X_{d(M)})$. On the other hand, $R_0(X_{d(M)}) \subseteq H_0(X_{d(M)})$. Therefore, $R_0(X_d)$ is contained in $Lip(X, M, \alpha)$. □

Corollary 2.3. *If $d(M) = 0$, then $R_0(X)$ ic contained in $Lip(X, M, \alpha)$.*

Abtahi and Honary in [2] proved that if X is a perfect compact plane set such that $(D(X, M), \|\cdot\|_{D(X,M)})$ is complete, then

(i) $H_0(X_d) \subseteq D(X, M)$, when $d = d(M)$,
(ii) $R_0(X_d) \subseteq D(X, M)$, if and only if $d(M) \le d$,
(iii) $R_0(X_{d(M)}) \subseteq D(X, M)$ if and only if $d(M) = 0$.

They also defined the closed subalgebras $D_{P,d}(X, M), D_{R,d}(X, M)$, and $D_{H,d}(X, M)$ of $D(X, M)$ to be the closure of $P_0(X_d), R_0(X_d)$ and $H_0(X_d)$ in $D(X, M)$, respectively (see [2, Definition 2.4]). Similarly, we introduce certain subalgebras of $Lip(X, M, \alpha)$ as the following.

Definition 2.4. Let $d = d(M)$ and let X satisfying the $(*)$-condition. We define $Lip_{P,d}(X, M)$,$Lip_{R,d}(X, M)$, and $Lip_{H,d}(X, M)$ to be the closure of $P_0(X_d)$, $R_0(X_d)$, and $H_0(X_d)$ in $Lip(X, M, \alpha)$, respectively.

It is clear that $Lip_{P,d}(X, M, \alpha)$, $Lip_{R,d}(X, M, \alpha)$, and $Lip_{H,d}(X, M, \alpha)$ are Banach function algebras on X with the norm $\|\cdot\|_{Lip(X,M,\alpha)}$.

Definition 2.5. Let Y be a plane set and let $z_0 \in Y$. We say that Y is *star-shaped* with respect to z_0, if for each $z \in X$ the closed segment $[z_0, z]$ is contained in X.

Lemma 2.6. *Let $z_0 \in \hat{X}$ and let $\hat{X}$ be star-shaped with respect to z_0. Suppose that $\rho > 0$. If $f \in H_0(\widehat{X_\rho})$, then there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials such that*

$$\lim_{n\to\infty} \|p_n - f\|_{X_\rho} = 0, \quad \textit{and} \quad \lim_{n\to\infty} \|{p_n}' - f'\|_{X_\rho} = 0.$$

Proof . Let $f \in H_0(\widehat{X_\rho})$. Then $f' \in H_0(\widehat{X_\rho})$. By Runge's theorem, there exists a sequence $\{q_n\}_{n=1}^{\infty}$ of polynomials such that

$$\lim_{n\to\infty} \|q_n - f'\|_{X_\rho} = 0.$$

Let $\{p_n\}_{n=1}^{\infty}$ be the sequence of polynomials with ${p_n}' = q_n$ on X_ρ and $p_n(z_0) = f(z_0)$ for all $n \in \mathbb{N}$. Let $z \in X_\rho$. Then there exists $w_z \in X$ such that $|w_z - z| = \rho$. We assume that $C_z = [z_0, w_z] + [w_z, z]$. Then

$$\begin{aligned}
|p_n(z) - f(z)| &= |[\int_{C_z} q_n(\zeta)\, d\zeta + p_n(z_0)] - [\int_{C_z} f'(\zeta)\, d\zeta + f(z_0)]| \\
&= |\int_{C_z} [q_n(\zeta) - f'(\zeta)]\, d\zeta| \\
&\le (|w_z - z_0| + |z - w_z|)\, \|q_n - f'\|_{X_\rho} \\
&\le \left(\rho + diam\hat{X}\right) \|q_n - f'\|_{X_\rho},
\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that $||p_n - f||_{X_\rho} \le (\rho + diam\hat{X})||q_n - f'||_{X_\rho}$ and so $\lim_{n\to\infty} \|p_n - f\|_{X_\rho} = 0$. Hence, the proof is complete. □

Theorem 2.7. *Suppose that X satisfies the $(*)$-condition, $z_0 \in \hat{X}$ and $\hat{X}$ is star-shaped with respect to z_0. If $d = d(M)$, then $H_0(\widehat{X_d})$ is contained in $Lip_{P,d}(X, M, \alpha)$.*

Proof . For $f \in H_0(\widehat{X_d})$, there exists $\rho > d$ such that $f \in H_0(\widehat{X_\rho})$. By Lemma 2.6, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials such that

$$\lim_{n\to\infty} \|p_n - f\|_{X_\rho} = 0, \quad \textit{and} \quad \lim_{n\to\infty} \|{p_n}' - f'\|_{X_\rho} = 0.$$

Let $n \in \mathbb{N}$. Since $H_0(\widehat{X_\rho}) \subseteq H_0(X_\rho)$ and X satisfies the $(*)$-condition, we conclude that $p_n - f \in Lip(X, M, \alpha)$ by Proposition 2.1 and

$$\|p_n - f\|_{Lip(X,M,\alpha)} \le \eta_n \sum_{m=0}^{\infty} \frac{m!}{\rho^m M_m},$$

by given argument in the proof of Proposition 2.1, where

$$\eta_n = \|p_n - f\|_{X_\rho} + C(diamX)^{1-\alpha}(||p_n - f||_{X_\rho} + \|{p_n}' - f'\|_{X_\rho}).$$

Since $\lim_{n\to\infty} \eta_n = 0$, we conclude that

$$\lim_{n\to\infty} \|p_n - f\|_{Lip(X,M,\alpha)} = 0,$$

and so $f \in Lip_{P,d}(X, M, \alpha)$. Therefore,

$$H_0(\widehat{X_d}) \subseteq Lip_{P,d}(X, M, \alpha),$$

and the proof is complete. □

3. Extensions of infinitely differentiable Lipschitz functions

Throughout this section, we assume that X is a perfect compact plane set, $M = \{M_n\}_{n=0}^{\infty}$ is an analytic sequence with $d = d(M) > 0$.

Let $f \in D(X, M)$ and $z \in X$. By Lemma 1.2,we have

$$\frac{f^{(k)}(z)}{k!}(\zeta - z)^k \leq \frac{\|f^{(k)}\|_X}{k!}d^k \leq \frac{\|f^{(k)}\|_X}{M_k},$$

for all $\zeta \in \Delta(z, d)$ and for all $k \in \mathbb{N} \cup \{0\}$. This implies that the power series $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!}(\zeta - z)^k$ is uniformly convergent on $\Delta(z, d)$. Therefore, the function $F_z : \Delta(z, d) \to \mathbb{C}$ defined by

$$F_z(\zeta) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!}(\zeta - z)^k, \tag{3.1}$$

is continuous on $\Delta(z, d)$ and analytic on $\mathbb{D}(z, d)$.

Abtahi and Honary obtained the following result in [2].

Theorem 3.1 (see [2, Theorem 3.2]). *Let X be a perfect, compact plane set such that $(D(X, M), \|\cdot\|_{D(X,M)})$ is complete. Then every $f \in D_{H,d}(X, M)$ has a unique extension F in $A(X_d)$ and $||F||_{X_d} \leq ||f||_{D(X,M)}$.*

Theorem 3.2 (see [2, Corollary 3.3]). *Let X be a perfect, compact plane set such that $(D(X, M), \|\cdot\|_{D(X,M)})$ is complete. Then every $f \in D_{P,d}(X, M)$ has a unique extension F in $A(\widehat{X_d})$.*

Theorem 3.3 (see [2, Corollary 3.5]). *Let X be a regular set such that $(D(X, M), \|\cdot\|_{D(X,M)})$ is complete. Then $D(X, M)$ is contained in $H_0(X)$.*

Since $Lip^{\infty}(X, \alpha) \subseteq D^{\infty}(X)$ and $||f^{(n)}||_X \leq ||f^{(n)}||_X + p_{\alpha}(f^{(n)})$ for each $f \in Lip^{\infty}(X, \alpha)$ and for all $n \in \mathbb{N} \cup \{0\}$, we conclude that $Lip(X, M, \alpha) \subseteq D(X, M)$. Hence, $Lip_{H,d}(X, M, \alpha) \subseteq D_{H,d}(X, M)$ and $Lip_{P,d}(X, M, \alpha) \subseteq D_{P,d}(X, M)$, when X satisfies the $(*)$-condition. Therefore, we give the following results.

Theorem 3.4. *Suppose that X satisfies the $(*)$-condition. Then every $f \in Lip_{H,d}(X, M, \alpha)$ has a unique extension F in $A(X_d)$ and*

$$||F||_{X_d} \leq ||f||_{D(X,M)} \leq ||f||_{Lip(X,M,\alpha)}.$$

Theorem 3.5. *Suppose that X satisfies the $(*)$-condition. Then, every $f \in Lip_{P,d}(X, M, \alpha)$ has a unique extension F in $A(\widehat{X_d})$ and*

$$||F||_{\widehat{X_d}} \leq ||f||_{D(X,M)} \leq ||f||_{Lip(X,M,\alpha)}.$$

Theorem 3.6. *Let X be a regular set satisfying the $(*)$-condition. Then $Lip(X, M, \alpha)$ is contained in $H_0(X)$.*

4. Maximal ideal space

Throughout this section, we assume that X is a perfect compact plane set, $M = \{M_n\}_{n=0}^{\infty}$ is an algebra sequence and $\alpha \in (0, 1]$.

Dalse and Davie in [6] proved that if $d(M) = 0$, then $D(X, M)$ is complete and $\mathcal{M}(D_R(X, M)) \approx X$ and $\mathcal{M}(D_P(X, M)) \approx \widehat{X}$. Honary and Mahyar proved [8] that if $d(M) = 0$, then $Lip(X, M, \alpha)$ is complete and $Lip_{R,d}(X, M, \alpha)$ is natural and $\mathcal{M}(Lip_{p,d}(X, M, \alpha)) \approx \widehat{X}$.

Abtahi and Honary obtained the following result in [2].

Theorem 4.1 (see [2, Theorem 4.1]). *Let X be a perfect, compact plane set such that $D(X, M)$ is complete. Then $\mathcal{M}(D_{R,d}(X, M)) \approx X_d$ and $\mathcal{M}(D_{P,d}(X, M)) \approx \widehat{X_d}$, where $d = d(M)$.*

We now determine the maximal ideal space of the Banach function subalgebras $Lip_{R,d}(X, M, \alpha)$ and $Lip_{P,d}(X, M, \alpha)$ of $Lip(X, M, \alpha)$.

Theorem 4.2. *Suppose that X satisfies the $(*)$-condition. If $d = d(M)$, then $\mathcal{M}(Lip_{R,d}(X, M, \alpha)) \approx X_d$.*

Proof . Let $\zeta \in X_d$. We define $h_\zeta : Lip_{R,d}(X, M, \alpha) \to \mathbb{C}$ by $h_\zeta(f) = F(\zeta)$, where F is the unique extension of f in $A(X_d)$ given by Theorem 3.4. We deduce that $h_\zeta \in \mathcal{M}(Lip_{R,d}(X, M, \alpha))$ by Theorem 3.4.

Let $h \in \mathcal{M}(Lip_{R,d}(X, M, \alpha))$ and let $\zeta = h(Z)$, where Z is the coordinate functional on X. We claim that $\zeta \in X_d$. Since $\zeta 1 - Z \in Lip_{R,d}(X, M, \alpha)$ and $h(\zeta 1 - Z) = \zeta h(1) - h(Z) = \zeta - \zeta = 0$, we conclude that the function $\zeta 1 - Z$ is not invertible in $Lip_{R,d}(X, M, \alpha)$. Now, we assume that $\zeta \in \mathbb{C} \setminus X_d$. Then $\frac{1}{\zeta 1 - Z} \in R_0(X_d) \subseteq Lip_{R,d}(X, M, \alpha)$. Hence $\zeta 1 - Z$ is invertible in $Lip_{R,d}(X, M, \alpha)$. This contradiction implies that our claim is justified. It is easy to see that $h(f|_X) = f(\zeta)$ for all $f \in R_0(X_d)$. Now, let $f \in Lip_{R,d}(X, M, \alpha)$. Then there is a sequence $\{f_n\}_{n=1}^{\infty}$ in $R_0(X_d)$ such that

$$\lim_{n\to\infty} ||f_n|_X - f||_{Lip(X,M,\alpha)} = 0. \tag{4.1}$$

This implies that $f_n|_X \to f$ uniformly on X. Since $f \in Lip_{H,d}(X, M, \alpha)$, we conclude that f has a unique extension F in $A(X_d)$ and $||F||_{X_d} \le ||f||_{Lip(X,M,\alpha)}$, by Theorem 3.4. Then $f_n - F$ is the unique extension $f_n|_X - f$ in $A(X_d)$ and so $||f_n - F||_{X_d} \le ||f_n|_X - f||_{Lip(X,M,\alpha}$, for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} f_n(\zeta) = F(\zeta), \tag{4.2}$$

by (4.1). From (4.1), continuity of h on $Lip_{R,d}(X, M, \alpha)$ and (4.2), we deduce that

$$h(f) = \lim_{n\to\infty} h(f_n|_X) = \lim_{n\to\infty} f_n(\zeta) = F(\zeta) = h_\zeta(f).$$

Since f assumed that an arbitrary element of $Lip_{R,d}(X, M, \alpha)$, we conclude that $h = h_\zeta$. Therefore, $\mathcal{M}(Lip_{R,d}(X, M, \alpha)) \subseteq \{h_\zeta : \zeta \in X_d\}$. Hence, $\mathcal{M}(Lip_{R,d}(X, M, \alpha)) \approx X_d$. □

Theorem 4.3. *Suppose that X satisfies the $(*)$-condition, $z_0 \in \hat{X}$, and $\hat{X}$ is star-shaped with respect to z_0. If $d = d(M)$, then $\mathcal{M}(Lip_{P,d}(X, M, \alpha)) \approx \widehat{X_d}$.*

Proof . Let $\zeta \in \widehat{X_d}$. We define $h_\zeta : Lip_{P,d}(X, M, \alpha) \to \mathbb{C}$ by $h_\zeta(f) = F(\zeta)$, where F is the unique extension of f in $A(\widehat{X_d})$ given by Theorem 3.5. We deduce that $h_\zeta \in \mathcal{M}(Lip_{P,d}(X, M, \alpha))$ by Theorem 3.5. Let $h \in \mathcal{M}(Lip_{P,d}(X, M, \alpha))$ and let $\zeta = h(Z)$, where Z is the coordinate functional on X. We claim that $\zeta \in \widehat{X_d}$. Since $\zeta 1 - Z \in Lip_{P,d}(X, M, \alpha)$ and $h(\zeta 1 - Z) = \zeta h(1) - h(Z) = \zeta - \zeta = 0$, we conclude that the function $\zeta 1 - Z$ is not invertible in $Lip_{P,d}(X, M, \alpha)$. Now, we assume that $\zeta \in \mathbb{C}\backslash\widehat{X_d}$. Then $\frac{1}{\zeta 1 - Z} \in H_0(\widehat{X_d})$ and so $\frac{1}{\zeta 1 - Z} \in Lip_{P,d}(X, M, \alpha)$ by Theorem 2.7. This contradiction implies that our claim is justified. It is easy to see that $h_\zeta(p) = p(\zeta)$ for all $p \in P_0(\widehat{X_d})$. Now let $f \in Lip_{P,d}(X, M, \alpha)$. Then there is a sequence $\{p_n\}_{n=1}^{\infty}$ in $P_0(\widehat{X_d})$ such that

$$\lim_{n\to\infty} \|p_n|_X - f\|_{Lip(X,M,\alpha)} = 0. \tag{4.3}$$

This shows that $p_n|_X \to f$ uniformly on X. Since $f \in Lip_{P,d}(X, M, \alpha)$, we conclude that f has a unique extension F in $A(\widehat{X_d})$ and $||F||_{\widehat{X_d}} \le ||f||_{Lip(X,M,\alpha)}$ by Theorem 3.5. Therefore, $p_n|_{\widehat{X_d}} - F$ is the unique extension $p_n|_X - f$ in $A(\widehat{X_d})$ and so that $||p_n|_{\widehat{X_d}} - F||_{\widehat{X_d}} \le ||p_n|_{\widehat{X_d}} - f||_{Lip(X,M,\alpha)}$, for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} p_n(\zeta) = F(\zeta), \tag{4.4}$$

by (4.3). From (4.3), continuity of f on $Lip_{P,d}(X, M, \alpha)$ and (4.4), we deduce that

$$h(f) = \lim_{n\to\infty} h(p_n|_X) = \lim_{n\to\infty} p_n(\zeta) = F(\zeta) = h_\zeta(f).$$

Since f assumed that an arbitrary element of $Lip_{P,d}(X, M, \alpha)$, we conclude that $h = h(\zeta)$. Therefore, $\mathcal{M}(Lip_{P,d}(X, M, \alpha)) \subseteq \{h_\zeta : \zeta \in \widehat{X_d}\}$. Hence, $\mathcal{M}(Lip_{P,d}(X, M, \alpha)) \approx \widehat{X_d}$. $\square$

To continue we need some formulae from combinatorial analysis. For $m, n \in \mathbb{N}$ with $n \ge m$, we take $S(m, n)$ as the set of all $\underline{a} = (a_1, ..., a_n) \in (\mathbb{N} \cup \{0\})^n$ such that $\sum_{k=1}^{n} a_k = m$ and $\sum_{k=1}^{n} k a_k = n$. For any $m \in \mathbb{N}$ and any sequence $\{A_k\}_{k=1}^{\infty}$ of positive numbers, by [1, Formula B3, P. 823],

$$(\sum_{k=1}^{\infty} A_k)^m = m! \sum_{n=m}^{\infty} \sum_{\underline{a}\in S(m,n)} \frac{1}{\prod\limits_{k=1}^{n} a_k!} \prod_{k=1}^{n} (A_k)^{a_k}. \tag{4.5}$$

The following equality for higher derivatives of composite functions is known as the Faa di Bruno's Formula (See[1, P. 823]):

$$(Fof)^{(n)} = \sum_{m=1}^{n} (F^{(m)} of) \sum_{\underline{a}\in S(m,n)} \frac{n!}{\prod\limits_{k=1}^{n} a_k!} \prod_{k=1}^{n} (\frac{f^{(k)}}{k!})^{a_k} \quad (n \in \mathbb{N}). \tag{4.6}$$

The following lemma is useful and found in [2].

Lemma 4.4 (see [2, Lemma 4.2]). *Let $K > 0$ and let $\{\varepsilon_m\}_{m=0}^{\infty}$ be a sequence of positive numbers with $\lim\limits_{m\to\infty} \varepsilon_m = 0$. Then*

$$\limsup_{p\to\infty} \left(\sum_{m=0}^{p} \binom{p}{m} (\varepsilon_m)^m K^{p-m} \right)^{\frac{1}{p}} \le K.$$

Lemma 4.5. *Let X be a perfect, compact plane set such that $Lip(X, M, \alpha)$ is complete. Suppose that $A_L = \{f \in Lip^\infty(X, \alpha) : f' \in Lip(X, M, \alpha)\}$.*
(i) The set A_L is a subalgebra of $Lip(X, M, \alpha)$ containing 1_X and separating the points of X.
(ii) If $d(M) = 0$, then $||\widehat{f}||_{\mathcal{M}(Lip(X,M,\alpha))} = ||f||_X$ for all $f \in A_L$.

Proof . (i) Clearly, $\mathbb{C}1 \subseteq A_L$. Let $f \in A_L$ and $g = f'$. Since $\frac{n+1}{M_{n+1}} \leq \frac{1}{M_1 M_n}$ for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{||f^{(n)}||_\alpha}{M_n} &= \frac{||f||_\alpha}{M_0} + \sum_{n=1}^{\infty} \frac{||f^{(n)}||_\alpha}{M_n} \\
&= \frac{||f||_\alpha}{M_0} + \sum_{n=0}^{\infty} \frac{||f^{(n+1)}||_\alpha}{M_{n+1}} \\
&= \frac{||f||_\alpha}{M_0} + \sum_{n=0}^{\infty} \frac{n+1}{M_{n+1}} \frac{||g^{(n)}||_\alpha}{n+1} \\
&\leq \frac{||f||_\alpha}{M_0} + \sum_{n=0}^{\infty} \frac{1}{M_1 M_n} \frac{||g^{(n)}||_\alpha}{n+1} \\
&= \frac{||f||_\alpha}{M_0} + \frac{1}{M_1} \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{||g^{(n)}||_\alpha}{M_n} \\
&\leq \frac{||f||_\alpha}{M_0} + \frac{1}{M_1} \sum_{n=0}^{\infty} \frac{||g^{(n)}||_\alpha}{M_n} \\
&< \infty.
\end{aligned}$$

Hence, $f \in Lip(X, M, \alpha)$ and so that $A_L \subseteq Lip(X, M, \alpha)$. It is clear that $Z \in A_L$. Thus, A_L separates the points of X. Moreover, A_L contains the constant functions on X. We now show that A_L is a subalgebra of $Lip(X, M, \alpha)$. Let $f, g \in A_L$. Clearly, A_L is closed under the addition and scalar multiplication. Let $f, g \in A_L$. Since $f', g' \in Lip(X, M, \alpha)$ and $A_L \subseteq Lip(X, M, \alpha)$, we deduce that $f, g \in Lip(X, M, \alpha)$. Thus, $f'g + fg' \in Lip(X, M, \alpha)$ and so $(fg)' \in Lip(X, M, \alpha)$. Therefore, $fg \in A_L$ and so A_L is a subalgebra of $Lip(X, M, \alpha)$.
(ii) Let $d(M) = 0$. We deduce that $Lip(X, M, \alpha)$ is a Banach function algebra on X. Let $f \in A$ and $j \in \mathbb{N}$. Suppose that $F(z) = z^j$ $(z \in \mathbb{C})$. Then $f^j = F \circ f$. By the Faa di Bruno's formula (4.6), we have

$$\begin{aligned}
& ||f^j||_{Lip(X,M,\alpha)} \\
= \; & ||F \circ f||_{Lip(X,M,\alpha)} \\
= \; & \frac{||F \circ f||_\alpha}{M_0} + \sum_{n=1}^{\infty} \frac{||(F \circ f)^{(n)}||_\alpha}{M_n} \\
= \; & \frac{||f^j||_\alpha}{M_0} + \sum_{n=1}^{\infty} \| \frac{1}{M_n} \sum_{m=1}^{\infty} \left(F^{(m)} o f\right) \sum_{a \in S(m,n)} \frac{n!}{\prod_{k=1}^{n} a_k!} \prod_{k=1}^{n} \left(\frac{f^{(k)}}{k!} \right)^{a_k} \|_\alpha \\
\leqslant \; & \frac{||f^j||_\alpha}{M_0} + \sum_{n=1}^{\infty} \sum_{m=1}^{\min\{j,n\}} \| m! \begin{pmatrix} j \\ m \end{pmatrix} \frac{f^{j-m}}{M_n} \sum_{\underline{a} \in S(m,n)} \frac{n!}{\prod_{k=1}^{n} a_k!} \prod_{k=1}^{n} \left(\frac{f^{(k)}}{k!} \right)^{a_k} \|_\alpha \\
\leqslant \; & \frac{||f^j||_\alpha}{M_0} + \sum_{n=1}^{\infty} \sum_{m=1}^{\min\{j,n\}} m! \begin{pmatrix} j \\ m \end{pmatrix} \frac{\|f\|_\alpha^{j-m}}{M_n} \sum_{a \in S(m,n)} \frac{n!}{\prod_{k=1}^{n} a_k!} \prod_{k=1}^{n} \left(\frac{\|f^{(k)}\|_\alpha}{k!} \right)^{a_k} .
\end{aligned}$$

After interchanging the order of summation, we have

$$
\begin{aligned}
&||f^j||_{Lip(X,M,\alpha)}\\
&\le \frac{||f^j||_\alpha}{M_0} + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} m! \sum_{n=m}^{\infty} \sum_{\underline{a}\in S(m,n)} \frac{1}{\prod_{k=1}^{n} a_k!} \frac{n!}{M_n} \prod_{k=1}^{n} \left(\frac{\left\|f^{(k)}\right\|_\alpha}{k!}\right)^{a_k}\\
&\le \frac{||f^j||_\alpha}{M_0} + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} m! \sum_{n=m}^{\infty} \sum_{\underline{a}\in S(m,n)} \frac{1}{\prod_{k=1}^{n} a_k!} \frac{\prod_{k=1}^{n} (P_{k-1})^{a_k}}{P_n} \prod_{k=1}^{n} \left(\frac{\left\|f^{(k)}\right\|_\alpha}{kM_{k-1}}\right)^{a_k}\\
&\le \frac{||f^j||_\alpha}{M_0} + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} m! \sum_{n=m}^{\infty} \sum_{\underline{a}\in S(m,n)} \frac{1}{\prod_{k=1}^{n} a_k!} \frac{P_{n-m}}{P_n} \prod_{k=1}^{n} \left(\frac{\left\|f^{(k)}\right\|_\alpha}{kM_{k-1}}\right)^{a_k}\\
&\le \frac{||f^j||_\alpha}{M_0} + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} \frac{m!}{P_m} \sum_{n=m}^{\infty} \sum_{\underline{a}\in S(m,n)} \frac{1}{\prod_{k=1}^{n} a_k!} \prod_{k=1}^{n} \left(\frac{\left\|f^{(k)}\right\|_\alpha}{kM_{k-1}}\right)^{a_k}\\
&\le \frac{||f^j||_\alpha}{M_0} + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} m! \sum_{n=m}^{\infty} \sum_{\underline{a}\in S(m,n)} \frac{1}{\prod_{k=1}^{n} a_k!} \prod_{k=1}^{n} \left(\frac{\left\|(f')^{(k-1)}\right\|_\alpha}{(P_m)^{\frac{1}{m}} M_{k-1}}\right)^{a_k}.
\end{aligned}
$$

Choosing $A_k = \frac{||(f')^{(k-1)}||_\alpha}{(P_m)^{\frac{1}{m}} M_{k-1}}$ for all $k \in \mathbb{N}$ and applying Formula (4.5), we have

$$
\begin{aligned}
||f^j||_{Lip(X,M,\alpha)} &\le \frac{||f^j||_\alpha}{M_0} + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} \left(\sum_{k=1}^{\infty} \frac{||(f')^{(k-1)}||_\alpha}{(P_m)^{\frac{1}{m}} M_{k-1}}\right)^m\\
&= \frac{||f^j||_\alpha}{M_0} + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} \left(\sum_{k=0}^{\infty} \frac{||(f')^{(k)}||_\alpha}{(P_m)^{\frac{1}{m}} M_{k}}\right)^m\\
&= \frac{||f^j||_\alpha}{M_0} + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} \frac{1}{P_m} (||f'||_{Lip(X,M,\alpha)})^m.
\end{aligned}
$$

If $f' = 0$, then $||f^j||_{Lip(X,M,\alpha)} \le ||f^j||_\alpha$ and so

$$||\widehat{f}||_{M(Lip(X,M,\alpha))} \le \lim_{j\to\infty} (||f^j||_\alpha)^{\frac{1}{j}} = \|\hat{f}\|_{\mathcal{M}(Lip(X,\alpha))} = ||f||_X.$$

Suppose that $f' \neq 0$. Take $K = ||f||_X$ and define the sequence $\{\varepsilon_m\}_{m=0}^{\infty}$ by

$$\varepsilon_0 = 1, \quad and \quad \varepsilon_m = \left(\frac{m!}{M_m}\right)^{\frac{1}{m}} ||f'||_{Lip(X,M,\alpha)} \quad (m \in \mathbb{N}).$$

Then $\lim_{m\to\infty} \varepsilon_m = d(M)||f'||_{Lip(X,M,\alpha)} = 0$. By Lemma 4.4,

$$\limsup_{j\to\infty} \left(\sum_{m=0}^{j} \binom{j}{m} (\varepsilon_m)^m K^{j-m}\right)^{\frac{1}{j}} \le K.$$

Hence,

$$\limsup_{j\to\infty} \left((||f||_\alpha)^j + \sum_{m=1}^{j} \binom{j}{m} \frac{1}{P_m} (||f'||_{Lip(X,M,\alpha)})^m ||f||_\alpha^{j-m}\right)^{\frac{1}{j}} \le K.$$

On the other hand, for all $j \in \mathbb{N}$ we have

$$||f^j||_{Lip(X,M,\alpha)} \le ||f^j||_\alpha + \sum_{m=1}^{j} \binom{j}{m} ||f||_\alpha^{j-m} \frac{1}{P_m} (||f'||_{Lip(X,M,\alpha)})^m.$$

Therefore, $\limsup_{j\to\infty}(||f^j||_{Lip(X,M,\alpha)})^{\frac{1}{j}} \le K = ||f||_X$, and so

$$||\widehat{f}||_{\mathcal{M}(Lip(X,M,\alpha))} \le ||f||_X.$$

Since $Lip(X,M,\alpha)$ is a Banach function algebra on X, we have $||f||_X \le ||\widehat{f}||_{\mathcal{M}(Lip(X,M,\alpha))}$. Consequently, $||\widehat{f}||_{\mathcal{M}(Lip(X,M,\alpha))} = ||f||_X$. □

Theorem 4.6. *Let $M = \{M_n\}_{n=0}^{\infty}$ be a non-analytic algebra sequence, and let X be a uniformly regular set. Suppose that $A_L = \{f \in Lip^{\infty}(X,\alpha) : f' \in Lip(X,M,\alpha)\}$, and $B_L = \overline{A_L}^L$, the closure of A_L in $Lip(X,M,\alpha)$. Then B_L is a natural Banach function subalgebra of $Lip(X,M,\alpha)$ on X.*

Proof . By Lemma 4.5, A_L is a subalgebra of $Lip(X,M,\alpha)$ containing 1_X and separating the points of X. Since $M = \{M_n\}_{n=0}^{\infty}$ is a non-analytic algebra sequence, $Lip(X,M,\alpha)$ is a Banach function algebra on X. Thus, B_L is a Banach function algebra on X with the norm $||.||_{Lip(\widehat{X},M,\alpha)}$ and so

$$||\widehat{g}||_{\mathcal{M}(Lip(X,M,\alpha))} = ||\widehat{g}||_{\mathcal{M}(B_L)}, \qquad \forall g \in B_L. \tag{4.7}$$

We now show that B_L is natural. Using Theorem 1.1, it is enough to show that $\overline{B_L}$, the uniform closure of B_L in $C(X)$, is natural and $||\hat{f}||_{\mathcal{M}(B_L)} = ||f||_X$ for all $f \in B_L$. Since $M = \{M_n\}_{n=0}^{\infty}$ is a non-analytic sequence, $R_0(X) \subseteq A_L$. It is clear that $B_L \subseteq D^1(X)$. Since X is a uniformly regular set, $D^1(X) \subseteq R(X)$ by [4,Theorem 1.5(iv)]. Thus, $R_0(X) \subseteq B_L \subseteq R(X)$ so that $\overline{B_L} = R(X)$, and $\overline{B_L}$ is natural by naturality of $R(X)$. Let $f \in B_L$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in A such that $\lim_{n\to\infty} ||f_n - f||_{Lip(X,M,\alpha)} = 0$. Since

$$||f_n - f||_X \le ||\widehat{f_n} - \widehat{f}||_{\mathcal{M}(B_L)} = ||\widehat{f_n} - \widehat{f}||_{\mathcal{M}(Lip(X,M,\alpha))} \le ||f_n - f||_{Lip(X,M,\alpha)},$$

for all $n \in \mathbb{N}$, we conclude that

$$\lim_{n\to\infty} ||\widehat{f_n}||_{\mathcal{M}(Lip(X,M,\alpha))} = ||\widehat{f}||_{\mathcal{M}(Lip(X,M,\alpha))} \;\; \textit{and} \;\; \lim_{n\to\infty} ||f_n||_X = ||f||_X.$$

Since $f_n \in A_L$ for all $n \in \mathbb{N}$, we have $||\widehat{f_n}||_{\mathcal{M}(Lip(X,M,\alpha))} = ||f_n||_X$ for all $n \in \mathbb{N}$ by Lemma 4.5. Therefore, $||\widehat{f}||_{\mathcal{M}(Lip(X,M,\alpha))} = ||f||_X$. Consequently, $||\widehat{f_n}||_{\mathcal{M}(B_L)} = ||f||_X$ by (4.7). This completes the proof. □

Question. Is the subalgebra $A_L = \{f \in Lip^{\infty}(X,\alpha) : f' \in Lip(X,M,\alpha)\}$ dense in $Lip(X,M,\alpha)$?

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, U. S. Department of Commerce, Washington, 1964.

[2] M. Abtahi and T. G. Honary, *Properties of certain subalgebras of Dalse-Davie algebras*, Glasgow Math. J. 49 (2007) 225-233.

[3] D. Alimohammadi, *Non-weakly amenable subalgebras of $A(K)$ and $D^1(K)$*, For East J. Math. Sci. 2(5)(2000), 739-766.

[4] W. J. Bland and J. F. Feinestein *Completion of normed algebras of differentiable functions*, Studia Mathematica 170 (2005), 89-111.

[5] J. B. Conway, *Functions of One Complex Variable*, Second Edition Springer-Verlag, New York, 1995.

[6] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs. New Series, Volume 24, The Clarendon Press, Oxford, 2000.

[7] H. G. Dales and A. M. Davie, *Quasi-analytic Banach function algebras*, J. Functional Analysis 13 (1973), 28-50.
[8] T. W. Gamelin, *Uniform Algebras*, Second Edition, Chelsea Press, New York, 1984.
[9] T. G. Honary, *Relation between Banach function algebras and their uniform closures*, Proc. Amer. Math. Soc. 109 (1990), 337-342.
[10] T. G. Honary and H. Mahyar, *Approximation in Lipschitz algebras of infinitely differentiable functions*, Bull. Korean Math. Soc. 36 (1999), 629-636.
[11] H. Mahyar, *Compact endomorphisms of infinitely differentiable Lipschitz algebras*, Rocky Mountain J. Math. 39(1)(2009), 193-217.
[12] H. Mahyar, *Quasicompact and Riesz endomorphisms of infinitely differentiable Lipscgitz algebras*, Southeast Asian Bull. Math. 35 (2011), 249-259.
[13] D. R. Sherbert, *Banach algebras of Lipschitz functions*, Pacific J. Math. 13 (1963), 1387-1399.
[14] D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. 111 (1964), 240-272.